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THE NEUMANN LAPLACIAN OF A JELLY ROLL

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ABSTRACT. We consider the Laplacian with Neumann boundary conditions of a bounded connected region obtained by removing a suitable infinite spiral from an annulus. We show that the spectrum has an absolutely continuous component.

This note is a contribution to the study of the spectral properties of Neumann Laplacians, a subject of several recent papers [2–4]. Consider the curve, Γ, in $\mathbb{R}^2$ given in polar coordinates by

$$r(\theta) = [3\pi/2 + \operatorname{Arctan}(\theta)]/2\pi \qquad -\infty < \theta < \infty,$$

which is asymptotic to the circles $r = \frac{1}{2}$ (resp. $r = 1$) as $\theta \to -\infty$ (resp. $\theta \to \infty$).

Let Ω be the region

$$\{(x, y) \in \mathbb{R}^2 | \tfrac{1}{2} < r < 1\} \backslash \Gamma,$$

which is open, connected and bounded. Its boundary is $\Gamma \cup \{r = \frac{1}{2}\} \cup \{r = 1\}$. Let $H = -\Delta_N^\Omega$ the Neumann Laplacian for Ω. Since the circular parts of $\partial\Omega$ are singular points, we use the method of quadratic forms to define H. In fact, however, it could be defined by requiring classical $\partial\varphi/\partial n = 0$ boundary conditions on (both sides of) Γ and no boundary conditions on the circles because $\{\varphi \in D(H) | \operatorname{supp}\varphi \subset \{a < r < b\}$ with $\frac{1}{2} < a < b < 1\}$ is a core for H.

Our main result here is

Theorem. (a) $\sigma(H) = [0, \infty)$;

(b) $\sigma_{\text{ac}}(H) = [0, \infty)$ *of uniform multiplicity* 2;

(c) $\sigma_{\text{sc}}(H) = \varnothing$;

(d) *Any eigenvalue of* H *is of finite multiplicity and the only possible limit point of eigenvalues is* ∞.

What is interesting is that Ω is a bounded region but H still has absolutely continuous spectrum. It has been known, at least since the book of Courant–Hilbert [1], that even though Dirichlet Laplacians of bounded regions have

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purely discrete spectrum, there are bounded regions with σ_{ess} $(-\Delta_N^\Omega) \neq \varnothing$. But the Courant–Hilbert example has $\sigma_{\text{ess}} = \{0\}$ [3]. Recently Hempel, Seco, and Simon [3] constructed regions with $\sigma_{\text{ess}}(-\Delta_N^\Omega) = [0, \infty)$ but their examples have empty absolutely continuous spectrum.

In light of Davies–Simon [2] who discuss unbounded but finite volume regions whose $-\Delta_N^\Omega$ have absolutely continuous spectrum, our result here should not be surprising—in a real sense, our Ω here is just one of their regions "rolled up." That is why we think of Ω as a jelly roll, albeit one whose jelly, alas, is infinitely thin.

Proof of the theorem. We shift to polar coordinates θ, r with θ running from $-\infty$ to ∞. Explicitly, we let $\widetilde{\Omega}$ be $\{(\theta, r) | -\infty < \theta < \infty; r_-(\theta) < r < r_+(\theta)\}$ with $r_-(\theta) = r(\theta)$ and $r_+(\theta) = r(\theta + 2\pi)$. There is an obvious one-to-one map from Ω to $\widetilde{\Omega}$ under which $L^2(\Omega, d^2r)$ is unitarily equivalent to $L^2(\widetilde{\Omega}, r\,dr\,d\theta)$ and H is equivalent to the quadratic form, $\widetilde{H}$, given by

$$(g, \widetilde{H}g) = \int \left(\left| \frac{\partial g}{\partial r} \right|^2 r + \left| \frac{\partial g}{\partial \theta} \right|^2 \frac{1}{r} \right) dr\,d\theta.$$

As in [2], a special role is played by the functions $g(\theta, r) = g(\theta)$; then

$$\|g\|^2 = \int F(\theta) |g(\theta)|^2\,d\theta, \qquad (g, \widetilde{H}g) = \int G(\theta) \left| \frac{dg}{d\theta}(\theta) \right|^2 d\theta,$$

where $F(\theta) = \frac{1}{2}[r_+(\theta)^2 - r_-(\theta)^2]$ and $G(\theta) = \ln[r_+(\theta)/r_-(\theta)]$. Since $r'(\theta) \sim \theta^{-2}$ at infinity, $r_+(\theta) - r_-(\theta) \sim \theta^{-2}$ so, F, $G \sim \theta^{-2}$. Explicitly

$$\begin{aligned}
r'(\theta) &\sim \pi^{-1}[\theta^{-2} - \theta^{-4} + O(|\theta|^{-6})]; \\
r(\theta) - r(\pm\infty) &\sim \pi^{-1}\theta - \tfrac{1}{3}\pi^{-1}\theta^{-3} + O(|\theta|^{-5}); \\
F(\theta) &\sim r(\pm\infty)[\frac{2}{\theta^2} + \frac{\beta}{\theta^4} + O(|\theta|^{-6}]; \\
G(\theta) &= \frac{2}{\theta^2} + \frac{\alpha}{\theta^4} + O(|\theta|^{-6}); \\
G(\theta)/F(\theta) &= r(\pm\infty)^{-1}[1 + O(|\theta|^{-2})]. \qquad (1)
\end{aligned}$$

In the usual way, $\widetilde{H}$ is unitarily equivalent to $\widehat{H}$ on $L^2(\mathbb{R}, d\theta)$ where

$$\widetilde{H} = -\frac{1}{\sqrt{F}} \frac{d}{d\theta} \sqrt{F} \left(\frac{G}{F} \right) \sqrt{F} \frac{d}{d\theta} \frac{1}{\sqrt{F}} = -\frac{d}{d\theta} \left(\frac{G}{F} \right) \frac{d}{d\theta} + V(\theta),$$

where $V(\theta) \sim \theta^{-2}$. Except for the θ dependence G/F in $-d^2/d\theta^2$, the setup looks exactly like that in Davies–Simon [2]. Since (1) holds and the Enss theory easily accommodates principal part perturbations, our proof follows that in [2]. □

References

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