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Eigenvalue Asymptotics of the Neumann Laplacian of Regions and Manifolds with Cusps

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We study the eigenvalue asymptotics of a Neumann Laplacian $-\Delta_N^\Omega$ in unbounded regions Ω of $\mathbf{R}^2$ with cusps at infinity (a typical example is $\Omega = \{(x, y) \in \mathbf{R}^2 : x > 1,\ |y| < e^{-x^2}\}$) and prove that $N_E(-\Delta_N^\Omega) \sim N_E(H_V) + E/2\,\mathrm{Vol}(\Omega)$, where H_V is the canonical one-dimensional Schrödinger operator associated to the problem. We establish a similar formula for manifolds with cusps and derive the eigenvalue asymptotics of a Dirichlet Laplacian $-\Delta_D^\Omega$ for a class of cusp-type regions of infinite volume.

1. INTRODUCTION

Let Ω be a region in $\mathbf{R}^d$. We recall that the Neumann Laplacian $H_N = -\Delta_N^\Omega$ is the unique self-adjoint operator whose quadratic form is

$$g(f, f) = \int_\Omega |\nabla f|^2 \, dx \tag{1.1}$$

on the domain $H^1(\Omega) = \{f \in L^2(\Omega) \mid \nabla f \in L^2(\Omega)\}$, where the gradient is

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taken in the distributional sense. One similarly defines the Dirichlet Laplacian $H_D = -\Delta_D^\Omega$ as the unique self-adjoint operator whose quadratic form is given by the closure of (1.1) on the domain $C_0^\infty(\Omega)$. If Ω is a bounded region with a smooth boundary it is well known that both H_N and H_D have compact resolvents and that their eigenvalue distributions are given by Weyl's law

$$N_E(H_N) \sim N_E(H_D) \sim \frac{\tau_d}{(2\pi)^d} \operatorname{Vol}(\Omega)\, E^{d/2}, \tag{1.2}$$

where by $f(E) \sim g(E)$ we mean $\lim_{E \to \infty} f(E)/g(E) = 1$. We denote by τ_d the volume of a unit ball in $\mathbf{R}^d$, by $\operatorname{Vol}(\Omega)$ the Lebesgue measure of Ω, and by $N_E(A)$ the number of eigenvalues of the operator A which are less than E. If one drops the condition that Ω have a smooth boundary, nothing dramatic happens with the Dirichlet Laplacian H_D. As long as $\operatorname{Vol}(\Omega) < \infty$, H_D will have a compact resolvent and (1.2) remains true [17]. On the other hand, the spectrum of H_N can undergo rather spectacular changes. The following theorem was proved in [10]:

THEOREM. *Let S be a closed subset of the positive real axis. Then there exists a bounded domain Ω for which*

$$\sigma_{\text{ess}}(H_N) = S.$$

In the theorem, Ω can be chosen in such a way that its boundary has a singularity at exactly one point. We will be interested in the other extreme, namely when the domain Ω retains a nice boundary, but is unbounded, and in particular is of the form

$$\Omega = \{(x, y) \in \mathbf{R}^2 : x > 1,\ |y| < f(x)\}. \tag{1.3}$$

Through the rest of the paper we will suppose that f is $C^\infty[1, \infty)$ and strictly positive, and that its first three derivatives are bounded (although less regularity could be required). If $f(x) \to 0$, the Dirichlet Laplacian still has a compact resolvent [13, 16], but if $f(x) = x^{-1}$, or even if $f(x) = \exp(-x)$, (so $\operatorname{Vol}(\Omega) < \infty$), Davies and Simon [6] showed that $\sigma_{\text{ac}}(H_N)$ is nonempty. The difference in the spectral behavior is again striking, and one feels that a rather rapid decay of f should be required to ensure compactness of the resolvent of H_N. The following beautiful theorem was proven in [7]:

THEOREM. *If Ω is given by* (1.3), *H_N has a compact resolvent if and only if*

$$\lim_{x \to \infty} \left(\int_1^x \frac{1}{f(t)}\, dt \right) \left(\int_x^\infty f(t)\, dt \right) = 0. \tag{1.4}$$

In this paper we study the large E asymptotics of the eigenvalue distribution of H_N in the regions (1.3). As in [6], the main role is played by the one-dimensional Schrödinger operator

$$H_V = -\frac{d^2}{dx^2} + V(x), \qquad V(x) = \frac{1}{4}\left(\frac{f'}{f}\right)^2 + \frac{1}{2}\left(\frac{f'}{f}\right)', \tag{1.5}$$

acting on $L^2[1, \infty)$, and with the Dirichlet boundary condition at 1. We make the following two hypotheses:

$$V(x) \to \infty, \qquad f''(x) \to 0 \qquad \text{as} \quad x \to \infty; \tag{H1}$$

$$\text{if } 0 < \varepsilon < 1, \qquad N_E((1 \pm \varepsilon) H_V) = N_E(H_V)(1 + O(\varepsilon)). \tag{H2}$$

Our main result is

THEOREM 1.1. *If* (H1) *and* (H2) *are satisfied, we have*

$$N_E(H_N) \sim N_E(H_V) + \frac{E}{2} \operatorname{Vol}(\Omega). \tag{1.6}$$

Remark 1. Hypothesis (H1) implies that

$$f(x) + f'(x)^2/f(x) \to 0 \qquad \text{as} \quad x \to \infty \tag{1.7}$$

(see Section 2.1). In turn, Davies and Simon [6] showed that if (1.7) is satisfied, H_N will have a compact resolvent if and only if H_V does. Consequently, both sides in (1.6) are finite, and in particular (H1) implies that $\operatorname{Vol}(\Omega) < \infty$. Hypothesis (H2) prevents $N_E(H_V)$ from growing too rapidly (e.g., exponentially), which is needed to make our perturbation argument work. For example, it is satisfied if V is a convex function [21] or if $V(x) \sim x^\alpha (\ln x)^\beta$, $\alpha > 0$. On the other hand, if $V(x) \sim \ln x$ (e.g., $f(x) = \exp(-x \ln x)$), it is not, and our argument does not apply.

Remark 2. The fact that Ω is symmetric is irrelevant. If $\Omega = \{(x, y) : x > 1, \ -f_1(x) < y < f_2(x)\}$, (1.6) remains valid provided that $f_1'' \to 0$, $f_2'' \to 0$, and H_V, defined with $f = (f_1 + f_2)/2$, satisfies (H1), (H2). Also, if (H1) is replaced by a more involved hypothesis, the result extends (as usual [6, 17]) to the case when $\mathbf{R}^2$ is replaced by $\mathbf{R}^{d+1}$, $(x_1, x_\perp) \in \mathbf{R}^{d+1}$, $x_\perp \in \mathbf{R}^d$, $\Omega = \{x \mid x_\perp / f(x_1) \in G, \ 1 \leqslant x_1 \leqslant \infty\}$, where G is a bounded connected set. The asymptotics is given by (2.6) (replacing M with Ω).

EXAMPLE 1. Let $f(x) = \exp(-x^\alpha)$. H_N has a compact resolvent if and only if $\alpha > 1$. One calculates

$$V(x) = \frac{\alpha^2}{4} x^{2(\alpha-1)} - \frac{\alpha(\alpha-1)}{2} x^{(\alpha-2)}.$$

The semiclassical formula [21] yields

$$N_E(H_V) \sim \frac{1}{4(\alpha-1)\sqrt{\pi}} \left(\frac{\alpha}{2}\right)^{1/(1-\alpha)} \frac{\Gamma(1/(2(\alpha-1)))}{\Gamma(\frac{3}{2}+1/(2(\alpha-1)))} E^{1/2+1/(2(\alpha-1))}, \tag{1.8}$$

and thus both (H1) and (H2) are satisfied. Formulas (1.6) and (1.8) imply that the asymptotics of $N_E(H_N)$ satisfies Weyl's law if $\alpha > 2$; it is given by (1.8) if $1 < \alpha < 2$. If $\alpha = 2$ we have $N_E(H_N) \sim E/2(\mathrm{Vol}(\Omega) + \frac{1}{2})$. The leading order is the same as that in Weyl's law but the constant is larger. We observe a phase transition in the eigenvalue asymptotics for the value $\alpha_c = 2$. In [6] it was shown that for $\alpha = 1$, $\sigma_{\mathrm{ac}}(H_N) = [\frac{1}{4}, \infty)$, and for $0 < \alpha < 1$, $\sigma_{\mathrm{ac}}(H_N) = [0, \infty)$. In both cases $\sigma_{\mathrm{sing}}(H_N) = \varnothing$, and $\sigma_{\mathrm{pp}}(H_N)$ consists of a discrete set $0 = \lambda_1 < \lambda_2 \leqslant \cdots \leqslant \lambda_n \to \infty$ of embedded eigenvalues of finite multiplicity (see also [12]).

EXAMPLE 2. Let $f(x) = \exp(-x^2 g(x))$, $g(x) = 1 + \cos^2(\sqrt{\ln(1+x)})$. We have $V(x) \sim x^2 g(x)^2$ and the semiclassical formula yields

$$N_E(H_V) \sim \frac{E}{4g(E)}.$$

Hypotheses (H1), (H2) are satisfied, and we observe that $N_E(H_N)/E$ stays bounded above and below but $\lim_{E\to\infty} N_E(H_N)/E$ does not exist.

The simplest way to understand the result of Theorem 1.1 is to consider a subspace P of $L^2(\Omega)$ consisting of functions u which depend on the x variable only. On $C_0^2(\bar{\Omega}) \cap P$ the form (1.1) acts as

$$\int_1^\infty \left|\frac{d}{dx} u(x)\right|^2 2f(x)\, dx,$$

and viewed as a form on $L^2([1, \infty), 2f(x)\, dx)$ yields an operator which is (up to a change of boundary condition at $x = 1$) unitarily equivalent to H_V. It is now immediate that $N_E(-\Delta_N^\Omega) \succcurlyeq N_E(H_V)$ ($f(E) \succcurlyeq g(E)$ means that $\liminf_{E\to\infty} f(E)/g(E) \geqslant 1$), but in an equally simple way we can say even more. Denote $\Omega_L = \{(x, y) : 1 < x < L,\ |y| < f(x)\}$ and put an additional Dirichlet boundary condition along the line $x = L$. Dirichlet–Neumann bracketing yields

$$N_E(-\Delta_N^\Omega) \succcurlyeq N_E(H_V) + \frac{E}{2} \mathrm{Vol}(\Omega_L).$$

Letting $L \to \infty$, we obtain the one-sided inequality in (1.6), which is obviously true under the sole condition that f is a $C^2[1, \infty)$ function. It is the other, nontrivial direction of (1.6) which forces us to place conditions on f and V, and which could be proven using techniques developed in [6, 18]. The main technical point in such an approach is to obtain control of H_N on the subspace orthogonal to P. Here we will adopt a different strategy which, we believe, sheds some new light on the problem. Let $M = (-1, 1) \times (1, \infty)$ be a strip with the metric

$$ds_M^2 = dx^2 + f(x)^2\, dy^2, \tag{1.9}$$

and denote by H_N the Lapalace–Beltrami operator on M with the Neumann boundary condition. Separating the variables we obtain that H_N is unitarily equivalent to the operator $\bigoplus_{n \geqslant 0} H_n$ acting on $\bigoplus_{n \geqslant 0} L^2([1, \infty), dx)$, where

$$H_n = H_V + \left(\frac{n\pi}{2f(x)}\right)^2$$

with a boundary condition $\psi'(1) = (f'(1)/2f(1))\, \psi(1)$. The main technical ingredient in this approach is to show that $N_E(\bigoplus_{n \geqslant 1} H_n)$ satisfies Weyl's law. Then the analog of Theorem 1.1 for H_N is immediate. After a suitable coordinate change, region Ω is transformed into the strip M, with a metric which is (under the conditions of the Theorem 1.1) asymptotically of the form (1.9). At this point, a relatively easy perturbation argument will yield (1.6).

Finally, we remark that the above approach appears useful in studying eigenvalue asymptotics of a Dirichlet Laplacian in a region Ω given by (1.3), with $f(x) \to 0$ and $\mathrm{Vol}(\Omega) = \infty$. While we can recover most of the known results on the asymptotics of $N_E(H_D)$ in such regions (but not all; e.g., we cannot treat the case $f(x) = (\ln(1+x))^{-1}$, see [1, 17]), here we restrict ourselves to giving a new proof of the well-known [1, 17, 18, 20])

THEOREM 1.2. *Let $\Omega = \{(x, y) : |x|^\alpha\, |y| \leqslant 1\}$. Then*

$$N_E(H_D) \sim \frac{1}{\sqrt{\pi}} \left(\frac{2}{\pi}\right)^{1/\alpha} \zeta(1/\alpha)\, \frac{\Gamma(1 + 1/2\alpha)}{\Gamma(\frac{3}{2} + 1/2\alpha)}\, E^{1/2 + 1/2\alpha}, \qquad \text{if} \quad 0 < \alpha < 1,$$

$$N_E(H_D) \sim \frac{1}{\pi}\, E \ln E, \qquad \text{if} \quad \alpha = 1,$$

where ζ is the standard zeta function. The case $\alpha > 1$ follows by symmetry.

2. Neumann Laplacians on Manifolds and Regions with Cusps

We begin by studying the eigenvalue distribution of a Laplace–Beltrami operator on a Riemannian manifold of the form $M = N \times [1, \infty)$, with a metric ds_M^2, $\mathrm{Vol}(M) < \infty$. Here, N is a compact, oriented Riemannian manifold (with or without boundary), $\dim(N) = d$, with a metric ds_N^2, and a volume element dm_N. We remark that the boundary of M does not have to be C^∞, but it is certainly piecewise C^∞ and therefore causes no problem in the discussion below (see, e.g., [2]). In Sections 2.1 and 2.2 we treat the case when the metric on M has a warped product form. Perturbations are studied in Section 2.3. Finally, in Section 2.4, we derive Theorem 1.1 as an easy consequence of the results obtained for manifolds.

2.1. *Preliminaries*

We suppose that the metric on M is given by

$$ds_M^2 = dx^2 + f(x)^2\, ds_N^2, \tag{2.1}$$

where f is a positive, $C^\infty[1, \infty)$ function, and that

$$\mathrm{Vol}(M) = \mathrm{Vol}(N) \int_1^\infty f(x)^d\, dx < \infty. \tag{2.2}$$

If $d = 1$, (2.2) is a consequence of (H1). H_N, the Laplace–Beltrami operator on M with Neumann boundary conditions, acts on a Hilbert space $L^2(M, dm_M)$ and is the unique self-adjoint operator whose quadratic form is given by the closure of

$$q(\phi, \phi) = \int_M |\nabla\phi|^2\, dm_M \tag{2.3}$$

on $C_0^2(\bar{M})$. In (2.2), $dm_M = f^d\, dm_N\, dx$ and ∇ is the gradient on M. Of equal importance for us is the Laplace–Beltrami operator $H_{N,D}$ on M with the Dirichlet boundary condition along $\{1\} \times N$, and the Neumann one on the rest of the boundary. It is defined as a closure of the form (2.3) on a subspace of $C_0^2(\bar{M})$ consisting of functions which vanish along $\{1\} \times N$. The analog of (1.5) is the one-dimensional Schrödinger operator of the form

$$H_V = -\frac{d^2}{dx^2} + V(x), \qquad V(x) = \frac{d^2}{4}\left(\frac{f'}{f}\right)^2 + \frac{d}{2}\left(\frac{f'}{f}\right)', \tag{2.4}$$

with the Dirichlet boundary condition at 1. Denote by

$$C_d = ((4\pi)^{(d+1)/2}\, \Gamma((d+3)/2))^{-1}. \tag{2.5}$$

The following lemma, which we prove in Section 2.2, is the main technical ingredient of our approach.

LEMMA 2.1. *Suppose that* $V(x) \to \infty$, $f(x)^2 V(x) \to 0$ *as* $x \to \infty$. *Then*

$$N_E(H_N) \sim N_E(H_{N,D}) \sim N_E(H_V) + E^{(d+1)/2} C_d \operatorname{Vol}(M). \tag{2.6}$$

In the sequel we collect, for the reader's convenience, a few simple results which will be needed later. Let

$$D_N = \{\phi : \phi \in C_0^2(\bar{M}), \nu\phi = 0\},$$

where ν is the outward unit normal vector field on ∂M. H_N acts on D_N as

$$H_N(\phi) = -\frac{1}{f(x)^d} \frac{\partial}{\partial x} f(x)^d \frac{\partial}{\partial x} \phi + \frac{1}{f(x)^2} H_N^N(\phi),$$

where H_N^N is a Laplace–Beltrami operator of N. H_N^N has a compact resolvent [2]; its spectrum consists of discrete eigenvalues $0 = \lambda_0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$, $\lambda_n \to \infty$, and we denote by ϕ_n^N the corresponding eigenfunctions. Introducing

$$L_n^2(M) = \left\{ g : g(x, t) = \psi(x)\, \phi_n^N(t), \int_1^\infty |\psi(x)|^2 f(x)^d \, dx < \infty \right\},$$

we obtain the decomposition

$$L^2(M) = \bigoplus_{n \geqslant 0} L_n^2(M) = \bigoplus_{n \geqslant 0} L_n^2([1, \infty), f(x)^d \, dx).$$

The operator H_N splits accordingly,

$$H_N = \bigoplus_{n \geqslant 0} H_{N,n},$$

where $H_{N,n}$ acts on $L_N^2([1, \infty), f(x)^d \, dx)$ as

$$H_{N,n} = -\frac{1}{f(x)^d} \frac{\partial}{\partial x} f(x)^d \frac{\partial}{\partial x} + \frac{\lambda_n}{f^2(x)}.$$

Under the unitary map

$$U \colon L^2([1, \infty), dx) \to L_N^2([1, \infty), f(x)^d \, dx), \qquad U(\phi) = f^{-d/2}\phi$$

$H_{N,n}$ transforms as

$$H_n = U^{-1} H_{N,n} U = -\frac{d^2}{dx^2} + V(x) + \frac{\lambda_n}{f^2(x)}, \tag{2.7}$$

and (if V is bounded below) is essentially self-adjoint on

$$D_V = \{\psi : \psi \in C_0^2[1, \infty), \psi'(1) = d/2(f'(1)/f(1))\, \psi(1)\}. \tag{2.8}$$

H_N is unitarily equivalent to the operator $\bigoplus_{n\geq 0} H_n$ acting on $\bigoplus_{n\geq 0} L^2([1, \infty), dx)$. Similarly, $H_{N,D}$ is unitarily equivalent to $\bigoplus_{n\geq 0} H_n^D$, where H_n^D is the operator (2.7) with the Dirichlet boundary condition at 1. The spectral analysis of H_N and $H_{N,D}$ reduces to the spectral analysis of the one-dimensional Schrödinger operators H_n, H_n^D. We will need

LEMMA 2.2. *If $V(x)\to\infty$ as $x\to\infty$, we have*

$$N_E(H_n^D)\leqslant N_E(H_n)\leqslant 1+N_E(H_n^D) \tag{2.9}$$

for all $n\geqslant 0$.

Proof. We just sketch the well-known argument. $C_0^\infty[1, \infty)$ is a form core for H_n^D, and thus $N_E(H_n^D)\leqslant N_E(H_n)$ follows from the min–max principle [15]. Let

$$D=\{\psi : \psi\in C_0^2[1, \infty), \psi(1)=0\}.$$

If L stands for an arbitrary vector subspace of $L^2[1, \infty)$, the min–max principle yields

$$N_E(H_n)=\sup_{\substack{L\subset D_V\\(H_n\psi,\psi)<E\\ \psi\in L,\ \|\psi\|=1}}\dim L, \qquad N_E(H_n^D)=\sup_{\substack{L\subset D\\(H_n^D\psi,\psi)<E\\ \psi\in L,\ \|\psi\|=1}}\dim L.$$

Fix $L\subset D_V$ and let $L_0=\{\psi\in L:\psi(1)=0\}$. Observing that $\dim L/L_0\leqslant 1$ we derive (2.9). ∎

We finish with the following

LEMMA 2.3. *If $d=1$ and (H1) is satisfied, we have*

$$f(x)+|f'(x)|+f'(x)^2/f(x)+f(x)\,V(x)\to 0 \qquad \text{as} \quad x\to\infty. \tag{2.10}$$

Furthermore, for large x, f is convex and strictly decreasing.

Proof. The result follows from

$$f''(x)=2f(x)\,V(x)+\frac{f'(x)^2}{2f(x)}. \quad ∎$$

2.2. *Proof of Lemma* 2.1

Denote $A_N=\bigoplus_{n\geqslant 1} H_n$, $A_D=\bigoplus_{n\geqslant 1} H_n^D$. Relations (2.6) will follow if we prove that

$$\lim_{E\to\infty}\frac{N_E(A_N)}{E^{(d+1)/2}}=\frac{N_E(A_D)}{E^{(d+1)/2}}=C_d\,\mathrm{Vol}(M). \tag{2.11}$$

Let

$$m = \max_{x>1} f(x)^2, \qquad M = \max_{x>1} |V(x) f(x)^2|.$$

We have that $N_E(H_n^D) = 0$ if $\lambda_n > mE + M$, and thus

$$\begin{aligned} N_E(A_D) \leqslant N_E(A) &\leqslant N_E(A_D) + \#\{\lambda_n : \lambda_n \leqslant mE + M\} \\ &\leqslant N_E(A_D) + O(E^{d/2}), \end{aligned}$$

since Weyl's law applies for H_N^N. Consequently, it suffice to prove (2.11) for A_D. By the Karamata–Tauberian theorem [19], (2.11) will follow if we prove

$$\lim_{t \to 0} t^{(d+1)/2} \operatorname{Tr}(\exp(-tA_D)) = (4\pi)^{-(d+1)/2} \operatorname{Vol}(M). \tag{2.12}$$

First, note that

$$\lim_{t \to 0} t^{d/2} \sum_{k \geqslant 1} \exp(-t\lambda_k) = (4\pi)^{-d/2} \operatorname{Vol}(N) \tag{2.13}$$

[2], and in addition

$$t^{d/2} \sum_{k \geqslant 1} \exp(-t\lambda_k) < L \tag{2.14}$$

for a uniform constant L and for all $t > 0$. The Golden–Thompson inequality [19] yields

$$\operatorname{Tr}(\exp(-tH_k^D)) \leqslant \frac{1}{2\sqrt{\pi t}} \int_1^\infty \exp(-t \cdot (V(x) + \lambda_k / f(x)^2))\, dx.$$

Fix $\varepsilon > 0$, $\lambda_1 > \varepsilon > 0$, and let $R > 0$ be big enough that $|f^2(x)\, V(x)| < \varepsilon$ if $x > R$. Let $c = \inf_{x \in [1, R]} V(x)$. We have

$$\operatorname{Tr}(\exp(-tH_k)) \leqslant \frac{e^{-ct}}{2\sqrt{\pi t}} \int_1^\infty \exp(-t \cdot (\lambda_k - \varepsilon)/f(x)^2)\, dx$$

$$\begin{aligned} t^{(d+1)/2} \operatorname{Tr}(\exp(-tA_D)) &= \sum_{k \geqslant 1} t^{(d+1)/2} \operatorname{Tr}(\exp(-tH_k)) \\ &\leqslant \frac{e^{-ct}}{2\sqrt{\pi}} \int_1^\infty f(x)^d \cdot \sum_k (t/f(x)^2)^{d/2} \\ &\quad \times \exp(-t(\lambda_k - \varepsilon)/f(x)^2)\, dx. \end{aligned}$$

Using (2.14) and the Lebesgue dominated convergence theorem we obtain

$$\limsup_{t \to 0} t^{(d+1)/2} \operatorname{Tr}(\exp(-tA_D)) \leqslant \frac{\operatorname{Vol}(N)}{(4\pi)^{(d+1)/2}} \int_1^\infty f(x)^d \, dx.$$

It remains to show

$$\liminf_{t \to 0} t^{(d+1)/2} \operatorname{Tr}(\exp(-tA_D)) \geqslant (4\pi)^{-(d+1)/2} \operatorname{Vol}(M). \tag{2.15}$$

Let $R > 1$ be a positive number; make a partition of $[1, R]$ into k intervals I_m of equal size. Denote by H_D^m the Dirichlet Laplacian on I_m, and let

$$d_m = \sup_{x \in I_m} f(x)^{-2}, \qquad c = \sup_{x \in [1, R]} |V(x)|.$$

Dirichlet–Neumann bracketing yields

$$\operatorname{Tr}(\exp(-tH_n)) \geqslant \sum_{m=1}^{k} \exp(-t(c + \lambda_n d_m)) \operatorname{Tr}(\exp(-tH_D^m)).$$

Obviously, $\operatorname{Tr}(\exp(-tH_D^m)) = \operatorname{Tr}(\exp(-tH_D^1))$ for all m, and

$$\lim_{t \to 0} t^{1/2} \operatorname{Tr} \exp(-tH_D^1) = \frac{1}{2\sqrt{\pi}} \frac{R}{k}.$$

We have

$$\begin{aligned} &\liminf_{t \to 0} t^{(d+1)/2} \sum_{n>0} \operatorname{Tr}(\exp(-tH_n)) \\ &\quad \geqslant \frac{1}{2\sqrt{\pi}} \sum_{m=1}^{k} \frac{R}{k} d_m^{-d/2} \liminf_{t \to 0} (td_m)^{d/2} \sum_{n>0} \exp(-t\, d_m \lambda_n) \\ &\quad = \operatorname{Vol}(N)(4\pi)^{-(d+1)/2} \sum_{m=1}^{k} \frac{R}{k} d_m^{-d/2}. \end{aligned}$$

Using that f is continuous and passing to the limit $k \to \infty$ we have

$$\liminf_{t \to 0} t^{(d+1)/2} \operatorname{Tr}(\exp(-tA_D)) \geqslant (4\pi)^{-(d+1)/2} \operatorname{Vol}(N) \int_1^R f(x)^d \, dx.$$

Letting $R \to \infty$ we obtain (2.15) and (2.12).

2.3. *Metric Perturbations*

In this section we suppose that a metric on M is given by

$$ds_M^2 = \alpha(t, x)^2 \, dx^2 + \beta(t, x)^2 \, ds_N^2, \tag{2.16}$$

where α, β are two positive, C^∞ functions on M. We also suppose that

$$\text{Vol}(M) = \int_M \alpha \cdot \beta^d \, dm_N \, dx < \infty.$$

After a suitable coordinate change, the region Ω, given by (1.3) (if (H1) is satisfied), transforms into $(-1, 1) \times (1, \infty)$ with a metric of the form (2.16). That is the reason why we choose to discuss (2.16), even if a much larger class of perturbations can be treated along the same lines (see [9, 14] for related discussions). H_N, $H_{N,D}$ are defined, as in the previous section, via the closure of the quadratic form (2.3) (with $dm_M = \alpha \cdot \beta^d \, dm_N \, dx$) on the appropriate subspace. If there exists a function f, satisfying the condition of Lemma 2.1, such that $\alpha \to 1$, $\beta \to f$ as $x \to \infty$, one expects that $N_E(H_N)$ should not be too far from $N_E(\hat{H}_N)$, where $\hat{H}_N$ is the Laplace–Beltrami operator on M for the metric

$$d\hat{s}_M^2 = dx^2 + f(x)^2 \, ds_N^2. \tag{2.17}$$

It is indeed the case. Denote $M_L = N \times [L, \infty)$,

$$\|g\|_L = \sup_{(t,x) \in M_L} |g(t, x)|,$$

and let

$$v(L) = \|\alpha - 1\|_L + \|f/\beta - 1\|_L + \| \, |\hat{\nabla}\alpha| \, \|_L + \| \, |\hat{\nabla}(f/\beta)| \, \|_L, \tag{2.18}$$

where $\hat{\nabla}$ is the gradient on M with the metric (2.17). For H_V given by (2.4) we have

LEMMA 2.4. *Suppose that $v(L) \to 0$ as $L \to \infty$, that f and V satisfy the conditions of Lemma 2.1, and that $N_E(H_V)$ satisfies (H2). Then*

$$N_E(H_N) \sim N_E(H_{N,D}) \sim N_E(H_V) + E^{(d+1)/2} C_d \, \text{Vol}(M).$$

We remark that, while $\text{Vol}(M)$ is calculated in the metric (2.16), the operator H_V arises from the metric (2.17).

Proof. We will consider only H_N. A virtually identical argument applies for $H_{N,D}$. For $L > 1$ denote by $H_{L,D}^-$, $H_{L,D}^+$ the Laplace–Beltrami operators acting on $N \times [1, L]$, M_L, with metric (2.16) and the Dirichlet boundary condition along $N \times \{L\}$; on the rest of boundary we take the Neumann one. Denote by $\hat{H}_{L,D}^+$ the Laplace–Beltrami operator on M_L with the metric (2.17) and with same boundary condition as $H_{L,D}^+$. Let

$$U: L^2(M_L, \alpha\beta^d \, dm_N \, dx) \to L^2(M_L, f^d \, dm_N \, dx)$$

be a unitary mapping defined as

$$U(\phi)=(\alpha\cdot(\beta/f)^d)^{1/2}\,\phi=(1/g)\cdot\phi.$$

The operator $H^+_{L,D}$ is then unitarily equivalent to the operator acting on $L^2(M_L, f^d\,dm_N\,dx)$, which we again denote by $H^+_{L,D}$ and whose quadratic form is given by the closure of

$$\int_{M_L} |\nabla(g\phi)|^2\,\alpha\beta^d\,dm_N\,dx \tag{2.19}$$

on the subspace

$$C^2_{0,L}(\bar{M}_L)=\{\phi : \phi\in C^2_0(\bar{M}_L), \phi(t, L)=0\}.$$

Vector fields $\nabla\phi$, $\hat{\nabla}\phi$ are given as

$$\nabla\phi=\frac{1}{\alpha^2}\,\frac{\partial\phi}{\partial x}\,\frac{\partial}{\partial x}+\frac{1}{\beta^2}\,\nabla_N\phi,$$

$$\hat{\nabla}\phi=\frac{\partial\phi}{\partial x}\,\frac{\partial}{\partial x}+\frac{1}{f^2}\,\nabla_N\phi,$$

where ∇_N is the gradient on N. If $\phi\in C^2_{0,L}(\bar{M}_L)$ and has norm 1 as an element of the $L^2(M_L, f^d\,dm_N\,dx)$, we estimate

$$\begin{aligned}
&|((H^+_{L,D}-\hat{H}^+_{L,D})\phi, \phi)|\\
&\quad\leqslant\int_{M_L} \big|\,|\nabla(g\phi)|^2\,\alpha\beta^d-|\hat{\nabla}(\phi)|^2\,f^d\big|\,dm_N\,dx\\
&\quad\leqslant\int_{M_L} A(L)\,|\hat{\nabla}(g\phi)|^2+\big|\,|\hat{\nabla}(g\phi)|^2-|\hat{\nabla}(\phi)|^2\big|\,f^d\,dm_N\,dx,
\end{aligned}$$

where

$$A(L)=\|1/g\|^2_L\,(\|(1/\alpha)^2-1\|_L+\|(f/\beta)^2-1\|_L)+\|(1/g)^2-1\|_L.$$

Furthermore, we have

$$\begin{aligned}
|\hat{\nabla}(g\phi)|^2&\leqslant|\hat{\nabla}(\phi)|^2\cdot(\|g\|^2_L+\|g\|_L\cdot\|\,|\hat{\nabla}g|\,\|_L)\\
&\quad+|\phi|^2\cdot(\|\,|\hat{\nabla}g|\,\|^2_L+\|g\|_L\cdot\|\,|\hat{\nabla}g|\,\|_L),\\
\big|\,|\hat{\nabla}(g\cdot\phi)|^2-|\hat{\nabla}(\phi)|^2\big|&\leqslant|\hat{\nabla}(\phi)|^2\cdot(\|g^2-1\|_L+\|g\|_L\cdot\|\,|\hat{\nabla}g|\,\|_L)\\
&\quad+|\phi|^2\cdot(\|\,|\hat{\nabla}g|\,\|^2_L+\|g\|_L\cdot\|\,|\hat{\nabla}g|\,\|_L).
\end{aligned}$$

Because $v(L) \to 0$, all the constants in the above estimates are $O(v(L))$, and we conclude that

$$|((H^+_{L,D} - \hat{H}^+_{L,D})\phi, \phi)| \leqslant Dv(L)((\hat{H}^+_{L,D}\phi, \phi) + 1), \tag{2.20}$$

for an L-independent constant D. In the sequel we take L large enough that $Dv(L) < 1$, and then absorb D into $v(L)$. The inequality (2.20) and the min–max principle yield (recall that $f(E) \succcurlyeq g(E)$ means $\liminf_{E \to \infty} f(E)/g(E) \geqslant 1$)

$$N_E(H^+_{L,D}) \succcurlyeq N_{E-v(L)}((1 + v(L))\, \hat{H}^+_{L,D}) \sim N_E((1 + v(L))\, \hat{H}^+_{L,D}). \tag{2.21}$$

If $H_{V,L}$ is operator (2.4) acting on $L^2[L, \infty)$, we observe that the asymptotics of $N_E(H_{V,L})$ does not depend on the boundary condition at L, nor on L itself. Consequently, in the sequel we will deal only with H_V. Denote

$$C_1(L) = \liminf_{E \to \infty} \frac{N_E((1 - v(L))\, H_V)}{N_E(H_V)},$$

$$C_2(L) = \limsup_{E \to \infty} \frac{N_E((1 + v(L))\, H_V)}{N_E(H_V)}.$$

Hypotheses (H2) implies

$$\lim_{L \to \infty} C_1(L) = 1, \qquad \lim_{L \to \infty} C_2(L) = 1. \tag{2.22}$$

The relations (2.20), (2.22) were the two essential ingredients in the argument. Denote by $\mathrm{Vol}(\hat{M}_L)$ the volume of M_L in the metric (2.17). We have

$$\begin{aligned} N_E(H_N) &\succcurlyeq N_E(H^-_{L,D}) + N_E(H^+_{L,D}) \\ &\succcurlyeq E^{(d+1)/2} C_d \,\mathrm{Vol}(N \times [1, L]) + N_E((1 + v(L))\, \hat{H}^+_{L,D}) \\ &\succcurlyeq I(L)(E^{(d+1)/2} C_d (\mathrm{Vol}(N \times [1, L)) + \mathrm{Vol}(\hat{M}_L)) + N_E(H_V)), \end{aligned} \tag{2.23}$$

where

$$I(L) = \min\{(1 + v(L))^{-((1+d)/2}, C_2(L)\}.$$

Formula (2.23) follows from Dirichlet–Neumann bracketing, Lemma 2.1, and the fact that the eigenvalue distribution of a Laplace–Beltrami operator on a compact manifold with a piecewise smooth boundary and mixed boundary conditions satisfies Weyl's law [2]. Replacing the boundary condition along $N \times \{L\}$ with the Neumann b.c., we obtain the operators $H_{L,N}^{-}$ and $H_{L,N}^{+}$, and a completely analogous argument gives

$$\begin{aligned} N_E(H_N) &\leqslant N_E(H_{L,N}^{-}) + N_E(H_{L,N}^{+}) \\ &\leqslant S(L)(E^{(d+1)/2} C_d(\mathrm{Vol}(N \times [1, L)) + \mathrm{Vol}(\hat{M}_L)) + N_E(H_V)), \end{aligned} \tag{2.24}$$

where

$$S(L) = \max\{(1 - v(L))^{-(1+d)/2}, C_1(L)\}.$$

As $L \to \infty$, $I(L) \to 1$, $S(L) \to 1$, $\mathrm{Vol}(\hat{M}_L) \to 0$, $\mathrm{Vol}(N \times [1, L)) \to \mathrm{Vol}(M)$, and the lemma follows from (2.23), (2.24). ∎

It is now obvious why our argument fails in the case when $N_E(H_V)$ growths exponentially fast ($C_1(L) = C_2(L) = \infty$). It is natural to conjecture that in such cases $N_E(H_N) \sim N_E(H_V)$, but it is unlikely that the above argument can be modified to prove it.

2.4. *Proof of Theorem* 1.1.

One consequence of hypothesis (H1) (see Lemma 2.3) is that f is strictly decreasing function for large x. The familiar Dirichlet–Neumann bracketing argument, which will be repeated in detail once again below, implies that without loss of generality we can assume $f'(x) < 0$ for $x > 1$. We construct a change of variable as follows: Let

$$\varepsilon(x, y) = \frac{y}{f(x)}, \qquad -1 \leqslant \varepsilon \leqslant 1.$$

ε is the first integral of the equation

$$\frac{dy}{dx} = y \cdot \frac{f'}{f}.$$

The equation for the orthogonal lines is given by

$$y \cdot \frac{dy}{dx} = -\frac{f}{f'},$$

whose first integral is

$$\frac{y^2}{2} + \int_1^x \frac{f(t)}{f'(t)}\, dt = c.$$

Any C^1 function of this first integral is an orthogonal coordinate to ε. Let

$$F(x) = \int_1^x \frac{f(t)}{f'(t)}\, dt;$$

note that F is a decreasing function ($f' < 0$) and denote $a = \lim_{x \to \infty} F(x)$. The inverse function F^{-1} is well defined on $(a, 0]$, and for R large enough we have

$$\frac{y^2}{2} + F(x) \in (a, 0], \qquad x > R,\ (x, y) \in \Omega.$$

Let

$$\eta(x, y) = F^{-1}(y^2/2 + F(x)), \qquad (x, y) \in \Omega,\ x > R.$$

It is easy to check that (ε, η) is one–one, and that Jacobian $D(\varepsilon, \eta)/D(x, y) \sim 1/f(x) \neq 0$ for x large. Denoting (for $c > R$)

$$\Omega_1 = \{(x, y) : (x, y) \in \Omega,\ \eta(x, y) > c\}, \tag{2.25}$$

we conclude that, for a large c, (ε, η) is a C^∞-bijection between Ω_1 and half-strip $M = (-1, 1) \times (c, \infty)$ with a C^∞-inverse. The eigenvalue asymptotics of a Laplacian on a bounded region with piecewise C^∞ boundary and with mixed boundary conditions satisfies Weyl's law. Consequently, putting an additional Dirichlet or Neumann b.c. along $\eta(x, y) = c$ we observe that it is enough to prove the statement for H_N, $H_{N,D}$, the Laplacians on Ω_1 with respectively Neumann or Dirichlet b.c. along $\eta(x, y) = c$, and the Neumann b.c. on the rest of the boundary. The above change of variables transforms H_N, $H_{N,D}$ into Laplace–Beltrami operators on M, with the metric $ds_M^2 = dx(\varepsilon, \eta)^2 + dy(\varepsilon, \eta)^2$, and the Neumann or Dirichlet b.c. along $[-1, 1] \times \{c\}$ and the Neumann b.c. along $\{\pm 1\} \times [c, \infty)$. An easy calculation shows

$$dx^2 + dy^2 = \left(1 + y^2 \left(\frac{f'(x)}{f(x)}\right)^2\right)^{-1} \left(d\varepsilon^2 f^2(x) + d\eta^2 \left(\frac{f'(x)}{f(x)}\right)^2 \left(\frac{f(\eta)}{f'(\eta)}\right)^2\right). \tag{2.26}$$

In the notation of Section 2.3

$$\alpha(\varepsilon, \eta) = \left(\frac{f'(x)}{f(x)}\right)\left(\frac{f(\eta)}{f'(\eta)}\right) \cdot \left(1 + y^2 \left(\frac{f'(x)}{f(x)}\right)^2\right)^{-1/2}$$

$$\beta(\varepsilon, \eta) = f(x) \cdot \left(1 + y^2 \left(\frac{f'(x)}{f(x)}\right)^2\right)^{-1/2},$$

and it is a straightforward (but rather long) exercise in differentiation to show that

$$|\alpha(\varepsilon, \eta) - 1| = O(F(\eta)), \qquad |f(\eta)/\beta(\varepsilon, \eta) - 1| = O(F(\eta)),$$

$$|\hat{\nabla}\alpha(\varepsilon, \eta)| \leqslant \left| \frac{1}{f(\eta)} \frac{\partial \alpha(\varepsilon, \eta)}{\partial \varepsilon} \right| + \left| \frac{\partial \alpha(\varepsilon, \eta)}{\partial \eta} \right| = O(F(\eta)), \tag{2.27}$$

$$|\hat{\nabla}(f(\eta)/\beta(\varepsilon, \eta))| \leqslant \left| \frac{1}{f(\eta)} \frac{\partial (f/\beta)}{\partial \varepsilon} \right| + \left| \frac{\partial (f/\beta)}{\partial \eta} \right| = O(F(\eta)),$$

where

$$F(\eta) = |f(\eta)| + |f'(\eta)| + |f''(\eta)| + |f'(\eta)^2/f(\eta)|.$$

In obtaining (2.27) we have used the fact that the third derivative of f is bounded (recall (1.3)). Theorem 1.1. is now an immediate consequence of Lemmas 2.3, 2.4.

3. Dirichlet Laplacians on Regions with Cusps

3.1. *Some Generalities*

There have been quite a few results [1, 3, 4, 8, 11, 17, 18, 20] on the asymptotics of the eigenvalue distribution of H_D in regions Ω given by (1.3) when $f(x) \to 0$ and $\mathrm{Vol}(\Omega) = \infty$. Here we give a new treatment which, besides being elementary, seems to cover most of the interesting examples. We refer to the papers of Rosenbljum [17] and Davies [5] for a detailed discussion of the spectral properties of H_D in limit-cylindrical domains.

We suppose that f is convex and that

$$f(x) + f''(x) + f'(x)^2/f(x) \to 0 \qquad \text{as} \quad x \to \infty. \tag{3.1}$$

If $\mathrm{Vol}(\Omega) = \infty$, $\lim_{E \to \infty} N_E(H_D)/E = 0$ and we restrict ourselves to studying the operators H_D, $H_{D,N}$ on Ω_1 given by (2.25), with respectively Dirichlet or Neumann boundary condition on $\eta(x, y) = c$. Performing the same change of variable as that in the previous section, we obtain the Laplace–Beltrami operators on $M = (-1, 1) \times (c, +\infty)$ with the metric (2.26) and with the Dirichlet boundary conditions on $\{\pm 1\} \times [c, \infty)$ and the Dirichlet or Neumann b.c. on $[-1, 1] \times \{c\}$. Let us first analyze $H_{D,N}$. If $\hat{H}_{D,N}$ is the Laplace–Beltrami operator on M with metric (1.9) with the same boundary condition as $H_{D,N}$ we obtain as in Sections 2.3, 2.4 that for any $\varepsilon > 0$ we can find c big enough that

$$N_E((1-\varepsilon)\,\hat{H}_{N,D}) \geqslant N_E(H_{N,D}) \geqslant N_E((1+\varepsilon)\,\hat{H}_{N,D}). \tag{3.2}$$

Separating the variables, we obtain that $\hat{H}_{D,N}$ is unitarily equivalent to $\bigoplus_{n \geqslant 1} H_n$, given by (2.7), acting on $\bigoplus_{n \geqslant 1}^{\infty} L^2[c, \infty)$, and with the boundary conditions (2.8) at $x=c$. Formula (3.1) implies that $V(x)f(x)^2 \to 0$, and (eventually increasing ε in (3.2)) we can restrict ourselves to studying

$$A = \bigoplus_{n \geqslant 1} -\frac{d^2}{dx^2} + \left(\frac{n\pi}{2f(x)}\right)^2. \tag{3.3}$$

Starting with H_D on Ω_1, we end up with operator (3.3) with the Dirichlet boundary condition at c, which we denote by A_D. Formula (3.2) implies that $\lim_{E \to \infty} N_E(A)/E = \infty$, and, as in Section 2.2, we observe that the asymptotics of $N_E(A)$ does not depend on the boundary condition at c, nor on c itself. Consequently, we can restrict ourselves to studying A_D with $c=1$. The strategy is now clear: If we show that

$$\lim_{\varepsilon \to 0} \lim_{E \to \infty} \frac{N_E((1 \pm \varepsilon) A_D)}{N_E(A_D)} = 1, \tag{3.4}$$

we have $N_E(H_D) \sim N_E(A_D)$, and the asymptotics of the original problem follows.

To demonstrate the effectiveness of the above strategy, we prove Theorem 1.2.

3.2. *Proof of Theorem* 1.2

We can obviously restrict ourselves to studying only the horn $\Omega_1 = \{x : x > 1,\ |y| < x^{-\alpha}\}$, and multiplying the result by 2 if $0 < \alpha < 1$, or with 4 if $\alpha = 1$. Formula (3.1) is obviously valid. The operators A_D become

$$\bigoplus_{n \geqslant 1} -\frac{d^2}{dx^2} + \left(\frac{n\pi}{2}\right)^2 x^{2\alpha}$$

acting on $L^2[1, \infty)$. Suppose that we prove

$$\lim_{t \to 0} t^{1/2 + 1/2\alpha} \operatorname{Tr}(\exp(-tA_D)) = \frac{1}{2\sqrt{\pi}} \left(\frac{2}{\pi}\right)^{1/\alpha} \zeta\left(\frac{1}{\alpha}\right) \Gamma\left(\frac{1}{2\alpha} + 1\right), \quad \text{if} \quad 0 < \alpha < 1 \tag{3.5}$$

and

$$\lim_{t \to 0} t(\ln t^{-1})^{-1} \operatorname{Tr}(\exp(-tA_D)) = \frac{1}{4\pi}, \qquad \text{if} \quad \alpha = 1. \tag{3.6}$$

Then, by the Karamata–Tauberian theorem [18, 19]

$$N_E(A_D) \sim \frac{1}{2\sqrt{\pi}} \left(\frac{2}{\pi}\right)^{1/\alpha} \zeta(1/\alpha) \frac{\Gamma(1/(2\alpha)+1)}{\Gamma(\frac{3}{2} + 1/(2\alpha))} E^{1/2 + 1/(2\alpha)}, \qquad \text{if} \quad 0 < \alpha < 1,$$

$$N_E(A_D) \sim \frac{1}{4\pi} E \ln E \qquad \text{if} \quad \alpha = 1,$$

formula (3.4) is immediate and the theorem follows. It remains to prove (3.5), (3.6). It should not come as a suprise that the argument closely follows that of Section 2.2.

Case $0<\alpha<1$. The Gordon–Thompson inequality yields

$$\begin{aligned} t^{1/2+1/2\alpha}\operatorname{Tr}(\exp(-tA_D)) &\leqslant t^{1/2+1/2\alpha}\sum_{n>0}\frac{1}{2\sqrt{\pi t}}\int_1^\infty \exp(-t(n\pi/2)^2x^{2\alpha})\,dx \\ &\leqslant \frac{1}{2\sqrt{\pi}}\left(\frac{2}{\pi}\right)^{1/\alpha}\zeta\left(\frac{1}{\alpha}\right)\int_0^\infty \exp(-x^{2\alpha})\,dx \\ &\leqslant \frac{1}{2\sqrt{\pi}}\left(\frac{2}{\pi}\right)^{1/\alpha}\zeta\left(\frac{1}{\alpha}\right)\Gamma\left(\frac{1}{2\alpha}+1\right), \end{aligned}$$

and it is immediate that

$$\limsup_{t\to 0}\, t^{1/2+1/2\alpha}\operatorname{Tr}(\exp(-tA_D))\leqslant\frac{1}{\sqrt{\pi}}\left(\frac{2}{\pi}\right)^{1/\alpha}\zeta\left(\frac{1}{\alpha}\right)\Gamma\left(\frac{1}{2\alpha}+1\right).$$

It remains to prove

$$\liminf_{t\to 0}\, t^{1/2+1/2\alpha}\operatorname{Tr}(\exp(-tA_D))\geqslant\frac{1}{\sqrt{\pi}}\left(\frac{2}{\pi}\right)^{1/\alpha}\zeta\left(\frac{1}{\alpha}\right)\Gamma\left(\frac{1}{2\alpha}+1\right). \tag{3.7}$$

Make a partition of $[1,\infty)$ into intervals I_k of equal size $1/m$. Denote by H_k^D the Dirichlet Laplacian on I_k, and by

$$d_k=\sup_{x\in I_k} x^{2\alpha},\qquad Q_m(t)=mt^{1/2}\operatorname{Tr}(\exp(-tH_k^D)),\qquad V_m(x)=\sum_{k>0} d_k\cdot\chi_k(x), \tag{3.8}$$

where χ_k is the characteristic function of the interval I_k. Putting additional Dirichlet boundary conditions at the end points of intervals I_k we obtain

$$\begin{aligned} &t^{1/2+1/2\alpha}\operatorname{Tr}(\exp(-tA_D)) \\ &\quad\geqslant t^{1/2+1/2\alpha}\sum_{k>0}\operatorname{Tr}(\exp(-tH_k^D))\sum_{n>0}\exp(-t(n\pi/2)^2 d_k) \\ &\quad\geqslant Q_m(t)\,t^{1/2\alpha}\sum_{n>0}\int_1^\infty \exp(-t(n\pi/2)^2 V_m(x))\,dx \\ &\quad= Q_m(t)\left(\frac{2}{\pi}\right)^{1/\alpha}\sum_{n>0}\left(\frac{1}{n}\right)^{1/\alpha}\int_{t^{1/2\alpha}(n\pi/2)^{1/\alpha}}^\infty \exp(-V_m(x))\,dx \\ &\quad\geqslant Q_m(t)\left(\frac{2}{\pi}\right)^{1/\alpha}\sum_1^N\left(\frac{1}{n}\right)^{1/\alpha}\int_{t^{1/2\alpha}(N\pi/2)^{1/\alpha}}^\infty \exp(-V_m(x))\,dx. \end{aligned}$$

Using that

$$\lim_{t \to 0} Q_m(t) = \frac{1}{2\sqrt{\pi}}$$

we obtain

$$\liminf_{t \to 0} t^{1/2 + 1/2\alpha} \operatorname{Tr}(\exp(-tA_D))$$
$$\geqslant \frac{1}{\sqrt{\pi}} \left(\frac{2}{\pi}\right)^{1/\alpha} \sum_1^N \left(\frac{1}{n}\right)^{1/\alpha} \int_0^\infty \exp(-V_m(x))\, dx.$$

Letting $N \to \infty$ and $m \to \infty$ we obtain (3.7) and (3.5).

Case $\alpha = 1$. As before

$$\operatorname{Tr}(\exp(-tA_D)) \leqslant \frac{1}{2\sqrt{\pi t}} \sum_{n>0} \int_1^\infty \exp(-t(n\pi/2)^2 x^2)\, dx$$
$$= \frac{1}{t\pi\sqrt{\pi}} \sum_{n>0} \frac{1}{n} \int_{\sqrt{t}n\pi/2}^\infty \exp(-x^2)\, dx. \tag{3.9}$$

Split the positive integers into two sets, $I_1 = \{n : n\sqrt{t} \leqslant 2/\pi\}$ and $I_2 = \{n : n\sqrt{t} \geqslant 2/\pi\}$. We have

$$\sum_{n \in I_2} \frac{1}{n} \int_{\sqrt{t}n\pi/2}^\infty \exp(-x^2)\, dx \leqslant \frac{\pi\sqrt{t}}{2} \sum_{n \in I_2} \exp(-\sqrt{t}n\pi/2)$$
$$= O(1) \quad \text{as} \quad t \to 0, \tag{3.10}$$

$$\sum_{n \in I_1} \frac{1}{\pi\sqrt{\pi}} \int_{\sqrt{t}n\pi/2}^\infty \exp(-x^2)\, dx \leqslant \frac{1}{2\pi} \sum_{n \in I_1} \frac{1}{n}$$
$$\sim \frac{1}{4\pi} \ln t^{-1} \quad \text{as} \quad t \to 0. \tag{3.11}$$

In (3.11) we used that $1 + \frac{1}{2} + \cdots + 1/n - \ln n \to \gamma$, as $n \to \infty$, where γ is the Euler constant. From (3.9), (3.10), (3.11) we obtain

$$\limsup_{t \to 0} t(\ln t^{-1})^{-1} \operatorname{Tr}(\exp(-tA_D)) \leqslant \frac{1}{4\pi}.$$

To prove that

$$\liminf_{t \to 0} t(\ln t^{-1})^{-1} \operatorname{Tr}(\exp(-tA_D)) \geqslant \frac{1}{4\pi} \tag{3.12}$$

we proceed as follows. Let $I=\{n : n\sqrt{t}<2\varepsilon/\pi\}$, and with notation (3.8) we have

$$\begin{aligned} &t(\ln t^{-1})^{-1}\operatorname{Tr}(\exp(-tA_D)) \\ &\quad\geqslant Q_m(t)\, t^{1/2}(\ln t^{-1})^{-1} \sum_{n>0} \int_1^\infty \exp(-t(n\pi/2)^2\, V_m(x))\, dx \\ &\quad= Q_m(t)\, \frac{2}{\pi} \sum_{n>0} \frac{1}{n} (\ln t^{-1})^{-1} \int_{\sqrt{t}n\pi/2}^\infty \exp(-V_m(x))\, dx \\ &\quad\geqslant Q_m(t)\, \frac{2}{\pi} (\ln t^{-1})^{-1} \sum_{n\in I} \frac{1}{n} \int_\varepsilon^\infty \exp(-V_m(x))\, dx. \end{aligned}$$

As $t\to 0$,

$$Q_m(t)\to\frac{1}{2\sqrt{\pi}}, \qquad (\ln t^{-1})^{-1}\sum_{n\in I}\frac{1}{n}\to\frac{1}{2},$$

and consequently,

$$\liminf_{t\to 0}\, t(\ln t^{-1})^{-1}\operatorname{Tr}(\exp(-tA_D))\geqslant\frac{1}{2\pi\sqrt{\pi}}\int_\varepsilon^\infty \exp(-V_m(x))\, dx.$$

Letting $\varepsilon\to 0$ and $m\to\infty$ we obtain (3.12) and (3.6).

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