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LARGE TIME BEHAVIOR OF THE HEAT KERNEL: ON A THEOREM OF CHAVEL AND KARP

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ABSTRACT. We show that a theorem of Chavel and Karp follows from the spectral theorem and elliptic regularity

Recently Chavel and Karp [1] proved the following:

Let M be a noncompact Riemannian manifold with Laplace-Beltrami operator Δ acting on functions on M, $\lambda =: \lambda(M)$ the bottom of $\operatorname{spec}(-\Delta)$, and attendant minimal positive heat kernel $p(x, y, t)$ (where (x, y, t) is an element of $M \times M \times (0, +\infty)$).

Theorem. *For all x, y in M we have the existence of the limit*

$$\lim_{t \uparrow +\infty} e^{\lambda t} p(x, y, t) =: \mathscr{F}(x, y), \tag{1}$$

for which we have the following alternative:

Either $\mathscr{F}$ vanishes identically on all of $M \times M$, in which case λ possesses no L^2 eigenfunctions, or $\mathscr{F}$ is strictly positive on all of $M \times M$, in which case λ possesses a positive normalized L^2 eigenfunction ϕ (normalized in the sense that its L^2 norm is equal to 1) for which

$$\lim_{t \uparrow +\infty} e^{\lambda t} p(x, y, t) = \phi(x)\phi(y) \tag{2}$$

locally uniformly on all of $M \times M$.

Our goal here is to show that this result is essentially an immediate consequence of the spectral theorem and elliptic regularity.

The following well-known lemma follows directly form the spectral theorem and the Lebesgue monotone convergence theorem.

Lemma. *Let A be a selfadjoint operator and let $f(x, t)$ be a measurable function on $\sigma(A) \times [0, \infty]$ so that $f(x, \cdot)$ is monotone for each fixed x and $f(x, \infty) = \inf_t f(x, t) = \lim_t f(x, t)$. Then, $s\text{-}\lim_{t \to \infty} f(A, t) = f(A, \infty)$.*

For $t < \infty$, let $f(x, t) = e^{-t(x-\lambda)}$ and let $f(x, \infty) = \delta_\lambda(x)$, the characteristic function of $\{\lambda\}$. Then $f(-\Delta, \infty)$ is the projection P onto the space S of

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all L^2 eigenfunctions with eigenvalue λ. Since $p(x, y, t)$ is strictly positive, the Perron-Frobenius theorem (see [2, §XIII.12]) implies that either $S = \{0\}$ or is one-dimensional with a unique element φ so that $\varphi(x) > 0$ and $\|\varphi\|_2 = 1$. Thus, either $f(-\Delta, \infty) = 0$ or $f(-\Delta, \infty) = (\varphi, \cdot)\varphi$ as operators.

Equation (1) therefore holds from the lemma if convergence is intended in the L^2 sense. To turn this into pointwise convergence (even local C^∞), we need only appeal to elliptic regularity.

By elliptic regularity, $C^\infty(H) \equiv \bigcap_n D(\Delta^n) \supset \operatorname{Ran}(e^{t\Delta})$ consists of C^∞ functions. Thus, $f \mapsto (e^{+\Delta} f)(x)$ is a bounded functional on L^2. By duality $g_x(y) \equiv (e^{(\Delta+\lambda)})(x, y)$ is in L^2. Thus, by the strong L^2 convergence and the semigroup property,

$$e^{\lambda(t+2)} p(x, y, t+2) = \int g_x(z) e^{\lambda t} p(z, w, t) g_y(w)\, dt\, dw$$

converges to $(g_x, P g_y) = P(x, y)$. This proves the theorem.

We close with several remarks:

1. Since elliptic regularity implies that $C^\infty(H)$ consists of C^∞ functions, it is not hard to see that the convergence is in the C^∞ topology.

2. We did not provide a proof of the last statement in the main theorem of [1] that $\lim_{x\to\infty} \varphi(x) = 0$ if M is noncompact Riemannian with bounded geometry. This should follow by a general subsolution estimate that bounded geometry implies that

$$|\varphi(x)| \le c \int_{\rho(x,y)\le 1} |\varphi(y)|\, dy.$$

3. By the proof, the operators $A_t = e^{\lambda t} e^{\Delta t}$ are monotone decreasing in t. This implies that $(\delta_x, A\delta_x) = A(x, x)$ is monotone as noted by Chavel-Karp but also that $A(x, x) + A(y, y) \pm 2A(x, y) = (\delta_x \pm \delta_y, A(\delta_x \pm \delta_y))$ is monotone, providing a direct proof of pointwise convergence.

References

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