THE WEYL TRANSFORM AND $L^p$ FUNCTIONS ON PHASE SPACE

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ABSTRACT. This is primarily a negative paper showing that a bound of the form $\|W(f)\|_{\text{operator norm}} \leq c\|f\|_p^p$ fails for the Weyl transform if $p > 2$. $L^p$ properties of Wigner distribution functions are discussed as well as Cwikel's theorem.

Trace ideal properties of operators of the form $f(x)g(-i\nabla)$ on $L^2(\mathbb{R}^n)$ have been an important element in the study of Schrödinger operators (both scattering theory and, via the Birman-Schwinger principle, bound state problems) and Yukawa quantum field theories (see [4, Chapter 4]). The main results here are

**Theorem 1.** If $f, g \in L^p(\mathbb{R}^n)$, $2 \leq p < \infty$, then $f(x)g(-i\nabla) \in \mathcal{F}_p$ and $\|f(x)g(-i\nabla)\|_p \leq \|f\|_p \|g\|_p$.

**Theorem 2** (Cwikel). If $f \in L^p(\mathbb{R}^n)$ and $g \in L^p_w(\mathbb{R}^n)$, $2 < p < \infty$, then $f(x)g(-i\nabla) \in \mathcal{F}^w_p$ and $\|f(x)g(-i\nabla)\|_{p,w} \leq c_p\|f\|_p \|g\|_{p,w}$.

It is natural to try to extend this to some nonproduct functions. Define for $F \in \mathcal{S}(\mathbb{R}^{2n})$

$$W(F) = (2\pi)^{-n} \int \tilde{F}(k, y) e^{i(kx + y^p)} dk dy,$$

the Weyl quantization, and its asymmetrical form

$$A(F) = (2\pi)^{-n} \int \tilde{F}(k, y) e^{i(kx)e^{iyp}} dk dy.$$

Then $A(f(x)g(p)) = f(x)g(-i\nabla)$, so one might expect that Theorem A extends to a result of the form

$$\|A(F)\|_p \leq c\|F\|_p \quad \text{or} \quad \|W(F)\|_p \leq c\|F\|_p.$$  

At first sight this might seem incompatible with the fact that $f, g \in L^p_w$ does not imply that $f(x)g(-i\nabla)$ is even compact (consider $f(x) = g(x) = |x|^{-n/p}$); but in fact, it is consistent, for $f \in L^p_w$ and $g \in L^p_w$ does not imply that $f(x)g(y)$ on $\mathbb{R}^{2n}$ is in weak $L^p$, e.g., $f(x) = g(x) = |x|^{-n/p}$ where $\|(x, y)|f(x)g(y)|^2\|_1 = \infty$. But if $f \in L^p$ and $g \in L^p_w$ imply that $f(x)g(y) \in L^p_w(\mathbb{R}^{2n})$ by a simple argument. Indeed, for $f$ fixed, $f(x)g(y) \in L^p_w(\mathbb{R}^{2n})$
for all $g \in L^p_0(\mathbb{R}^n)$ if and only if $f \in L^p(\mathbb{R}^n)$. Thus if (1) holds, a simple consequence would be Cwikel's theorem via a standard application of weak interpolation theorems.

On the other hand, since it is natural to view the map $F \mapsto A(F)$ as an operator-valued Fourier transform [2], one would not expect Theorem A to extend to a result of the form (1). This is confirmed in the following theorem.

**Theorem 1.** Let $p \geq 2$. Then (1) holds if and only if $p = 2$.

That (1) holds if $p = 2$ follows formally from

$$\text{Tr}(e^{i(kX+yP)}) = (2\pi)^{n/2} \delta(k) \delta(y) = \text{Tr}(e^{ikX} e^{iyP}),$$

which yields

$$\text{Tr}(W^*(F)W(F)) = \text{Tr}(A^*(F)A(F)) = (2\pi)^{-n} \int |F(x, k)|^2 \, dx \, dk.$$

A proof of this well-known fact [1, 3] follows from the Plancherel theorem and the explicit form of the integral kernels for $W$ and $A$:

$$W(F)(x, z) = (2\pi)^{-n} \int \hat{F}(k, z-x) e^{ikx} e^{ik(z-x)/2} \, dk,$$

$$A(F)(x, z) = (2\pi)^{-n} \int \hat{F}(k, z-x) e^{ikx} \, dk.$$

So we turn to proving that (1) fails for $p > 2$. Indeed, we will even prove that

$$\|A(F)\|_{\infty} \leq c\|F\|_p, \quad \|W(F)\|_{\infty} \leq c\|F\|_p$$

both fail where $\| \cdot \|_\infty$ is the operator norm.

This will follow from the simple duality argument. If we define for $\psi \in L^2(\mathbb{R}^n)$

$$\rho^A(\psi)(x, p) = (2\pi)^{-2n} \int e^{-ixk} e^{-ipy} (e^{ikX} e^{iyP} \psi) \, dk \, dy,$$

$$\rho^W(\psi)(x, p) = (2\pi)^{-2n} \int e^{-ixk} e^{-ipy} (e^{i(kX+yP)} \psi) \, dx \, dy$$

(with $\langle \psi | B | \psi \rangle = (\psi, B\psi)$); then

$$\langle \psi | A(F) | \psi \rangle = \int F(x, p) \rho^A(\psi)(x, p) \, dx \, dp,$$

$$\langle \psi | W(F) | \psi \rangle = \int F(x, p) \rho^W(\psi)(x, p) \, dx \, dp.$$

From this we conclude:

**Proposition.** If $\|A(F)\|_{\infty} \leq c\|F\|_p$ (resp., $\|W(F)\|_{\infty} \leq c\|F\|_p$), then for $p' = p/p - 1$ we have that $\|\rho^A(\psi)\|_{p'} \leq c$ (resp., $\|\rho^W(\psi)\|_{p'} \leq c$) for all $\psi \in L^2(\mathbb{R}^n)$.

By straightforward calculation,

$$\rho^A(\psi)(x, p) = (2\pi)^{-n/2} e^{ipx} \psi^*(x) \psi(p),$$

$$\rho^W(\psi)(x, p) = (2\pi)^{-n} \int e^{ipy} \psi^*(x - \frac{1}{2}y) \psi(x + \frac{1}{2}y) \, dy.$$
From (2) we see that $p^A(\psi) \in L^q$ if and only if both $\psi$ and $\hat{\psi}$ lie in $L^q$. Since $\psi$ is arbitrary in $L^2$, there are $\psi$ with $p^A(\psi) \in L^q$ if and only if $q = 2$; so by the proposition, $\|A(f)\|_\infty \leq c\|f\|_p$ only if $p = 2$.

The Weyl case is a little more subtle. Note first that if $\psi$ is supported in $\{x| |x| \leq 1\}$, then by (3), $p^W(\psi)(x, p) \neq 0$ only if there is a $y$ with $|x + \frac{1}{2}y| \leq 1$; so $|x| \leq \frac{1}{2}|(x + \frac{1}{2}y) + (x - \frac{1}{2}y)| \leq 1$, i.e., $p^W(\psi)(x, p) = 0$ if $|x| \geq 1$.

Suppose that $\int |p^W|^q dx dp < \infty$. Then it follows, since the characteristic function of the unit ball is in $L^{4/q-1}$, that we have for any $\theta(x, p)$

$$\int p^W(x, p)e^{i\theta(x, p)} dx \in L^q(dp).$$

We will find $\psi$ and $\theta$ in (4) false if $1 \leq q < 2$. Pick $\theta(x, p) = 2px$. Then by (3)

$$\int e^{i\theta(x, p)} p^W(x, p) dx = (2\pi)^{-n} \int e^{2ip(x-y)/2} \psi^*(x - \frac{1}{2}y)\psi(x + \frac{1}{2}y) dy dx$$

$$= (2\pi)^{-n} \int e^{2ipu} \psi(u)\psi^*(z) dy dz$$

$$= (2\pi)^{-n/2} \bar{\psi}(0)\psi(2p).$$

Now take

$$\psi(x) = \begin{cases} |x|^{-\alpha}, & |x| < 1, \\ 0, & |x| < 1 \end{cases}$$

with $2\alpha < n$. Then, $\psi(0) \neq 0$ and $\hat{\psi}(p) \sim |p|^{-(n-\alpha)}$ for $p$ large; so $p^W \notin L^q$ if $q(n-\alpha) < n$. Since $\alpha$ can be arbitrarily closer to $n/2$, $q$ can be chosen anywhere in $[1, 2)$. This concludes the proof of Theorem 1.

Along the way, we proved the following of independent interest:

Theorem 2. In general, $p^A(\psi)$ may not lie in any $L^p$, $p \neq 2$. In general, $p^W(\psi)$ may not lie in $L^p$, $1 \leq p < 2$.

It is easy to show $p^W \in L^\infty$ so in $L^p$, $2 \leq p \leq \infty$. Since $p^W$ is a "density", $\int |p^W| dx dp = \infty$ is notable!

References


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