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Charge Deficiency, Charge Transport and Comparison of Dimensions

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Dedicated to Elliott Lieb on the occasion of his 60th birthday

Abstract: We study the relative index of two orthogonal infinite dimensional projections which, in the finite dimensional case, is the difference in their dimensions. We relate the relative index to the Fredholm index of appropriate operators, discuss its basic properties, and obtain various formulas for it. We apply the relative index to counting the change in the number of electrons below the Fermi energy of certain quantum systems and interpret it as the charge deficiency. We study the relation of the charge deficiency with the notion of adiabatic charge transport that arises from the consideration of the adiabatic curvature. It is shown that, under a certain covariance, (homogeneity), condition the two are related. The relative index is related to Bellissard's theory of the Integer Hall effect. For Landau Hamiltonians the relative index is computed explicitly for all Landau levels.

1. Introduction

An interesting observation that emerged in the last decade is that charge transport in quantum mechanics, in the absence of dissipation, often lends itself to geometric interpretation. A good part, but not all, of this research has been motivated by, and applied to, the integer and fractional Hall effect [2, 8, 11, 17, 20, 26, 32, 34, 35, 38, 44].

The framework that will concern us here is that of (non-relativistic) quantum mechanics. Within this framework common models of the integer Hall effect are Schrödinger operators associated with non-interacting electrons in the plane, with (constant) magnetic field perpendicular to the plane and random (or periodic) potential. The Hall conductance has been related to a Fredholm Index by Bellissard [5], and to a Chern number by Thouless, Kohmoto, Nightingale and den-Nijs [40]. The Fractional

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Hall effect is associated with electron-electron interaction and this goes beyond what we do here.

Quantum field theory is another framework where transport properties and geometry are related. The focal point here has been the Fractional Hall effect and the associated Chern-Simons field theories [7, 8, 18, 26, 42, 44]. We shall not address these issues.

The Chern number approach to quantum transport has been extended to a large class of quantum mechanical systems, including models of the integer Hall effect [17, 24, 25, 29–31, 41], to models with electron-electron interactions [3, 23, 30] and to other systems that bear only little resemblance to the Integer Hall effect [8, 14, 29, 35, 38]. The Index approach has not been as popular, and has not been substantially extended beyond the one electron setting considered by Bellissard for the integer Hall effect [5, 11, 28, 45].

We have two main purposes in this work. The first is to develop the Index approach from the physical point of view of "charge deficiency": Consider a quantum system of (non-interacting) electrons where the Fermi energy is in a gap. We allow an infinitely large number of electrons below the Fermi energy. Now consider taking this system through a cycle, so that at the end of the cycle the Schrödinger operator is unitarily related to the one at the outset. The examples we shall focus on here are where the initial and final systems are related by a *singular* gauge transformation corresponding to piercing the system with an infinitesimally thin flux tube, carrying one unit of quantum flux. Because of the unitary equivalence, at the end of the cycle we can put the Fermi energy in the same gap as at the outset, and can ask for the difference in the number of electrons below the Fermi energy. This deficiency of charge counts the charge transported in or out of the system as a result of the additional flux quantum. In interesting cases this difference is $\infty - \infty$. For non-interacting electrons, such a difference is the difference in dimensions of a pair of two infinite dimensional Hilbert space projections. This is the relative index. It turns out to be related to an index of an appropriate Fredholm operator. In particular, it is an integer. (The charge deficiency introduced here is reminiscent of a charge that enters in computing the vacuum polarization in Fock space. See [27].)

The identification of charge deficiency with an index implies integral charge transport. This holds for a wide class of two dimensional quantum system, including the conventional models of the integer Hall effect mentioned above. But it also holds for more general models whose geometries and background potentials may be far removed from the Integer Hall effect.

The theory described below appears to be restricted, at the moment at least, to non-interacting electrons. This is consistent with the common wisdom because electron-electron interaction will, in general, lead to fractional transport.

Our second purpose is to examine the relation of the charge deficiency (associated with an index) and the notion of charge transport that arises in theories of linear and adiabatic response. The latter is associated with Kubo's formulae, Chern numbers and adiabatic curvatures. These two notions are distinct in general. They turn out to be related for homogeneous systems. These are the kind of systems relevant to the Integer Hall effect.

This relation between charge deficiency and charge transport is reminiscent of known identities in related contexts: Štreda's formula (which is relating the Hall conductance with a gap label) [39] and certain Ward identities in Chern-Simons fields theories giving rise to relations between transport coefficients in linear response theory [18, 42].

2. Comparing Dimensions

In this section we describe various formulas for comparing dimensions of two orthogonal projections, P and Q. The index for two projectors of finite rank is just the difference of their dimensions.

$$\mathrm{Index}(P,Q) \equiv \dim P - \dim Q = \mathrm{Tr}(P-Q)\,. \tag{2.1}$$

A possible and, as we shall see, natural generalization of (2.1) to the infinite dimensional case is:

Definition 2.1. Let P and Q be orthogonal projections so that $P-Q$ is compact, then

$$\mathrm{Index}(P,Q) \equiv \dim(\mathrm{Ker}(P-Q-1)) - \dim(\mathrm{Ker}(Q-P-1))\,. \tag{2.2}$$

This Index is a well defined finite integer since $\dim(\mathrm{Ker}(P-Q\pm 1))$ are both finite by the compactness of $P-Q$. (One could take a broader perspective and define the left-hand side of (2.2) by the right-hand side whenever the latter makes sense). Before we discuss in what sense (2.2) is a generalization of (2.1) we note that the relative index indeed has some of the natural properties of an object that compares dimensions of two projections:

$$\begin{gathered}\mathrm{Index}(P,Q) = -\,\mathrm{Index}(Q,P) = -\,\mathrm{Index}(P_\perp,Q_\perp) = \mathrm{Index}(UPU^{-1},UQU^{-1})\,,\\ P_\perp \equiv 1-P\,, \qquad Q_\perp \equiv 1-Q\,,\end{gathered} \tag{2.3}$$

for any linear and invertible map U. The basic formulas for computing the relative Index is:

Proposition 2.2. *Suppose that $(P-Q)^{2n+1}$ is trace class for a natural number n, then*

$$\mathrm{Index}(P,Q) = \mathrm{Tr}(P-Q)^{2n+1}\,. \tag{2.4}$$

It follows that the right-hand side of (2.4) is independent of n for n large enough, and that it reduces to (2.1) in the finite dimensional case. We shall return to the proof of this proposition shortly.

To see where (2.4) comes, we start by noting an algebraic identity for any pair of projections P and Q:

$$(P-Q)^2P = P - PQP = PQ_\perp P = P(P-Q)^2\,. \tag{2.5}$$

In particular this says that $(P-Q)^2$ commutes with P and Q. This leads to:

Proposition 2.3. *Let n be a nonnegative integer so that $(P-Q)^{2n+1}$ is trace class, then:*

$$\mathrm{Tr}(P-Q)^{2n+3} = \mathrm{Tr}(P-Q)^{2n+1} \tag{2.6}$$

Proof. Subtracting the two equations below from each other

$$\begin{aligned}(P-Q)^{2n+2}P &= (P-Q)^{2n}(P-PQP)\,,\\ (P-Q)^{2n+2}Q &= (P-Q)^{2n}(Q-QPQ)\,,\end{aligned} \tag{2.7}$$

gives

$$(P-Q)^{2n+3} = (P-Q)^{2n+1} - (P-Q)^{2n}[PQ,QP]\,. \tag{2.8}$$

Since:

$$[PQ,QP] = [PQ,[Q,P]] = [PQ,[Q,P-Q]]\,, \tag{2.9}$$

we get, due to Eq. (2.5), the identity:

$$(P-Q)^{2n+3} = (P-Q)^{2n+1} - [PQ, B], \qquad B \equiv [Q, (P-Q)^{2n+1}]. \tag{2.10}$$

PQ is bounded and B is trace class, so $\mathrm{Tr}[PQ, B] = 0$. Tracing (2.10) gives (2.6). □

In the applications we never go beyond the trace class situation discussed above, in fact the case $n = 1$ covers all the cases we shall consider.

Proof of Proposition 2.2. (2.6) implies that $\mathrm{Tr}((P-Q)^{2m+1})$ is independent of m for $m \geq n$. As m goes to infinity, this trace converges to $\mathrm{Index}(P, Q)$ since $-1 \leq P - Q \leq 1$. Thus (2.4) is proven. □

In future work we'll examine this result further providing several other proofs which illuminate it.

In the applications we consider projectors P and Q on subspaces with energies below some fixed Fermi energy. $\mathrm{Index}(P, Q)$ then counts the difference in the number of electrons, which we identify with the charge deficiency. Physical considerations, that we shall describe in the following sections, motivate considering P and Q which are related by a unitary U:

$$Q = UPU^*. \tag{2.11}$$

In the finite dimensional case P and Q are related by a unitary if and only if their dimensions coincide. In the infinite dimensional case of a separable Hilbert space with $\dim P = \dim P_\perp = \dim Q = \dim Q_\perp = \infty$ such a U always exists, and does not force $\mathrm{Index}(P, Q) = 0$.

In the case that P and Q are related by a unitary map the index of the pair can be related to a Fredholm index of one single operator:

Proposition 2.4. *Let $Q = UPU^*$, P an orthogonal projections and U unitary and suppose that $(P-Q)^{2n+1}$ is trace class. Then,* $\mathrm{Tr}(P-PQP)^{n+1}$ *and* $\mathrm{Tr}(Q-QPQ)^{n+1}$ *are trace class; PUP is a Fredholm operator in range P and*

$$\begin{aligned} \mathrm{Index}(P, Q) &= \mathrm{Tr}([P, U]U^*)^{2n+1} = \mathrm{Tr}(P - PQP)^{n+1} - \mathrm{Tr}(Q - QPQ)^{n+1} \\ &= -(\dim \mathrm{Ker}(U \mid \mathrm{Range}\, P) - \dim \mathrm{Ker}(U^* \mid \mathrm{Range}\, P)) \\ &\equiv -\mathrm{Index}(PUP). \end{aligned} \tag{2.12}$$

Proof. The first identity is a rewrite of (2.4) upon noting that

$$P - Q = [P, U]U^*. \tag{2.13}$$

The second identity follows from (2.5) which gives:

$$\begin{aligned} (P - PQP)^{n+1} &= ((P-Q)^2 P)^{n+1} = (P-Q)^{2n+2} P, \\ (Q - QPQ)^{n+1} &= ((P-Q)^2 Q)^{n+1} = (P-Q)^{2n+2} Q, \end{aligned} \tag{2.14}$$

(proving our trace class assertion), subtracting and tracing using (2.4) and (2.6) gives the second identity. To get the third identity note that:

$$\begin{aligned} P - PQP &= P - PUPU^*P, \\ Q - QPQ &= U(P - PU^*PUP)U^*, \end{aligned} \tag{2.15}$$

using the unitary invariance of the trace we see that the third term in (2.12) can be written as:

$$\mathrm{Tr}(P - PUPU^*P)^{n+1} - \mathrm{Tr}(P - PU^*PUP)^{n+1}\,. \tag{2.16}$$

Since both terms are finite the oprators (PUP) and (PU^*P) are inverses of each other in range P up to compacts. A formula of Fedosov [19, 16] then says that under such circumstances (2.16) is a formula for $\mathrm{Index}(PU^*P)$ respectively $-\mathrm{Index}(PUP)$. □

We can now use the relation $\mathrm{Index}(P, Q) = -\mathrm{Index}(PUP)$, to transfer known facts about the Fredholm Index to the relative index, and vice versa.

Proposition 2.5. *Let P, Q, R be orthogonal projections, which differ by compacts. Then*

$$\mathrm{Index}(P, R) = \mathrm{Index}(P, Q) + \mathrm{Index}(Q, R)\,. \tag{2.17}$$

This identity is, of course, trivial in the situation where P, Q, R differ by trace class operators. When interpreted as charge deficiency, it is a statement of charge (or particle) conservation.

Proof. For simplicity we suppose that P, Q and R are unitarily related. Elsewhere we shall give a proof of the general case.

Equation (2.17) equivalent to:

$$\mathrm{Index}(P(U_2U_1)P) = \mathrm{Index}(PU_1P) + \mathrm{Index}(QU_2Q)\,. \tag{2.18}$$

Now we rewrite all expressions in terms of Q and the necessary unitaries:

$$\begin{aligned}\mathrm{Index}(PU_2U_1P) &= \mathrm{Index}(U_1^{-1}QU_1U_2QU_1)\\ &= \mathrm{Index}(QU_1U_2Q)\\ \mathrm{Index}(PU_1P) &= \mathrm{Index}(U_1^{-1}QU_1QU_1)\\ &= \mathrm{Index}(QU_1Q)\,.\end{aligned}$$

Hence it remains to show

$$\mathrm{Index}(QU_1U_2Q) = \mathrm{Index}(QU_1Q) + \mathrm{Index}(QU_2Q)\,. \tag{2.19}$$

The left-hand side can be replaced by $\mathrm{Index}(QU_1QU_2Q)$ because the difference of the corresponding operators is compact,

$$QU_1QU_2Q - QU_1U_2Q = Q[U_1, Q]U_1^{-1}U_1U_2Q\,.$$

This follows from the compactness of $[U_1, Q]U_1^{-1}$ and the fact that all the remaining terms are bounded. By a basic result of stability theory for indices [22] the index is invariant under perturbations by compacts. Furthermore by the product formula for Fredholm indices one gets

$$\mathrm{Index}(QU_1QU_2Q) = \mathrm{Index}(QU_1Q) + \mathrm{Index}(QU_2Q)\,, \tag{2.20}$$

This prove the proposition. □

Related questions are addressed in [9, 12, 15].

3. Gauge Transformations and Computations with Integral Kernels

In this section we introduce additional structure into the general operator theoretic framework of the previous section, which will accompany us throughout. It is motivated by the applications we have in mind, and involves conditions on the kind of projections we consider and the unitaries that relate them. In particular, the unitary that relates the orthogonal projections P and Q will be associated with a (singular) gauge transformation which corresponds to piercing the quantum system with a flux tube carrying an integral number of flux quanta. That is, U is a unitary multiplication operator whose winding is the number of flux quanta carried by the flux tube. (More precise conditions will be stated shortly.) This naturally forces us into considering two dimensional quantum systems. Furthermore, it turns out, that for Index$(P, Q) \neq 0$ the orthogonal projection P has to be infinite dimensional and time reversal invariance must be broken.

We describe this additional structure under

Hypothesis 3.1. (a) *The Hilbert space is $L^2(\Omega)$, where $\Omega \subseteq \mathbb{R}^2$ is a two dimensional domain in $\mathbb{R}^2$ with smooth (possibly empty) boundary $\partial\Omega$. In particular, the orthogonal projections P and Q of the previous section are projections in $L^2(\Omega)$.*
(b) *The projection P has integral kernel $p(x, y)$, $x, y \in \Omega$, which is jointly continuous in x and y and decays away from the diagonal, so that:*

$$|p(x,y)| \leq \frac{C}{1 + (\mathrm{dis}(x,y))^\eta} \tag{3.1}$$

with $\eta > 2$ and $\mathrm{dis}(x, y)$ is the distance between x and y.
(c) *U is a multiplication operator on $L^2(\Omega)$ by a complex valued function $u(x)$, with $|u(x)| = 1$, and $u(x)$ is differentiable away from a single point which we take to be $x = 0$. The derivative is $\mathcal{O}\left(\frac{1}{|x|}\right)$. More precisely, we assume that there are constants C_1 and C_2 such that:*

$$|u(x+y) - u(y)| \leq C_1 \frac{|x|}{|y|} \tag{3.2}$$

for $|x| \leq C_2 |y|$. The winding number of U about the singularity is denoted by $N(U)$. This is the number of magnetic flux quanta carried by the flux tube associated with U.

Example 3.2. Let $\Omega = \mathbb{R}^2$, and let $z = x + iy$.

$$u_\alpha(z) = \begin{cases} \dfrac{z^\alpha}{|z|^\alpha}, & z \in \mathbb{R}^2/[0,\infty] \\ 1, & z \in [0,\infty) \end{cases} \tag{3.3}$$

are unitaries which, for integer α, are smooth away from the origin and have winding number α. Such unitaries are associated with an infinitesimally thin flux tubes through the origin carrying α units of quantum flux. In particular, for $\alpha = 1$ condition (c) above holds with $C_1 = C_2^{-1} = 2$. This follows from the elementary inequality $|u_1(z) - u_1(z')| \leq |z - z'| \max\left(\frac{1}{|z|}, \frac{1}{|z'|}\right)$, which implies (3.2). On the other hand, if $\alpha \notin \mathbb{Z}$, condition (c) clearly fails near the positive real axis.

The fact that U is a gauge transformation distinguishes coordinate space, and in the rest of this section the integral kernel of $P - Q$ will play a role. In particular, we'd like to know that an object like $\mathrm{Tr}(P-Q)^3$ can be computed from the integral kernel of $P-Q$ by integrating on the diagonal. This somewhat technical issue is guaranteed by the following preparatory result:

Proposition 3.3. *Let K be trace class with integral kernel $K(x,y)$, $x,y \in \mathbb{R}^n$, which is jointly continuous in x and y away from a finite set of points (x_i,y_i) so that $K(x,x) \in L^1$ in neighborhoods of these points, then:*

$$\mathrm{Tr}\, K = \int_{\mathbb{R}^n} K(x,x)\, dx\,. \tag{3.4}$$

Sketch of proof. Let E_ε, F_ε, G_ε be the characteristic functions of the union of ε-balls about the singular points, the exterior of a $1/\varepsilon$ ball and the complement of these two sets. Then

$$\mathrm{Tr}(K) = \mathrm{Tr}(E_\varepsilon K) + \mathrm{Tr}(F_\varepsilon K) + \mathrm{Tr}(G_\varepsilon K G_\varepsilon)\,, \tag{3.5}$$

where we used cyclicity of the trace to get the last term. Since E_ε and F_ε converge strongly to zero as ε goes to 0, $E_\varepsilon K$ and $F_\varepsilon K$ go to zero in trace norm (as can be seen by writing K as a finite rank plus small trace norm), and since a result in [36] says that $\mathrm{Tr}(G_\varepsilon K G_\varepsilon)$ is the integral over G_ε of $K(x,x)$ the result follows by taking the limit using the fact that $K(x,x)$ is L^1. This proves Proposition 3.3. □

Proposition 3.3 could be replaced by the following statement which is a kind of a Lebesgue integral version of Proposition 3.3 [6]. Its application to the concrete cases we have in mind requires however somewhat more care.

Remark 3.4. Let K be trace class on $L^2(\mathbb{R}^n)$. Then, its integral kernel $K(x,y)$ may be chosen so that the function $L(x,y) \equiv K(x,x+y)$ is a continuous function of y with values in $L^1(\mathbb{R}^n)$. Furthermore if $l(y) \equiv \int L(x,y)\,dx$ then $\mathrm{Tr}\, K = l(0)$.

Our first application is the following result that guarantees that $\mathrm{Index}(P-Q)=0$ if $P-Q$ is trace class:

Proposition 3.5. *Suppose $P-Q$ is trace class with $Q = UPU^{-1}$, U and P satisfying Hypothesis 3.1. Then* $\mathrm{Index}(P,Q) = \mathrm{Tr}(P-Q) = 0$.

Proof. The integral kernel of $P-Q$ is:

$$(P-Q)(x,y) = p(x,y)\left(1 - \frac{u(x)}{u(y)}\right). \tag{3.6}$$

By Proposition 3.3, $\mathrm{Tr}(P-Q)$ is the integral of (3.6) on the diagonal with $x=y$. But the kernel of $(P-Q)$ vanishes on the diagonal. Hence the trace is zero. □

This means the trace class situation is like the finite dimensional case, i.e. unitary equivalence of P and Q implies equality of dimensions in the generalized sense. In particular, for $\mathrm{Index}(P,Q) \neq 0$, $(P-Q)$ must not be trace class, so $\dim P = \dim Q = \infty$.

The following proposition is central.

Proposition 3.6. *Under Hypothesis* 3.1 *$(P-Q)^p$ is trace class for $p>2$. In particular $\mathrm{Tr}(P-Q)^3$ is an integer and*

$$-\,\mathrm{Index}(PUP) = \int_{\Omega^3} dx\,dy\,dz\, p(x,y)p(y,z)p(z,x) \times \left(1-\frac{u(x)}{u(y)}\right)\left(1-\frac{u(y)}{u(z)}\right)\left(1-\frac{u(z)}{u(x)}\right). \tag{3.7}$$

Remarks 3.7. 1. In the case where $p(x,y)$ is C_0^∞ the proposition is in [11].
2. The index is real, of course. Under complex conjugation the first triple product in (3.7) becomes $p(y,x)p(z,y)p(x,z)$, since, by self adjointness $\bar{p}(x,y) = p(y,x)$. The second triple product transforms to $\left(1-\frac{u(y)}{u(x)}\right)\left(1-\frac{u(z)}{u(y)}\right)\left(1-\frac{u(x)}{u(z)}\right)$ by the unimodularity of $u(x)$. This reduces to the original integrand upon interchanging x and z.
3. If we were to consider, for example, $\mathbb{R}^3$, then the integrand in (3.7) under Hypothesis 3.1 would lack decay in the direction of the magnetic flux tube, and (3.7) would become meaningless, in general.
4. Flux tubes that carry fractional fluxes are associated with unitaries of Example 3.2 with $\alpha \notin \mathbb{Z}$. For such U's, the integrand in (3.7) lacks decay along the cut, and the integral is divergent in general.
5. This proposition also tells us that, as far as Sect. 2 is concerned, $n = 1$ is all we have to consider.

Proof. By Hypothesis 3.1 $P-Q$ is an integral operator with kernel $p(x,y)\left(1-\frac{u(x)}{u(y)}\right)$. To prove that $(P-Q)^p$, $p > 2$, is trace class it is enough to show that the function

$$g(y) \equiv \int \left|p(x+y,y)\left(1-\frac{u(x+y)}{u(y)}\right)\right|^q dx \in L^{p-1}(\mathbb{R}^2)\,, \qquad 1/p+1/q=1\,, \tag{3.8}$$

because of Russo's theorem [33]. To prove (3.8) notice that close to the diagonal $x = 0$ the second term of the integrand in (3.8) is small, off the diagonal it is the first one which is small. To put this in analytic form we note firstly that it is enough to prove (3.8) in the following situation: Replace in (3.8) $p(x+y,y)\left(1-\frac{u(x+y)}{u(y)}\right)$ by the function

$$f(x,y) \equiv \frac{1}{1+|x|^\eta}\,\mathrm{Min}\left\{C_2, \frac{|x|}{|y|}\right\}, \tag{3.9}$$

where C_2 is the constant introduced in Hypothesis 3.1; i.e. it is enough to prove

$$F(y) \equiv \int (f(x,y))^q\, dx \in L^{p-1}(\mathbb{R})\,, \tag{3.10}$$

because, by construction, $\left|p(x+y,y)\left(1-\frac{u(x+y)}{u(x)}\right)\right|$ is pointwise dominated by a constant multiple of $f(x,y)$. Secondly we show that F is uniformly bounded in y. This follows from the y independent bound on $f(x,y)$,

$$(f(x,y))^q \le \mathrm{const}\left(\frac{1}{1+|x|^\eta}\right)^q \tag{3.11}$$

together with $\eta q - 2 > 0$ (use $\eta > 2$ and $q > 1$). Hence the right-hand side of (3.10) is integrable. Thirdly we analyze the behavior of F for large y. To do that we split the defining integral into two pieces and prove that each term is in $L^{p-1}(\mathbb{R})$. The first term is defined by

$$F_1(y) \equiv \int_{I(y)} (f(x,y))^q\, dx\,, \tag{3.12}$$

where $I(y) \equiv \{x \mid |x| \leq C_2|y|\}$ denotes the domain close to the diagonal $x = 0$. By construction it satisfies the estimate

$$F_1(y) \leq \frac{1}{|y|^q} \int\limits_{I(y)} \frac{|x|^q}{(1+|x|^\eta)^q}\, dx\,. \tag{3.13}$$

Cutting out the unit ball B in $I(y)$ we get the inequality

$$F_1(y) \leq \frac{\pi}{|y|^q} + \frac{1}{|y|^q} \int\limits_{I(y)\setminus B} \frac{|x|^q}{(1+|x|^\eta)^q}\, dx\,. \tag{3.14}$$

The second term is bounded up to a constant 2π by

$$\frac{1}{|y|^q} \int\limits_1^{|y|} r^{q+1-\eta q} dr = \frac{1}{|y|^q}\left(\frac{1}{|y|^{\eta q-q-2}} - 1\right). \tag{3.15}$$

Hence one gets the inequality

$$F_1 \leq \text{const}\, \frac{1}{|y|^q} + \text{const}\, \frac{1}{|y|^{\eta q-2}}\,. \tag{3.16}$$

Because $(p-1)q-2 = p-2 > 0$ and $(p-1)(\eta q-2)-2 = (\eta-2)p > 0$ both terms on the right-hand side of (3.16) are in $L^{p-1}(\mathbb{R}^2_y)$. The second term in the decomposition of F is

$$F_2(y) \equiv \int\limits_{I(y)^c} (f(x,y))^q\, dx = C_2 \int\limits_{|x|\geq C_2|y|} \frac{1}{(1+|x|^\eta)^q}\, dx\,. \tag{3.17}$$

The integrand has no decay in y, however the domain of integration shrinks for increasing y. An explicit computation proves

$$F_2(y) \leq \text{const}\, \frac{1}{|y|^{\eta q-2}}\,. \tag{3.18}$$

Hence F is again in $L^{p-1}(\mathbb{R})$, and the theorem is proved. □

We close with the following observations about Index(PUP). The first is a statement of stability of the relative index with respect to deformations of the flux tube such as translating and other local deformations, and is a consequence of the stability of the Fredholm index under compact perturbations. We state one special case only:

Proposition 3.8. *Let U be a gauge transformation as in Hypothesis* 3.1 *and let T be a translation, then:*

$$\text{Index}(PUP) = \text{Index}(PTUT^*P) = \text{Index}(P_T U P_T)\,, \qquad P_T \equiv TPT^*\,. \tag{3.19}$$

Sketch of Proof. $P(U - T^*UT)$ is a compact operator. This can be seen by an adaptation of the proof of Proposition 3.6 to the present case. The stability of the index under compact perturbations gives the first equality. The second one follows from the invertibility of T and the definition of the index. □

This makes the charge deficiency insensitive to the positioning of the flux tube, (and so a global property of the system).

There are classes of projections where the relative index is guaranteed to vanish. Experience with examples, such as the quantum Hall effect, have led to the recognition that nontrivial charge transport is intimately connected with breaking time reversal symmetry. Indeed:

Theorem 3.9. *Let U and P satisfy Hypothesis* 3.1 *and P be time reversal invariant, then* $\mathrm{Index}(PUP) = 0$.

Proof. Since the relative index is real, (3.7) is even under conjugation. On the other hand, time reversal invariance says that (3.7) is odd under conjugation, so the index must vanish. To see this, recall that time reversal says that (in the spinless case) the integral kernel of P is real [43]. It follows that the first triple product in (3.7), $p(x,y)p(y,z)p(z,x)$, is even under conjugation. The second triple product of (3.7), $\left(1-\frac{u(x)}{u(y)}\right)\left(1-\frac{u(y)}{u(z)}\right)\left(1-\frac{u(z)}{u(x)}\right)$, is odd under conjugation, since $u(x)$ is unimodular. It follows that the integrand in (3.7) is odd under conjugation. □

Remark 3.10. It is easy to extend this proof to the case of spin, and to generalized notions of time reversal.

The next triviality result has nothing to do with time reversal, but rather with the geometry of Ω. It states that one can not remove too much of $\mathbb{R}^2$ around the flux tube without making the relative index trivial. In particular, if Ω is contained in a wedge, and the flux is outside Ω, the index vanishes. More precisely:

Theorem 3.11. *Let U be a flux tube through the origin so that U and P satisfy Hypothesis* 3.1, *and let Ω be contained in a wedge excluding the origin,* i.e. $\Omega \subset \{z \mid z \in \mathbb{C}, \varepsilon < \arg z < 2\pi - \varepsilon, \varepsilon > 0\}$, *then,* $\mathrm{Index}(PUP) = 0$.

Proof. Suppose $\mathrm{Index}(PUP) = m$, $m \neq 0$. Take $V \equiv U^{1/2m}$ with cut along $[0, \infty)$, and so entire outside Ω. Since P is a projection in $L^2(\Omega)$, $p(x,y) = 0$ if either x or y is in Ω. Proposition 3.6 then can be adapted to this case with V replacing U, using the fact that near the edges of the wedge the decay in (3.1) replaces the decay in (3.2). It follows that $\mathrm{Index}(PVP)$ must be an integer. On the other hand a little argument, using Proposition 2.5 and Eq. (2.3) shows that $m = \mathrm{Index}(PUP) = 2m\,\mathrm{Index}(PVP)$. This is a contradiction. Hence $m = 0$. □

4. Covariant Projections

In this section we consider the relative index for projections arising in the study of homogeneous systems. Here we concentrate on the case of a single Hamiltonian. In Sect. 8 we shall consider families of Hamiltonians with random potentials which are covariant and ergodic under translation. The random case is of course much more interesting from the point of view of applications to real systems. Mathematically the case of one single covariant Hamiltonian is however the core of the matter as it will be seen latter.

The main result of this section, Theorem 4.2, gives a formula for $\mathrm{Index}(PUP)$ which holds for projections, which, in addition to the assumption on the decay of their integral kernel, (3.1), also satisfy a condition of covariance (or homogeneity): Projections that are translation invariant up to a gauge transofrmation. This formula plays a key role in relating the index to the adiabatic curvature and Kubo's formula, something we shall return to in the following sections.

Definition 4.1. We say that a projection P in $L^2(\mathbb{R}^n)$ is covariant if its integral kernel satisfies:

$$p(x,y) = \mathscr{U}_a(x)p(x-a,y-a)\mathscr{U}_a^{-1}(y) \qquad a,x,y \in \mathbb{R}^n\,. \tag{4.1}$$

$\mathscr{U}_a(x)$ denotes a family of unitary continuously differentiable multiplication operators, i.e. nonsingular gauge transformations.

This notion of covariance is motivated by the covariance for Schrödinger operators with constant magnetic fields [46].

It follows that the first triple product in the integrand in (3.7) is invariant under translation of all arguments x, y, z:

$$\begin{aligned} p(x,y)p(y,z)p(z,x) &= p(x-t,y-t)p(y-t,z-t)p(z-t,x-t) \qquad t\in\mathbb{R}^2\\ &= p(0,y-x)p(y-x,z-x)p(z-x,0)\,. \end{aligned} \tag{4.2}$$

This property can be used to reduce the six dimensional integral in the computation of Index(PUP) in (3.7) to a four dimensional one, provided we can say something about two dimensional integrals with the integrand $\left(1-\dfrac{u(x-a)}{u(x-b)}\right)\times\left(1-\dfrac{u(x-b)}{u(x-c)}\right)\left(1-\dfrac{u(x-c)}{u(x-a)}\right)$, where a, b and c are fixed points in $\mathbb{R}^2$. That such integrals can be evaluated explicitly, and have geometric significance is a result of Connes [11] and is a rather amazing fact. Lemma 4.4 is in part a simplification of the derivation and a generalization of the original observation of Connes to the case of singular gauge transformations (Connes proof works however also for the upper half plane).

Theorem 4.2. *Let P be a covariant projection in $L^2(R^2)$ satisfying the decay properties (3.1) and let U be a (singular) gauge transformation satisfying Hypothesis* 3.1, *with winding $N(U)$. Then:*

$$\mathrm{Index}(PUP) = -\,2\pi i N(U)\int_{\mathbb{R}^4} dx\,dy\,p(0,x)p(x,y)p(y,0)\,x\wedge y\,, \tag{4.3}$$

where $x\wedge y \equiv x_1y_2 - x_2y_1$, $x\equiv(x_1,x_2)$ and $y\equiv(y_1,y_2)$.

Remark 4.3. The self-adjointness of P gives $p(x,y)=\bar{p}(y,x)$, making the Index real. If $p(x,y)$ is real the index is manifestly 0, as it should (by Theorem 3.9).

The proof of the theorem needs an evaluation of an integral.

Lemma 4.4. *Let $N(U)$ denote the winding number of the multiplication operator U satisfying Hypothesis* 3.1. *Then:*

$$\begin{aligned} &\int_{\mathbb{R}^2} dx\left(1-\frac{u(x-a)}{u(x-b)}\right)\left(1-\frac{u(x-b)}{u(x-c)}\right)\left(1-\frac{u(x-c)}{u(x-a)}\right)\\ &\quad = 2\pi i N(U)\,\mathrm{Area}(a,b,c) \end{aligned} \tag{4.4}$$

with $a,b,c,\in\mathbb{R}^2$ and $\mathrm{Area}(a,b,c)\equiv a\wedge b + b\wedge c + c\wedge a$ *is twice the oriented area of the triangle with vertices a, b and c.*

Proof. Let

$$e(x,y) \equiv \left(\frac{u(x)}{u(y)} - \frac{u(y)}{u(x)}\right) = -\,e(y,x)\,. \tag{4.5}$$

Then

$$
\begin{aligned}
C(a,b,c) &\equiv \int_{\mathbb{R}^2} dx(e(x-a,x-b)+e(x-b,x-c)+e(x-c,x-a)) \\
&= -\int_{\mathbb{R}^2} dx\left(1-\frac{u(x-a)}{u(x-b)}\right)\left(1-\frac{u(x-b)}{u(x-c)}\right) \\
&\quad \times\left(1-\frac{u(x-c)}{u(x-a)}\right),
\end{aligned}
\tag{4.6}
$$

since the integrands of the two integrals are the same up to a minus sign. The integral converges absolutely since each of the 3 factors can be estimated by:

$$
\begin{aligned}
\left|1-\frac{u(x-a)}{u(x-b)}\right| &\leq \text{const}\,|a-b| \max\left\{\frac{1}{|x-a|}, \frac{1}{|x-b|}\right\} \\
&\leq \text{const}\,\frac{|a-b|}{|x|},
\end{aligned}
\tag{4.7}
$$

for $|x| \geq \text{const} \times (|a|+|b|)$.

$C(a,b,c)$ has several manifest properties that want to make it proportional to the oriented area of the triangle with vertices a,b,c: 1. It is even or odd under cyclic or anti-cyclic permutations of a,b,c respectively. 2. It is translation invariant:

$$
C(a+t,b+t,c+t) = C(a,b,c), \qquad a,b,c,t \in \mathbb{R}^2. \tag{4.8}
$$

This suggests looking at mixed second derivatives. There is a problem however with differentiability of the integrand in the vicinity of a, b and c and with convergence of the integral at infinity. For that reason this bad set is cut out. Let $B_\varepsilon(a)$ denote the ball of radius ε around a and let D_ε be defined by:

$$
D_\varepsilon \equiv B_{1/\varepsilon}(0)/(B_\varepsilon(a) \cup B_\varepsilon(b) \cup B_\varepsilon(c)). \tag{4.9}
$$

D_ε is a large disk punctured near the three points a, b and c. $C(a,b,c)$ is the $\varepsilon \to 0$ limit of:

$$
C_\varepsilon(a,b,c) \equiv \int_{D_\varepsilon} dx(e(x-a,x-b)+e(x-b,x-c)+e(x-c,x-a)). \tag{4.10}
$$

Since $C_\varepsilon(a,b,c)$ changes sign if two of its arguments are interchanged, it is enough to look at the anti-symmetric second derivatives, i.e.:

$$
\begin{aligned}
(\partial_{a_1}\partial_{b_2}-\partial_{a_2}\partial_{b_1})C_\varepsilon(a,b,c) &= \int_{D_\varepsilon}(\partial_{a_1}\partial_{b_2}-\partial_{a_2}\partial_{b_1})e(x-a,x-b) \\
&= \int_{D_\varepsilon}(\partial_2\bar{u}(x-b)\,\partial_1 u(x-a)-\partial_1\bar{u}(x-a)\,\partial_2 u(x-b)) \\
&\quad -(1\leftrightarrow 2), \qquad \varepsilon > 0.
\end{aligned}
\tag{4.11}
$$

Using the notation of differential forms and Stokes' theorem one gets in the limit $\varepsilon \to 0$:

$$
\begin{aligned}
(\partial_{a_1}\partial_{b_2} - \partial_{a_2}\partial_{b_1})C_\varepsilon(a,b,c) &= -\left(\int_{D_\varepsilon} d\bar{u}(x-a)\,du(x-b) - \text{c.c}\right) \\
&= -\int_{\partial D_e} (\bar{u}(x-b)\,du(x-a) - \text{c.c}) \\
&\to -4\pi i N(U)\,.
\end{aligned}
\tag{4.12}
$$

The boundary ∂D_ε is made of one large circle, and three tiny circles around the puncture at a, b and c. In the limit $\varepsilon \to 0$ the small circles around a, b, c do not contribute to the integral. The large circle however produces the winding number up to the factor $2(2\pi i)$.

An additional argument shows that the only non-vanishing second derivatives of $C(a,b,c)$ are the ones just considered (and their cyclic permutations) and that the limit $\varepsilon \to 0$ and derivation may be interchanged.

To reconstruct $C(a,b,c)$ from its second derivatives we integrate (4.12) twice and get:

$$
C(a,b,c) = \alpha + \beta(a,b,c) - 2\pi i N(U)\,\text{Area}(a,b,c)\,, \tag{4.13}
$$

where α is a constant and β a linear function. Since $C(0,0,0) = 0$, we learn that $\alpha = 0$. Since $C(a,b,c)$ and Area(a,b,c) are even/odd under permutations of a, b, c, so is $\beta(a,b,c)$. Since β is linear it must vanish identically. This finishes the proof of Lemma 4.4. □

Now we return to the proof of Theorem 4.2. Using the previously introduced notation (4.6) and translational invariance (4.2) in (3.7) one gets:

$$
\text{Index}(PUP) = \int dy\,dz\,p(0,y)p(y,z)p(z,0)C(0,-y,-z)\,. \tag{4.14}
$$

By Lemma 4.4 the proof is finished. □

5. Charge Deficiency and Charge Pumps

The wave function of n non-interacting fermions gives rise to an n-dimensional projection in the one particle Hilbert space. Therefore Index$(P,Q) = \dim P - \dim Q$ counts the difference of the corresponding number of fermions. We shall adopt the point of view that, with Definition 2.1, Index(P,Q) also correctly counts the difference in the number of Fermions associated with infinite dimensional projections P and Q.

Suppose we fix the Fermi energy in a gap in the spectrum of the Schrödinger operator, and consider the associated spectral projection P. We show in Appendix A that for a wide class of Schrödinger operators, the integral kernel of P satisfies the decay and regularity hypothesis in Sect. 3. (Presumably, these conditions are satisfied under weaker conditions, e.g. in the absence of an energy gap, but provided the Fermi energy is in a region of "localized states".) Let U be a singular gauge transformation which introduces $N(U)$ flux quanta into the system. $Q = UPU^*$ describes the spectral projection associated with the same Fermi energy, (also in a gap, by unitary invariance), with extra $N(U)$ units of quantum flux, piercing Ω at points. Hence Index(P, Q) counts the change in the number of electrons below the Fermi energy.

It is clear from Proposition 2.5, and is manifest in Theorem 4.2, that Index(PUP) is linear in the number of flux quanta carried by the a flux tube: If the flux tube U_1 adds charge q_1 and U_2 adds charge q_2, then U_1U_2 would add $(q_1 + q_2)$ charges. It is therefore natural to define the charge deficiency in terms of what a flux tube carrying one unit of quantum flux does. And, for the sake of concreteness we chose a specific (rotationally symmetric) flux tube:

Definition 5.1. For a spectral projection P of a Schrödinger operator in $L^2(\Omega)$, $\Omega \subseteq \mathbb{R}^2$, and $z = x + iy$, the charge deficiency is the Fredholm index Index $\left(P \frac{z}{|z|} P\right)$, whenever the latter is well defined.

In many simple cases the charge deficiency vanishes. Proposition 3.5 tells us that this is always the case for (reasonable Schrödinger operators associated with) compact domains where the number of electrons is finite. Nontrivial deficiency therefore requires an infinite number of Fermions. Theorem 3.10 tells us that even for non-compact domains with infinite number of Fermions, the deficiency vanishes whenever the flux tube is outside Ω and Ω is contained in a wedge. This leaves us with infinite domains that encircle the flux tube. Finally, even for these, Theorem 3.9 tells us that the deficiency vanishes whenever P is time reversal invariant. In particular, this is the case in the absence of gauge fields.

It is now natural to ask whether there are examples of Schrödinger operators whose spectral projections have non-trivial deficiencies. One way to break time reversal is with constant magnetic fields. As we shall see in Sect. 7, the simplest example of this kind, the Landau Hamiltonian associated with the Euclidean plane, has unit deficiency for each Landau level. It would be interesting to have aditional examples where the deficiency is computable and non-zero. In particular, it would be interesting to have examples where time reversal is broken in more subtle ways, for example, with Aharonov-Bohm fluxes.

Charge pumps are quantum mechanical devices which transfers an integer charge in each cycle. An interesting class of such pumps has been introduced by [29]. The kind of systems discussed in this paper are also charge pumps. They have a natural cycle of one unit of quantum flux and the periodicity is exact for non-interacting electrons. As real electrons are pumped, the pump charges. This may modify the effective potential in the one electron theory, and ultimately change the index, destroying the periodicity. Charging effects are, of course, smaller the larger the capacitance of Ω.

A pump of the kind discussed here is stable in the sense that deformations in the domain Ω, the potentials, the location of the flux tube or the Fermi energy would preserve the deficiency.

To clarify the concept of charge deficiency for the pair of projectors P and $Q = UPU^{-1}$ of the two Schrödinger operators H and UHU^{-1} let us introduce a canonical interpolation between the two:

$$H(t) = (-i\nabla - \phi(t)\,(\nabla \arg z) - A_0)^2 + V\,, \qquad t \in [0, 1]\,,$$

where $\phi(t)$ interpolates smoothly between zero and one. $\nabla \arg z$ denotes a vector field on the real two plane respectively the complex plane. $H(t)$ has, by definition, a *time independent* domain of definition. It is *not* unitary equivalent to H through conjugation with $U(t) = e^{it \arg z}$ because the domain of H is not invariant under $U(t)$ for t in the interior of the interval $[0, 1]$.

If we consider the time dependent dynamical system defined by the Schrödinger operator $H(t)$, it is evident, that in addition to the magnetic field $B = \nabla \times A_0$ there is an electric field $\dot{\phi}(t)\nabla \arg z$. It points in the azymuthal direction. Hence a charge experiences a Lorentz force and is pushed from the center of the flux tube to infinity. This motivates the interpretation of P and Q as physical states related through adiabatic dynamics of the time dependent Hamiltonian $H(t)$ and the terminology "charge deficiency."

Much of the discussion above has analogs in the analysis of the quantum Hall effect based on localization of wave functions [26, 20, 32, 41].

6. Adiabatic Curvature and Hall Conductance

In this section we discuss the Hall charge transport, which is a priori distinct from charge deficiency discussed in previous sections. This notion is related to adiabatic curvature, Chern numbers, and to Kubo's formula. We describe this in some detail. The main result, Theorem 6.8, says that under appropriate conditions the Hall charge transport and charge deficiency are related.

As in our discussion of charge deficiency, we consider a cycle of Schrödinger operators associated with a gauge transformation. However, the gauge transformation is not associated with a flux tube that pierces the system. Rather, it is associated with a (finite) voltage drop across the system whose time integral is a unit of quantum flux. This voltage drop is associated with a class of functions, which we call *switches* and which, roughly, look like the graphs of $\frac{1}{2}\tanh(x)$. More precisely:

Definition 6.1. $\Lambda(x)$, $x \in \mathbb{R}$, a function of one variable, is called a switch if it is a continuously differentiable, real valued, monotone, non-decreasing function such that the limits at $+\infty$ and $-\infty$ exist and

$$\Lambda(\infty) - \Lambda(-\infty) = 1\,. \tag{6.1}$$

The setting relevant to this section is described in the following:

Hypothesis 6.2. *Consider the family of, unitarily equivalent, magnetic Schrödinger operators in* $L^2(\mathbb{R}^2)$,

$$\begin{aligned} H(A,V) &\equiv (-\,\mathrm{i}d - A)^2 + V \\ &= e^{\imath(\Phi_1\Lambda_1+\Phi_2\Lambda_2)}((-\,\mathrm{i}d - A_0)^2 + V)\,e^{-\imath(\Phi_1\Lambda_1+\Phi_2\Lambda_2)}\,, \\ A &\equiv A_0 + \Phi_1 d\Lambda_1 + \Phi_2 d\Lambda_2\,, \end{aligned} \tag{6.2}$$

where:

a) A_0 *and* V, *the vector and scalar potentials, satisfy the (mild) regularity conditions in Appendix A;* $\Phi \equiv (\Phi_1, \Phi_2) \in \mathbb{R}^2$ *and* Λ_1, Λ_2 *are both switches.*

b)

$$P(\Phi) = e^{\imath(\Phi_1\Lambda_1+\Phi_2\Lambda_2)}P(0)\,e^{-\imath(\Phi_1\Lambda_1+\Phi_2\Lambda_2)} \tag{6.3}$$

is a family of spectral projections for $H(A,V)$ *associated with a Fermi energy in a gap in the spectrum.*

Remarks. 1. In Appendix A we show that b) of Hypothesis 6.2 implies that the integral kernel of P satisfies the regularity and decay properties in Hypothesis 3.1.
2. In the case where Φ is time dependent, $\dot{\Phi}_1$ is the voltage drop along the x-axis and $\dot{\Phi}_2$ is the voltage drop along the y-axis.

3. The monotonicity condition on the switch functions implies integrability of the derivative of switches in the absolute sense and enters in the proof of Proposition 6.9.

We recall:

Definition 6.3. The adiabatic curvature associated to P is:

$$\omega_{12} \equiv iP[\partial_{\Phi_1} P, \partial_{\Phi_2} P]P\,. \tag{6.4}$$

A direct calculation gives:

$$\begin{aligned}\omega_{12} &= -i[PA_1P, PA_2P] \\ &= iP[A_1P_\perp A_2 - A_2P_\perp A_1]P = i([P, A_1]P_\perp[P, A_2] - (1 \leftrightarrow 2))\,.\end{aligned} \tag{6.5}$$

Furthermore, since A_1 and A_2 are multiplication operators:

$$\omega_{12}(\Phi) = e^{i(\Phi_1 A_1 + \Phi_2 A_2)}\omega_{12}(0)\,e^{-i(\Phi_1 A_1 + \Phi_2 A_2)}\,. \tag{6.6}$$

It would be nice if Hypothesis 6.2 were to imply that the adiabatic curvature is trace class. Since we do not know if this is the case, we shall study traces by taking limits. To this end we introduce:

Notation 6.4. *Let $\Omega \subset \mathbb{R}^2$ denote the square box $[-L, L] \times [-L, L]$, and let χ_Ω be the characteristic function of the box. $|\Omega|$ denotes the area of the box.*

The unitary equivalence of the family in (6.2), makes the adiabatic curvature Φ independent in the following sense:

Proposition 6.5. *Let P be a spectral projection associated with a gap, then $\chi_\Omega\omega_{12}\chi_\Omega$ is trace class and its trace is independent of Φ.*

Proof. Since $A_1P_\perp A_2 - A_2P_\perp A_1$ is bounded it is enough to prove that $\chi_\Omega P$ is Hilbert-Schmidt (recall that all Schatten classes are ideals). By the theorem in Appendix A the integral kernel of P satisfies the decay properties (3.1). Consequently,

$$\int dx\,dy |\chi_\Omega(x)p(x,y)|^2 < \infty\,. \tag{6.7}$$

The Φ-independence is obvious from (6.6). □

For our purpose, the most convenient way of introducing charge transport in the Hall effect is to *define* it by:

Definition 6.6. The Hall charge transport, Q, is

$$Q \equiv -2\pi \lim_{L\to\infty} \operatorname{Tr} \chi_\Omega\omega_{12}\chi_\Omega\,. \tag{6.8}$$

Remark 6.7. a) Theorem 6.8 below guarantees the existence of the limit, under the conditions in Hypothesis 6.2.
b) In our units, the Hall conductance is $Q/2\pi$.
c) Our sign convention is such that the Hall conductance of a full Landau level is $1/2\pi$.

The physical interpretation of charge transport introduced here is the following. It is the charge that crosses the x_1 axis, in the positive direction, as the Hamiltonians in (6.2) undergo a cycle corresponding to adiabatically increasing Φ_1 from 0 to 2π. (Alternatively, it is minus the charge that crosses the x_2 axis as the Hamiltonians in (6.2) undergo a cycle corresponding to adiabatically increasing Φ_2 from 0 to 2π.) This is the transport in the Hall effect. For more on this the reader may want to consult [3, 20, 23, 30, 31].

The following theorem is the central result of this section. It says that the Hall conductance can sometimes be interpreted as an index. The strategy is to show that Definition 6.6 can be put into the form of Theorem 4.2.

Theorem 6.8. *Suppose P is a covariant projector, P and $\Lambda_{1,2}$ satisfy Hypothesis 6.2. Then the Hall charge transport Q equals the charge deficiency:*

$$Q = -2\pi i \int dy\, dz\, p(0,y)p_{\perp}(y,z)p(z,0)\, y \wedge z = -\operatorname{Index}\left(P\frac{z}{|z|}P\right). \tag{6.9}$$

The proof of the theorem, like that of Theorem 4.2 depends on an explicit evaluation of (another) area integral and this one too is related to areas of triangles. We start with this preparatory proposition:

Proposition 6.9. *For Λ a switch*

$$\int_{\mathbb{R}} dx(\Lambda(x+a) - \Lambda(x)) = a\,, \qquad a \in \mathbb{R}\,. \tag{6.10}$$

If both Λ_1 and Λ_2 are switches, then

$$\begin{aligned}\int_{\mathbb{R}^2} & dx_1 dx_2((\Lambda_1(x_1+a_1) - \Lambda_1(x_1)) \\ & \times (\Lambda_2(x_2+b_2) - \Lambda_2(x_2+a_2)) - (1 \leftrightarrow 2)) = a \wedge b\,,\end{aligned} \tag{6.11}$$

where $a \wedge b \equiv a_1 b_2 - a_2 b_1$. Both integrals converge absolutely.

Proof. a) Look at

$$\begin{aligned}\int_{-\infty}^{\infty} dx(\Lambda(x+a) - \Lambda(x)) &= \int_{-\infty}^{\infty} dx \int_{x}^{x+a} dt\, \Lambda'(t) = \int_{-\infty}^{\infty} dx \int_{0}^{a} dt\, \Lambda'(t+x) \\ &= \int_{0}^{a} dt \int_{-\infty}^{\infty} dx\, \Lambda'(t+x) = \int_{0}^{a} dt = a\,.\end{aligned} \tag{6.12}$$

Monotonicity of the switch implies absolute convergence.
b) From, (6.10)

$$\begin{aligned}\int_{\mathbb{R}^2} & dx_1 dx_2(\Lambda_1(x_1+a_1) - \Lambda_1(x_1)) \\ & \times (\Lambda_2(x_2+b_2) - \Lambda_2(x_2+a_2)) = a_1(b_2 - a_2)\,.\end{aligned} \tag{6.13}$$

And similarly with $1 \leftrightarrow 2$. Subtracting the two gives (6.11). □

Proof of Theorem 6.8. To compute the transport according to Definition 6.6 we look first at the integral kernel of the adiabatic curvature (the last identity in (6.5)) restricted to the diagonal

$$\begin{aligned}\omega_{12}(x,x) = i \int_{\mathbb{R}^4} & dy\, dz\, p(x,y)p_{\perp}(y,z)p(z,x) \\ & \times ((\Lambda_1(y_1) - \Lambda_1(x_1))(\Lambda_2(z_2) - \Lambda_2(y_2)) - (1 \leftrightarrow 2))\,.\end{aligned} \tag{6.14}$$

Due to translational invariance the integrand in (6.14) can be replaced by

$$ip(0,y)p_{\perp}(y,z)p(z,0) \\ \times ((A_1(y_1+x_1)-A_1(x_1))(A_2(z_2+x_2)-A_2(y_2+x_2))-(1\leftrightarrow 2))\,. \quad (6.15)$$

To compute the charge transport we have to integrate the above expression over the domain Ω and after that let $L \to \infty$. Since all integrations converge absolutely even for $\Omega = \mathbb{R}^2$ we are permitted to exchange the order of integration and the limit $L \to \infty$. Hence we integrate first over x, then we let $L \to \infty$ and then we integrate over y and z. The x integration can be done by b) of Proposition 6.9. Putting this into the definition of the Hall charge transport

$$\begin{aligned} Q &= -2\pi i \int_{\mathbb{R}^4} dy\,dz\,p(0,y)p_{\perp}(y,z)p(z,0)\,y\wedge z \\ &= 2\pi i \int_{\mathbb{R}^4} dy\,dz\,p(0,y)p(y,z)p(z,0)\,y\wedge z\,. \end{aligned} \quad (6.16)$$

This proves the first part of the theorem. The second part is a consequence of Theorem 4.2. □

To relate this expression to Kubo's formula is rather simple. We start from (6.9), multiplying the integral formula by $1 = \frac{1}{|\Omega|}\int_{\Omega} dx$, and use the covariance of the projectors (4.3) to get:

$$Q = -\frac{2\pi i}{|\Omega|}\int_{\Omega} dx \int_{\mathbb{R}^4} dy\,dz\,p(x,y)p_{\perp}(y,z)p(z,x)\,(y-z)\wedge(z-x)\,. \quad (6.17)$$

The terms arising from terms linear and quadratic in x again vanish. Hence, the conductance,

$$\begin{aligned} \frac{Q}{2\pi} &= -\frac{i}{|\Omega|}\int_{\Omega} dx \int_{\mathbb{R}^4} dy\,dz\,p(x,y)p_{\perp}(y,z)p(z,x)\,y\wedge z \\ &= -\frac{i}{|\Omega|}\,\mathrm{Tr}(\chi_{\Omega}(Px_1P_{\perp}x_2P - Px_2P_{\perp}x_1P))\,, \end{aligned} \quad (6.18)$$

which is Kubo's formula.

7. Landau Hamiltonians

It is instructive to consider an example where the theory of the previous section applies and, moreover, is non-trivial in the sense that it gives non-zero deficiency. Such an example is provided by Landau Hamiltonians and the spectral projections on Landau levels. The Landau Hamiltonian in $L^2(\mathbb{R}^2)$ is:

$$H(A) \equiv \tfrac{1}{2}(-\,\mathrm{i}d - A)^2\,, \quad (7.1)$$

where $dA = B\,dx\wedge dy$. $B > 0$ is a constant magnetic field. Spectrum$(H(A)) = \{\frac{1}{2}B(2n+1) \mid n \in \mathbb{N}\}$, and each point in the spectrum, a Landau level, is infinitely degenerate. We shall denote the spectral projection on the n^{th} Landau level by P_n.

Clearly, $\dim P_n = \infty$. We show below that projections on Landau levels satisfy Hypothesis 3.1, and that the charge deficiency of each Landau level is unity.

Proposition 7.1. *Let $H(A)$ be the Landau Hamiltonian with $B > 0$, A differentiable and P_n the projection on the n^{th} Landau level. Then $p_n(x, y)$ is covariant, jointly continuous in x and y, and decays like a gaussian in the variable $|x - y|$. In particular, Hypothesis 3.1 holds.*

Proof. a) Let T_a denote the translation by $a \in \mathbb{R}^2$. Since B is constant and $\mathbb{R}^2$ is simply connected, $A(x-a) - A(x) = d\Lambda_a(x) = i\mathscr{U}_a^* d\mathscr{U}_a$ with $\mathscr{U}_a(x) \equiv \exp -i\Lambda_a(x)$. It follows that

$$T_a H(A) T_{-a} = (-\operatorname{id} - A(x-a))^2 = (-\operatorname{id} - A(x) - d\Lambda_a(x))^2 = \mathscr{U}_a^* H(A) \mathscr{U}_a \,. \tag{7.2}$$

Hence $H(B)$ commutes with magnetic translations $\mathscr{Z}_a \equiv \mathscr{U}_a T_a$ [46]. The spectral projections are covariant in the sense of Sect. 4 and

$$p(x, y) = \mathscr{U}_x^{-1}(x) p(0, y - x) \mathscr{U}_x(y) \,. \tag{7.3}$$

b) With A and A' related by a (smooth) gauge transformation Λ, $A' = A + d\Lambda$, the corresponding integral kernels are related by $p_{A'}(x, y) = e^{i\Lambda(x)} p_A(x, y) e^{-i\Lambda(y)}$, and so $p_{A'}(x, y)$ is continuous in x and y if $p_A(x, y)$ is. It is therefore enough to check the regularity and decay for a specific choice of A. By scaling the coordinates, we may take $B = 2$. We shall now show that for $A_0 \equiv \frac{1}{2}(-y\,dx + x\,dy)$, $p_0(0, z) = \text{Polynomial}(z) \exp -|z|^2/2$, which proves the regularity and decay. The corresponding Landau Hamiltonian is:

$$H(A_0) = 2D^* D + 1 \,, \qquad D \equiv \left(\partial_{\bar{z}} + \frac{z}{2}\right), \qquad z = x + iy \,. \tag{7.4}$$

The lowest Landau level is spanned by:

$$\langle z \mid n, 0\rangle = (\pi n!)^{-1/2} z^n e^{-|z|^2/2} \,, \qquad n = 0, 1, \ldots \,, \tag{7.5}$$

and the n^{th} Landau level by

$$\langle z \mid n, m\rangle = (\pi n!(m+1)!)^{-1/2} (D^*)^m (z^n e^{-|z|^2/2}) \,. \tag{7.6}$$

Since $\langle 0 \mid n, m\rangle = 0$ for $m \neq n$ we have:

$$p_m(0, z) = \sum_n \langle 0 \mid n, m\rangle\langle n, m \mid z\rangle = \langle 0 \mid m, m\rangle\langle m, m \mid z\rangle \tag{7.7}$$

which is smooth and with gaussian decay. □

It follows that the results of the previous sections apply. In particular, the deficiency is a finite integer and the Hall conductance for the n^{th} Landau level is $-\frac{1}{2\pi}\,\text{Index}\left(P_n \frac{z}{|z|} P_n\right)$. It remains to compute the index. This computation depends on the following simple lemma:

Lemma 7.2. *Let M be a semi-infinite Fredholm matrix so that its non-zero entries lies on the i^{th} sub-principle diagonal,* i.e.*:*

$$(M)_{mn} = c_m \delta_{m+i,n} \,, \qquad n, m \in \mathbb{N} \,, \qquad i \in \mathbb{Z} \,, \tag{7.8}$$

then, Index $M = i$.

Proof. Suppose first that all the $c_m \neq 0$. The kernel of M is spanned by the projection on the first i dimensions. The kernel of M^* is empty. Consequently Index $M = i$. Now to the general case: Since M is Fredholm there is at most a finite number of $c_m = 0$. Deforming a finite number of c_m to zero, does not change the index by the stability under compact perturbations, and so Index $M = i$. □

That the Hall conductance of each full Landau level is $1/2\pi$ is known from 1001 different calculations and arguments. The following computation, via an index, gives the 1002 way of seeing that:

Proposition 7.3. *For the* m^{th} *Landau levels:*

$$\text{Index}\left(P_m \frac{z}{|z|} P_m\right) = -1 . \tag{7.9}$$

In particular, the charge transport and charge deficiency of each Landau level is unity.

Proof. From (7.6) one sees that the state $\langle z \mid n, m\rangle$ has angular momentum proportional to $n - m$. Consequently, the matrix elements of $\left(P_m \frac{z}{|z|} P_m\right)_{n,n'}$ are:

$$\left(P_m \frac{z}{|z|} P_m\right)_{n,n'} = \delta_{n,n'+1} c(m,n) . \tag{7.10}$$

The result now follows from Lemma 7.2. □

As we have discussed in previous sections, the charge deficiency may be thought of as the change of number of electrons in a cycle where a flux tube carrying one unit of quantum flux is introduced into the system. In the present situation one can follow this cycle by the spectral analysis of the Landau Hamiltonian with a flux tube carrying any real flux. One finds that as the flux increases by one unit, n states from the n^{th} Landau level descend to the $n-1$ Landau level, and one state is lost to infinity [1, 26].

8. The Ergodic Case

In this last section we extend the results of Sects. 4 and 6 about covariant families of projectors to the case of an ergodic family of Schrödinger operators, $H(A, V_\omega)$: ω is a point in probability space $\tilde{\Omega}$, the action of translations on $\tilde{\Omega}$ is ergodic and:

$$V_\omega(x + a) = V_{T_a\omega}(x) . \tag{8.1}$$

We shall denote integrals with respect to the probability measure by $\langle\cdot\rangle$. This family of Schrödinger operators is one of the canonical models for the integer Hall effect.

Proposition 8.1. *Let* P_ω *be a spectral projection for* $H(A, V_\omega)$ *satisfying Hypothesis* 3.1, $\omega \in \Omega$. *Then* $\text{Index}(P_\omega U P_\omega)$ *is measurable with values in* $\mathbb{Z}$. *In fact* $\text{Index}(P_\omega U P_\omega)$ *is integer and constant almost everyhwere.*

Proof. We prove first that $\text{Index}(P_\omega U P_\omega)$ is measurable. Due to Proposition 2.2 and 2.4 the index can be expressed in terms of a trace

$$\text{Index}(P_\omega U P_\omega) = \text{Tr}(P_\omega - Q_\omega)^{2n+1} , \qquad Q_\omega \equiv U_\omega P_\omega U_\omega^{-1} . \tag{8.2}$$

Hence it is enough to prove measurability of the operator as an operator valued function of ω, i.e. measurability of the scalar product $(f, (P_\omega - Q_\omega)^{2n+1} f)$, $f \in L^2(\mathbb{R}^2)$. But the resolvent and therefore the projector Q_ω, which by assumption can be expressed in terms of an integral over the resolvent, is measurable. This proves the assertion.

Secondly, the function $I(\omega) \equiv \mathrm{Index}(P_\omega U P_\omega)$ takes integer values. Hence $\tilde{\Omega} = I^{-1}(\mathbb{Z})$. Furthermore for every $k \in (\mathbb{Z})$, $I^{-1}(k)$ is an invariant set in $\tilde{\Omega}$ under the action of translations. This is seen as follows: Let $\mathscr{Z}_a$ denote again the magnetic translation. Since

$$(\mathscr{Z}_a V_\omega \mathscr{Z}_a^{-1})(x) = V_\omega(x-a) = V_{T_a\omega}(x)\,, \tag{8.3}$$

we have:

$$\mathscr{Z}_a P_\omega \mathscr{Z}_a^{-1} = P_{T_a\omega}\,. \tag{8.4}$$

Since the index is shift invariant (Proposition 3.8) we have:

$$\mathrm{Index}(P_\omega U P_\omega) = \mathrm{Index}(P_{T_a\omega} U P_{T_a\omega})\,. \tag{8.5}$$

So the index is constant on the orbits of translations. Due to ergodicity, the measure of $I^{-1}(k)$, is zero or one for all $k \in \mathbb{Z}$. Since

$$\mu(\tilde{\Omega}) = 1 = \sum_{k\in\mathbb{Z}} \mu(I^{-1}(k)) \tag{8.6}$$

it follows that there is just one $k_0 \in \mathbb{Z}$ for which $\mu(I^{-1}(k_0)) = 1$. $\square$

In the ergodic situation the analog of (4.1) is:

$$P_\omega(x,y) = \mathscr{U}_a(x) P_{T_a\omega}(x-a, y-a) \mathscr{U}_a^{-1}(y)\,. \tag{8.7}$$

This means that the analog of (4.2) is: The triple product that enters the basic formula, (3.7),

$$\langle P_\omega(x_1,x_2) P_\omega(x_2,x_3) P_\omega(x_3,x_1)\rangle\,, \tag{8.8}$$

is translation invariant, i.e. it does not change under the substitution $x_i \to x_i + a$, $\omega \to T_a\omega$, $a \in \mathbb{R}^2$.

We see that we get an analog of Theorem 4.1 at the price of averaging over probability space. Namely,

Theorem 8.2. *Let $H(A, V_\omega)$ be a family of ergodic Schrödinger operators and U a unitary operator with unit winding number satisfying Hypothesis 3.1 for all $\omega \in \Omega$, in particular $p_\omega(x,y)$ satisfies inequality (3.1). Then the average Hall charge transport $\langle Q\rangle$ satisfies,* a.e.*:*

$$\langle Q\rangle = -\ \mathrm{Index}(P_\omega U P_\omega)\,. \tag{8.9}$$

Proof. The proof of this statement is an adaptation of the one given in Sect. 4, Theorem 4.2; integrating the basic euality (3.7) for the index over probability space brings us into the situation we had encountered in the proof of Theorem 4.2 since the average of the triple product (8.8) is invariant under translations. $\square$

Appendix A

The purpose of this appendix is to show that Hypothesis 3.1 on the regularity and decay of the integral kernel of spectral projections is guaranteed whenever the Fermi energy is placed in a gap. Although we have not attempted to give optimal conditions on the vector potentials, the conditions are mild enough to cover the physically interesting models.

Theorem A.1. *Let $H(A,V)$ be a one particle Schrödinger operator in $n = 2,3$ dimensions with differentiable vector potential A and scalar potential V which is in the Kato class $K_{n=2,3}$ (which includes Coulombic singularities).*
a) *The integral kernel for spectral projections for $H(A,V)$, $p(x,y)$ is jointly continuous in x and y.*
b) *Suppose, in addition, that $H(A,V)$ has a gap in the spectrum. Then the spectral projection below the gap has integral kernel which decays exponentially with $|x-y|$.*

Remark. The two parts of the theorem have rather different proofs. The K_n condition is natural for (a). Part (b) only requires form boundedness of V which is slightly weaker than the K_n condition.

Proof. (a) $\exp(-tH)(x,y)$ has a jointly continuous integral kernel by the path integral (Ito) way of writing the kernel – see, e.g. [37]. Because H is bounded below and has a gap, $P = g(H)$, where g is a smooth function of compact support. Since $f(y) \equiv \exp(2y)g(y)$ can be approximated by polynomials $\exp(-y)$ uniformly, we can write

$$g(H) = \lim g_j(H)\,, \qquad g_j(H) \equiv \exp(-H) f_j(H) \exp(-H)\,, \tag{A.1}$$

where the operators f_j converge to f in norm as $L^2 \to L^2$ operators and each $f_j(H)$ is a polynomial in $\exp(-H)$. On general principles (see, e.g. [37]), $\exp(-H)$ is a bounded operator from L^1 to L^2 and from L^2 to L^∞. Thus the limit in (A.1) gives a bounded operator from L^1 to L^∞ and so in infinity norm for the integral kernel (see e.g. [37]). Since g_j has a continuous integral kernel the result follows.
b) Let $B_{\vec{a}} \equiv e^{i\vec{a}\cdot\vec{x}}$, $a \in \mathbb{C}$, be a complex boost. Then:

$$B_a H(A,V) B_{-a} = H(A,V) + \vec{a}\cdot\vec{a} + \vec{a}\cdot(-i\vec{\nabla} - \vec{A})\,. \tag{A.2}$$

This gives an analytic family of type B in the sense of Kato [21] if the form domain is independent of $\vec{a}$. In particular, this is the case if V is form bounded relative to the kinetic energy. By the diamagnetic inequality it is enough to check that V is bounded relative to the Laplacian. K_n implies form boundedness (see [37]). In particular, if P is a spectral projection associated with a gap, then the gap is stable and:

$$p_a(x,y) = e^{ia\cdot x} p(x,y) e^{-ia\cdot y} \tag{A.3}$$

is real analytic in $\vec{a}$ uniformly in x and y. In particular, (A.2) says that $p(x,y)$ is exponentially decaying in $|x-y|$. This is a version of the Combes-Thomas argument [10]. □

Remarks. 1. For potentials V which are perturbations of Landau Hamiltonian, an adaptation of the above method gives decay which is faster than any exponential.
2. It is easy to construct families of Schrödinger operators, with ergodic A and V so that $H(A,V)$ has gaps in the spectrum.

3. A central open question is whether the integral kernel of spectral projections for ergodic Schrödinger operators in two dimensions automatically satisfy the decay assumption of Hypothesis 3.1 for most Fermi energies.

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