PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 35, No. 1, September 1972

CONTINUUM EMBEDDED EIGENVALUES IN A SPATIALLY CUTOFF $P(\phi)_2$ FIELD THEORY

BARRY SIMON[1]

ABSTRACT. We prove the existence of nontrivial two-dimensional spatially cutoff self-coupled Boson field theory Hamiltonians with bound states embedded in the continuum.

The development of constructive quantum field theory by Glimm and Jaffe (and those of us who have followed in their footsteps) has raised a large number of mathematical questions, some with answers of physical interest—others of a purely mathematical nature. Among these is the analysis of the spectral properties of spatially cutoff Hamiltonians. In this brief note, we wish to exhibit nontrivial examples with eigenvalues embedded in the continuum.

Let H_0 be the free Hamiltonian of a Bose field of mass $m_0>0$ in two-dimensional space-time. Let $H_I^{(2)}(g)$ be the interaction $H_I^{(2)}(g)=\int g(x):\phi^2(x):dx$, where g is a smooth function of compact support with $0\leqq g\leqq 1$. If $\lambda>-m_0^2$, the model Hamiltonian $H^{(2)}(g)=H_0+\lambda H_I^{(2)}(g)$ is exactly solvable in the sense that it is unitarily equivalent to $d\Gamma(A)-E_g$ for a suitable operator A. Here $d\Gamma(\cdot)$ is the biquantization of $\cdot$, a symbolism introduced by Segal and discussed in [**1**]. A variety of authors (see [**2**] and references quoted therein) have discussed this model. Rosen [**2**] has analyzed the spectrum of $H^{(2)}(g)$ completely. One feature is that in some cases with $\lambda<0$, $H^{(2)}(g)$ has eigenvalues $E>m_0-E_g$ embedded in $\sigma_{\text{cont}}(H^{(2)}(g))=(m_0-E_g,\infty)$. These eigenvalues are states of two or more "dressed excitations". They seem to be artifacts of the linear nature of the equation of motion which leads to noninteracting excitations. As soon as one turns on any interaction of nontrivial nature, the excitations interact and one would expect these continuum embedded eigenvalues to dissolve. We think that this expectation is basically correct and in particular that there are no eigenvalues in $(2m_0-E_g,\infty)$ after an additional nontrivial interaction is turned on. What we wish to show is that some of the eigenvalues will remain because of a very simple mechanism.

Received by the editors December 13, 1971.

AMS 1970 *subject classifications*. Primary 81A18.

[1] Permanent address: Departments of Mathematics and Physics, Princeton University, Princeton, New Jersey 08540.

This mechanism is suggested by an analogy with the nonrelativistic Helium atom Hamiltonian in the limit of infinite nuclear mass and no Coulomb repulsion between electrons. The Hamiltonian in that case is just

$$H_0 = -\Delta_1 - \Delta_2 - 2/r_1 - 2/r_2$$

on $L^2(\boldsymbol{R}^6)$ and is exactly solvable. There are a continuous spectrum in $(-1, \infty)$ and eigenvalues at $E_{n,m}=-2/n^2-2/m^2$. If $n, m>1$, these eigenvalues are above the continuum limit. If one turns on a Coulomb repulsion $-\beta/|r_1-r_2|$ between the electrons, i.e. if one looks at $H(\beta)=H_0+\beta V$, one expects these eigenvalues to dissolve. A mechanism for proving this for most of the levels has recently been developed [**3**], but certain levels do not dissolve. Let $\mathscr{H}_{(n)}$ be the subspace of $L^2(\boldsymbol{R}^6)$ of states of natural parity, i.e. the subspace spanned by those simultaneous eigenvectors of total angular momentum J and total parity P obeying $(-1)^J=P$. Let $\mathscr{H}_{(u)}=\mathscr{H}_{(n)}^{\perp}$ be the states of unnatural parity. Both H_0 and $H_0+\beta V$ leave $\mathscr{H}_{(n)}$ invariant. $H_0\restriction\mathscr{H}_{(n)}$ and $H_0+\beta V\restriction\mathscr{H}_{(n)}$ have continuum $[-1, \infty)$ (if β is small) but $H_0\restriction\mathscr{H}_{(u)}$ and $H_0+\beta V\restriction\mathscr{H}_{(u)}$ have continuum only in $[-\frac{1}{4}, \infty)$. Any $E_{n,m}$ with $n=2$, $m\geqq 2$ has an associated eigenvector (of H_0) in $\mathscr{H}_{(u)}$. These eigenvectors of unnatural parity in $(-1, -\frac{1}{4})$ do not dissolve if β is small. In fact, it can be proven ([**4**], [**5**]) that the physical value $\beta=1$ leads to an infinite number of eigenvalues in $(-1, -\frac{1}{4})$.

In our field theory case, we only need to find the appropriate symmetry to replace $(-1)^J P$. The right choice is $(-1)^N$ where N is the number operator. (One can replace N with the dressed number operator N_g since $(-1)^N=(-1)^{N_g}$.) Our result then is:

THEOREM. *Let g, a smooth function of compact support on $\boldsymbol{R}$, be given. Let $P(X)=a_{2m}X^{2m}+a_{2m-2}X^{2m-2}+\cdots+a_0$ be an even polynomial with $a_{2m}>0$ and $m\geqq 2$. Let $H_I^{(2m)}(g)=\int g(x){:}P(\phi(x)){:}\,dx$ and let $H(g, \lambda, \beta)=H_0+\lambda H_I^{(2)}(g)+\beta H^{(2m)}(g)$. Then there is a $\Lambda>0$ and a B for each λ with $0>\lambda>-\Lambda$ so that $H(g, \lambda, \beta)$ has eigenvalues in the continuum if $0<\beta<B$.*

PROOF. (1) The Fock vacuum Ω_0 is an analytic vector for $H_I^{(2)}(g)$. This follows from a simple application of N_τ estimates [**6**].

(2) Let $V^{(2)}$ be the function on Q space ([**1**], [**6**]) with $H_I^{(2)}$ multiplication by $V^{(2)}$. Then $e^{\pm sV^{(2)}}\in L^1(Q, d\mu)$ if $|s|$ is sufficiently small. For (1) implies that $\sum_{m=0}^{\infty}\langle\Omega_0, H_0^{2m}\Omega_0\rangle s^{2m}/(2m)!<\infty$ if s is small so that

$$\sum_{m=0}^{\infty}(s^m/m!)\int |V|^m\, d\mu < \infty$$

if s is small. The monotone convergence theorem then implies $e^{+s|V|}\in L^1$ if $|s|$ is small which proves (2).

(3) There exist $p<2<q$ and $\Lambda>0$ so that $H_0+\lambda H_I^{(2)}(g)=H^{(2)}(g, \lambda)$ has the following properties if $0>\lambda>-\Lambda$:

(a) $\{e^{-tH}\}_{t>0}$ is an exponentially bounded semigroup on each L^r with $p<r<q$.

(b) For some $T>0$ and some r with $2<r<q$, e^{-TH} is a bounded map of L^2 into L^r. These results follow from (2) by a simple application of the method of hypercontractive semigroups ([**1**], [7]).

(4) As $\beta\to 0$ (β real), $H(g, \lambda, \beta)\to H(g, \lambda, 0)=H^{(2)}(g, \lambda)$ in norm resolvant sense if $0>\lambda>-\Lambda$. Given (3), this is also a simple application of the methods of hypercontractive semigroups.

(5) Let $T_{g,\lambda}$ be Rosen's dressing transformation [**2**] so that $H^{(2)}(g, \lambda)=Td\Gamma(\mu_g)T^{-1}-E_g$. Let $N_g=TNT^{-1}$. Then $(-1)^{N_g}=(-1)^N$. This follows from the fact that T commutes with $(-1)^N$ by its explicit form given in [**2**].

(6) Let $\mathscr{F}_e$ be the set of all vectors in $\mathscr{F}$ with an even number of particles and let $\mathscr{F}_o=\mathscr{F}_e^{\perp}$ be the set of vectors with an odd number of particles. T_g, H_0, $H^{(2)}(g, \lambda)$ and $H(g, \lambda, \beta)$ all leave $\mathscr{F}_0$ and $\mathscr{F}_e$ invariant since they all commute with $(-1)^N$.

(7) $\mu_g=(-\Delta+m_0^2+\lambda g(x))^{1/2}$ has at least one eigenvalue in $(0, m_0)$ if $-\Lambda<\lambda<0$ and $g\neq 0$. This is a basic property of one-dimensional quantum mechanics, that $-\Delta+V$ has a negative eigenvalue if V is nonpositive and goes to 0 at infinity. (This property follows from comparison with a square well.)

(8) Let $e_0<m_0$ be the lowest eigenvalue of μ_g. By decreasing Λ if necessary, suppose $e_0>m_0/2$. $H^{(2)}(g, \lambda)\restriction\mathscr{F}_o$ has continuous spectrum (m_0-E_g, ∞) while $H^{(2)}(g, \lambda)\restriction\mathscr{F}_e$ has continuous spectrum $(m_0+e_0-E_g, \infty)$. $H^{(2)}(g, \lambda)$ has an eigenvector in $\mathscr{F}_e$ with eigenvalue $2e_0-E_g$.

(9) By (4), for β small ($\beta>0$), $H(g, \lambda, \beta)$ has an eigenvector in $\mathscr{F}_e$ with eigenvalue larger than $\frac{1}{2}(m_0+2e_0)-E_g$, and the lowest (vacuum) state of $H(g, \lambda, \beta)$ has energy smaller than $(e_0-\frac{1}{2}m_0)-E_g$.

(10) By a result of Høegh-Krohn [**8**], $H(g, \lambda, \beta)$ has continuous spectrum in (Σ, ∞) with Σ smaller than $\frac{1}{2}(m_0+2e_0)-E_g$.

(11) Thus for β small, $H(g, \lambda, \beta)$ has an eigenvalue embedded in its continuum. Q.E.D.

These continuum embedded eigenvalues are present for a simple reason but they indicate how difficult it will be to control the point spectrum of spatially cutoff $P(\phi)_2$-Hamiltonians. For example, we expect that there will be no eigenvalues above the continuum limit in $\mathscr{F}_e$—the analogous conjecture can be made for the Helium atom, i.e. no eigenvalues above $-\frac{1}{4}$. It is still an open question whether the Helium atom Hamiltonian has any eigenvalues in $(-\frac{1}{4}, 0)$ and the difficulty of this question bodes ill for the analogous $P(\phi)_2$ question.

ACKNOWLEDGMENT. It is a pleasure to thank Professor S. Sternberg for the hospitality of the Mathematics Department of the Weizmann Institute and Professor Y. Ne'eman for the hospitality of the Physics Department of the Tel-Aviv University. I am indebted to Lon Rosen for making a copy of [2] available to me prior to its publication.

REFERENCES

1. B. Simon and R. Høegh-Krohn, *Hypercontractive semi-groups and self-coupled Bose fields in two dimensions*, J. Functional Analysis **9** (1972), 121–180.

2. L. Rosen, *Renormalization of the Hilbert space in the mass shift model*, J. Math. Phys. (to appear).

3. B. Simon, *Resonances for dilatation analytic potentials and the foundations of time dependent perturbation theory*, Ann. of Math. (to appear).

4. G. M. Zhislin and A. Sigalov, *On the spectrum of an energy operator for atoms with fixed nuclei in subspaces corresponding to irreducible representations of permutation groups*, Izv. Akad. Nauk SSSR Ser. Mat. **29** (1965), 853–860.

5. E. Balslev, *Spectral theory of Schrödinger operators of many-body systems with permutation and rotation symmetries*, Ann. Physics (to appear).

6. J. Glimm and A. Jaffe, "Field theory models," in the 1970 *Les Houches lectures*, C. DeWitt and R. Stora (editors), Gordon and Breach, New York, 1971.

7. I. Segal, *Construction of non-linear local quantum processes*. I, Ann. of Math. (2) **92** (1970), 462–481. MR **42** #7187.

8. R. Høegh-Krohn, *On the spectrum of the space cut-off: $P(\phi)$: Hamiltonian in two space-time dimensions*, Comm. Math. Phys. **18** (1970), 109.

DEPARTMENT OF PHYSICS, TEL-AVIV UNIVERSITY, TEL-AVIV, ISRAEL

DEPARTMENT OF MATHEMATICS, WEIZMANN INSTITUTE OF SCIENCE, REHOVOT, ISRAEL