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Spectral Properties of Random Schrödinger Operators with Unbounded Potentials

Y.A. Gordon[1], V. Jakšić[2]*, S. Molčanov[1], and B. Simon[3]**

[1] Department of Mathematics and Mechanics, Moscow State University, Moscow 119808, Russia

[2] Department of Mathematics, University of Toronto, Toronto, M5S-1A1, Canada

[3] Division of Physics, Mathematics and Astronomy, California Institute of Technology, Pasadena, CA 91125, USA

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Abstract. We investigate spectral properties of random Schrödinger operators $H_\omega = -\Delta + \xi_n(\omega)(1 + |n|^\alpha)$ acting on $l^2(Z^d)$, where ξ_n are independent random variables uniformly distributed on $[0, 1]$.

1. Introduction

It is already a part of folklore that multiplicative perturbations of the Anderson model show rather "unusual" spectral behavior. The basic paradigm is the discrete Schrödinger operator on $l^2(Z^1)$,

$$H_\omega u(n) = 2u(n) - u(n+1) - u(n-1) + V_\omega(n)u(n)\,,$$

$$V_\omega(n) = \lambda \xi_n(\omega)|n|^\alpha\,,$$

where $\xi_n(\omega)$ are independent random variables with a bounded, compactly supported density $r(x)$, and λ is a parameter. For $\alpha < 0$ the above model has been extensively studied in [5, 7, 8, 18] and their main results can be summarized as follows (note that for $\alpha < 0$, $V_\omega(n) \to 0$ as $|n| \to \infty$ and thus $\sigma_{\text{ess}}(H_\omega) = [0, 4]$).

Theorem. *With probability* 1:

(i) *For* $-1/2 < \alpha < 0$, *the spectrum in* $[0, 4]$ *is pure point with eigenfunctions decaying as* $\exp(-C|n|^{1+2\alpha})$.

(ii) *For* $\alpha < -1/2$, *the spectrum in* $[0, 4]$ *is purely absolutely continuous.*

(iii) *For* $\alpha = -1/2$ *and* λ *large, the spectrum in* $[0, 4]$ *is pure point with polynomially decaying eigenfunctions, while for* λ *small* H_ω *will have some singular continuous spectrum.*

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For $\alpha > 0$, $|n|^\alpha \to \infty$, but this does not imply that the spectrum is necessarily discrete: If $r(x)$ does not vanish in some neighborhood of 0, ξ_n can get arbitrarily small with positive probability and thus eventually compensate for the growth of $|n|^\alpha$ within infinitely many sites. That in turn can lead to nontrivial spectral behavior. Consider the simplest case when ξ_n are independent random variables uniformly distributed on [0, 1]: It was shown in [9] that for a.e. ω, H_ω will have a discrete spectrum if and only if $\alpha > 1$. Furthermore, if $1/k \geqq \alpha > 1/(k+1)$, $\sigma_{\text{ess}}(H_\omega) = [a_k, \infty)$, where a_k is a strictly decreasing nonrandom sequence of positive numbers, $\sigma(H_\omega) = \sigma_{\text{pp}}(H_\omega)$, and the eigenfunctions decay superexponentially. Thus, while for $\alpha < 0$ the essential spectrum is always [0, 4] with a transition in its nature at $\alpha_c = -1/2$, for $\alpha > 0$ the essential spectrum is always pure point but its end-point is piecewise constant function of the parameter α(!). In this paper we are interested in obtaining the multidimensional analog of the above results when $\alpha > 0$, namely we will study the operator

$$H_\omega = H_0 + \xi_n(\omega)(1 + |n|^\alpha), \quad \alpha > 0\,, \tag{1.1}$$

acting on $l^2(Z^d)$. In (1.1), $|n| = (\sum n_i^2)^{1/2}$, $\xi_n(\omega)$ are independent random variables uniformly distributed on [0, 1], and

$$H_0\phi(n) = \sum_{|n-m|_+ = 1} \phi(n) - \phi(m), \quad \|H_0\| = 4d\,, \tag{1.2}$$

where $|n|_+ = \sum |n_i|$. We view $\xi_n(\omega)$ as a random field on $\bigotimes_{n \in Z^d}[0, 1] = \Omega$, and denote by P the corresponding probability measure, and by $\mathbf{E}$ the mathematical expectation on Ω. Before stating our main results, we introduce some notation. For $X \subset Z^d$ denote by $C(X)$ the set of all $\phi \in l^2(Z^d)$ with support in X. Let $\mathscr{D}$ be the form associated to H_0,

$$\mathscr{D}(\phi) = \sum_{\substack{\langle n, m\rangle \\ |n-m|_+ = 1}} |\phi(m) - \phi(n)|^2\,, \tag{1.3}$$

where $\langle n, m\rangle$ reminds one that each pair appears in the summation only once. Denote

$$\Lambda(X) = \inf_{\substack{\|\phi\| = 1 \\ \phi \in C(X)}} \mathscr{D}(\phi)\,.$$

If $\# X < \infty$, $\Lambda(X)$ is the smallest eigenvalue of the spectral problem $H_0\phi = \lambda\phi$, $\phi(n) = 0$ if $n \in Z^d \backslash X$. A path between points $n, m \in Z^d$ is a sequence of sites

$$\tau = (n_1, n_2, \ldots, n_k), \quad n_1 = n, \quad n_k = m, \quad |n_{j+1} - n_j|_+ = 1\,. \tag{1.4}$$

Set $X \subset Z^d$ is connected if any two points in X can be connected with a path which lies within X. Following [10], we say that a set $A_k \subset Z^d$ is a k-animal (or just animal) if A_k is connected and $\# A_k = k$. Modulo translation, there are only finitely many animals of any given size k. Let

$$a_k = \inf_{A_k} \Lambda(A_k)\,. \tag{1.5}$$

It is obvious that a_k is a strictly decreasing sequence of positive numbers. The animals for which the infimum in (1.5) is attained we will denote by $A_{T,k}$ and call the tamed animals. As we will see later, taming the animals (namely, obtaining control over a_k, $A_{T,k}$) is not an easy task at all.

In the sequel, $f(x) \sim g(x)$ stands for $\lim_{x\to\infty} f(x)/g(x) = 1$, and τ_d for the volume of a unit ball in R^d. With the above notation, our main results are stated as follows.

Theorem 1.1. *H_ω has a discrete spectrum P-a.s. if and only if $\alpha > d$. Furthermore, if $N_\omega(E)$ denotes the number of eigenvalues of H_ω which are less than E, we have that for $\alpha > d$ and for* a.e. ω

$$N_\omega(E) \sim \frac{\tau_d}{\alpha - d} \cdot E^{d/\alpha} \quad \text{as } E \to \infty \ . \tag{1.6}$$

Theorem 1.2. *If $d/k \geqq \alpha > d/(k+1)$ for positive integer k, for* a.e. ω *we have*

(i) $\sigma(H_\omega) = \sigma_{\mathrm{pp}}(H_\omega)$ *and eigenfunctions of H_ω decay at least exponentially,*
(ii) $\sigma_{\mathrm{ess}}(H_\omega) = [a_k, \infty)$,
(iii) $\# \sigma_{\mathrm{disc}}(H_\omega) < \infty$.

Theorem 1.3. *Let λ_D be the lowest eigenvalue of a Dirichlet Laplacian of a unit ball in R^d. Then*

$$a_k \sim k^{-d/2} \cdot \lambda_D \quad \text{as } k \to \infty \ .$$

Remark 1. Only affecting the values of constants, we can suppose that the random variables ξ_n have common nonnegative absolutely continuous density which is of the form x^β on $[0, \delta)$ for some $\beta, \delta > 0$. On the other hand if ξ_n are uniformly distributed on $[-1, 1]$, Theorem 1.1 remains valid (with appropriate reformulation of (1.6)). Part (i) of Theorem 1.2 also holds, while part (ii) has to be replaced with $\sigma(H_\omega) = (-\infty, \infty)$.

Remark 2. By the exponential decay of the eigenfunctions we mean that the eigenfunction of H_ω corresponded to an eigenvalue E satisfies

$$|\phi_{\omega,E}(n)| \leqq C_{\omega,E} \exp(-a|n|) \tag{1.7}$$

for some fixed $a > 0$. Our argument can be easily modified to show that eigenfunctions decay as

$$|\phi_{\omega,E}(n)| \leqq C_{\omega,E} \exp(-a|n| \cdot \ln|n|) \ .$$

The constant λ_D, introduced in Theorem 1.3, coincides with the smallest zero of the Bessel function $J_{d/2-1}$. Thus $\lambda_D = 2.4048\ldots$ for $d = 2$, $\lambda_D = \pi$ for $d = 3$, and $\lambda_D \sim d/2$ for d large.

Remark 3. We will give two proofs that the spectrum of H_ω is pure point. The first one is rather simple and is based on the recent results on localization obtained in [14, 15]. The first proof does not yield the satisfactory decay of the eigenfunctions, nor can its strategy be used to investigate the discrete spectrum. The second proof, although more complicated and relying heavily on the particular structure of our potential, yields immediately (1.7) and part (iii) of Theorem 1.2.

For the model (1.1) when $\alpha \leqq d$, one can also investigate the natural analog of the integrated density of states. Let $C_L = \{n\colon \max|n_i| \leqq L\}$ be the box of side L centered at the origin. The form on $C(C_L)$ (recall (1.3))

$$\mathscr{D}(\phi) + (V_\omega \phi, \phi)$$

yields a finite matrix (it is just a restriction of H_ω to C_L with Dirichlet boundary condition) whose number of eigenvalues which are less than E we denote by $N_{\omega,L}(E)$. Set $a_0 = \infty$.

Theorem 1.4.
(i) *If $d/k > \alpha > d/(k+1)$ and $E \in (a_j, a_{j-1})$, $1 \leqq j \leqq k$, then*

$$\lim_{L\to\infty} \frac{N_{\omega,L}(E)}{L^{d-j\alpha}} = N_j(E) \tag{1.8}$$

exists for a.e. ω *and is a non-random function.*
(ii) *If $\alpha = d/k$ and $E \in (a_j, a_{j-1})$, $1 \leqq j < k$, (1.8) is valid. If $E \in (a_k, a_{k-1})$ then*

$$\lim_{L\to\infty} \frac{N_{\omega,L}(E)}{\ln L} = N_k(E) \tag{1.9}$$

exists for a.e. ω *and is a non-random function.*

In both cases, $N_j(E)$ is a continuous function on (a_j, a_{j-1}), and

$$N_j(E) \sim C_j(E - a_j)^j \quad \text{as } E \to a_j\,.$$

The constants C_j are of combinatorial nature.

The continuous analog of the model (1.1) is discussed in [11].

The paper is organized as follows. In Chapter 2 we prove Theorem 1.1, part (ii) of Theorem 1.2, and give a simple proof that $\sigma(H_\omega) = \sigma_{\mathrm{pp}}(H_\omega)$ P-a.s. In Chapter 3 we prove Theorem 1.3 and in Chapter 4 we finish the proof of Theorem 1.2. In Chapter 5 we prove Theorem 1.4. Finally, in the Appendix we prove the result (announced in [16]) which has been used in the simple proof of localization.

2. On the Discrete and Essential Spectrum

2.1. Proof of Theorem 1.1. A sufficient and necessary condition for $H_0 + V_\omega(n)$ to have discrete spectrum is that

$$|V_\omega(n)| \to \infty \quad \text{as } |n| \to \infty\,. \tag{2.1}$$

Let $c > 0$ be fixed and denote

$$A_n^c = \{\omega\colon \xi_n(\omega)\cdot(1 + |n|^\alpha) < c\}\,.$$

$P(A_n^c) = c/(1 + |n|^\alpha)$ and $\sum_n P(A_n^c)$ converges if and only if $\alpha > d$. The Borel–Cantelli lemma implies that (2.1) is valid for a.e. ω iff $\alpha > d$, and the first part of Theorem 1.1 follows.

To calculate the eigenvalue asymptotics, first note that monotonicity of $N_\omega(E)$ implies that it is enough to consider the case when $E \to \infty$ as an integer. We proceed as follows. Denote by

$$\chi_E^{\pm}(n, \omega) = \begin{cases} 1, & \text{if } \xi_n(\omega)\cdot(1 + |n|^\alpha) \leqq E \pm 4d\,; \\ 0, & \text{otherwise}\,. \end{cases}$$

and set $S_E^{\pm}(\omega) = \sum_{n\in Z^d} \chi_E^{\pm}(n, \omega)$. Because $\|H_0\| = 4d$, we have

$$S_E^-(\omega) \leqq N_\omega(E) \leqq S_E^+(\omega)\,.$$

Since

$$E(S_E^{\pm}) = \#\{n\colon (1+|n|^{\alpha}) \leqq E \pm 4d\} + \sum_{n\colon 1+|n|^{\alpha} > E \pm 4d} \frac{E \pm 4d}{1+|n|^{\alpha}},$$

one easily obtains

$$\begin{aligned}\lim_{E\to\infty} E^{-d/\alpha}\mathbf{E}(S_E^{\pm}) &= \tau_d + \int_{|x|>1} \frac{1}{|x|^{\alpha}}\,dx \\ &= \frac{\tau_d}{\alpha - d}. \end{aligned} \tag{2.2}$$

It remains to show that

$$E^{-d/\alpha}(S_E^{\pm} - \mathbf{E}(S_E^{\pm})) \to 0 \quad \text{as } E\to\infty \ P\text{-a.s.} \tag{2.3}$$

First note that

$$\begin{aligned}\mathbf{E}((S_E^{\pm})^k) \leqq 2^{k-1}\cdot\Bigg(&(\#\{n\colon (1+|n|^{\alpha}) < E \pm 4d\})^k \\ &+ \sum_{\substack{(n_1,\ldots,n_k)\\ n_i > E\pm 4d}} \frac{E \pm 4d}{\prod_{i=1}^k (1+|n_i|^{\alpha})}\Bigg),\end{aligned}$$

and thus $\limsup_{E\to\infty} E^{-dk/\alpha}\mathbf{E}((S_E^{\pm})^k) \leqq C$, where C is a uniform constant. Let $\eta_E^{\pm}(n,\omega) = \chi_E^{\pm}(n,\omega) - \mathbf{E}(\chi_E^{\pm}(n,\omega))$. $\eta_E^{\pm}(n)$ is a sequence of independent (but not identically distributed) random variables satisfying $|\eta_E^{\pm}(n)| \leqq 2$, $\mathbf{E}(\eta_E^{\pm}(n)) = 0$. We have

$$\begin{aligned}\mathbf{E}((S_E^{\pm} - E(S_E^{\pm}))^{2k}) &= \mathbf{E}\left(\left(\sum_{n\in\mathbf{Z}^d} \eta_E^{\pm}(n)\right)^{2k}\right) \\ &\leqq C\cdot\mathbf{E}((S_E^{\pm})^k) \\ &\leqq C\cdot E^{dk/\alpha}\end{aligned} \tag{2.4}$$

for a uniform constant C. The first inequality in (2.4) follows from simple combinatorics and observations that

$$\#\{n\colon \eta_E^{\pm}(n,\omega) \neq 0\} \leqq S_E^{\pm}, \quad \mathbf{E}\left(\sum_{n\in\mathbf{Z}^d} \eta_E^{\pm}(n)\right) = 0.$$

For $\varepsilon > 0$, Chebyshev's inequality yields

$$\begin{aligned}P_E = P\{E^{-d/\alpha}|S_E^{\pm} - \mathbf{E}(S_E^{\pm})| > \varepsilon\} &\leqq \frac{E^{-2dk/\alpha}\mathbf{E}((S_E^{\pm} - \mathbf{E}(S_E^{\pm}))^{2k})}{\varepsilon^{2k}} \\ &\leqq C\,\frac{E^{-dk/\alpha}}{\varepsilon^{2k}}.\end{aligned}$$

If $dk/\alpha > 1$, $\sum_{E=4d+1}^{\infty} P_E < \infty$, and the Borel–Cantelli Lemma yields (2.3).

2.2. On the Essential Spectrum. In this section we prove that for $d/k \geqq \alpha > d/(k+1)$, $\sigma_{\text{ess}}(H_\omega) = [a_k, \infty)$.

Denote $I = 4k \cdot Z^d$, and decompose $Z^d = \bigcup_{n \in I} \mathscr{T}_n$, where $\mathscr{T}_n$ is the cube of volume $(4k)^d$ centered at n. Let A_T be an arbitrary tamed k-animal which contains 0, and denote by A_T^n its translation for the vector $n \in I$. To show that

$$\sigma_{\text{ess}}(H_\omega) \supset [a_k, \infty) \quad P\text{-a.s.} \tag{2.5}$$

it suffices (using Weyl's criterion and fact that essential spectrum is closed set) to construct for every rational $\lambda > 0$ and for a.e. ω a sequence φ_i satisfying

$$\lim_{i \to \infty} \|(H_\omega - \lambda - a_k)\varphi_i\| = 0 . \tag{2.6}$$

Fix a sequence $b_i \to 0$, and denote for $n \in I$

$$A_n^{b_i} = \{\omega\colon \lambda - b_i < V_\omega(x) < \lambda + b_i \text{ for } x \in A_T^n\} .$$

$P(A_n^{b_i}) \geqq Cb_i/(1 + |n|^\alpha)^k$ for a suitable uniform constant C, and thus $\sum_{n \in I} P(A_n^{b_i})$ diverges. The Borel–Cantelli lemma and diagonal argument implies that there exist with probability 1 a strictly increasing (ω-dependent) sequence n_i such that

$$|V_\omega(n) - \lambda| < b_i \quad \text{for } n \in A_T^{n_i} .$$

If φ_0 is eigenfunction corresponded to a_k, set φ_0 to be zero outside A_T and denote $\varphi_i(n) = \varphi_0(n - n_i)$. It is trivial to check that (2.6) is satisfied and (2.5) follows. It remains to show that

$$\inf \sigma_{\text{ess}}(H_\omega) \geqq a_k \quad P\text{-a.s.} \tag{2.7}$$

Let $B(n, l) = \{m; |n - m|_+ \leqq l\}$. We will over and over again make use of the following

Lemma 2.1. *For $\gamma > 0$ let*

$$B_n = \{\omega\colon \textit{in the box } B(n, |n|^\gamma) \textit{ there exists } k + 1 \textit{ points } n_i \textit{ such that } V_\omega(n_i) \leqq |n_i|^\gamma\} . \tag{2.8}$$

Then, for γ small enough, with probability 1, only finitely many events B_n take place.

Proof. It is easy to show that for γ small enough

$$P(B_n) \leqq C \cdot |n|^{(k+1)(d\gamma + \gamma - \alpha)} \leqq C|n|^{-d-\varepsilon} ,$$

where C is a uniform constant. Thus, for such γ, $\sum_n P(B_n) < \infty$ and the Borel–Cantelli lemma yields the statement. ∎

The discrete version of Persson's theorem ([4], Theorem 3.12) states

$$\inf \sigma_{\text{ess}}(H_\omega) = \sup_{\substack{K \subset Z^d \\ \# K < \infty}} \inf_{\substack{\varphi \in C_{\text{fin}}(\mathbf{Z}^d \setminus K) \\ \|\phi\| = 1}} (\varphi, H_\omega \varphi) .$$

Let $K = K(\omega)$ be a box, centered at 0, of large enough side so that outside K no event B_n takes place for a.e. ω. Let

$$R = \inf_{n \notin K} |n|^\gamma ,$$

and let

$$A_\omega = \{n \in Z^d \setminus K : V_\omega(n) < 1\} .$$

For a.e. ω, A_ω is a disjoint union of animals of the size $\# \leqq k$. Let $B = Z^d \backslash (K \cup A_\omega)$, and for $\phi \in C_{\text{fin}}(Z^d \backslash K)$, $\|\phi\| = 1$ denote

$$\phi_1 = \begin{cases} \phi(n), & \text{if } n \in A_\omega; \\ 0, & \text{otherwise;} \end{cases} \quad \phi_2 = \begin{cases} \phi(n) & \text{if } n \in B_\omega; \\ 0, & \text{otherwise} . \end{cases}$$

Obviously,

$$(\phi_1, \phi_2) = 0, \quad \|\phi_1\|^2 + \|\phi_2\|^2 = 1 .$$

For $R > a_k$ we have

$$\begin{aligned} (H_\omega \phi, \phi) &= (H_\omega \phi_1, \phi_1) + (H_\omega \phi_2, \phi_2) + 2\mathrm{Re}(H_\omega \phi_1, \phi_2) \\ &\geqq a_k \|\phi_1\|^2 + R\|\phi_2\|^2 + 2\mathrm{Re}(H_\omega \phi_1, \phi_2) \\ &\geqq a_k + (R - a_k)\|\phi_2\|^2 - 2\|H_0\| \cdot \|\phi_1\| \cdot \|\phi_2\| \\ &\geqq a_k - \frac{16d^2}{R - a_k} . \end{aligned}$$

Consequently

$$\inf \sigma_{\text{ess}}(H_\omega) \geqq a_k - \frac{16d^2}{R - a_k} \quad P\text{-a.s.}$$

By taking K big enough, R can be made arbitrarily large and (2.7) follows.

2.3. A Simple Proof of Localization. We first fix some notation. Let ∂X be defined as

$$\partial X = \{n \in X : (\exists m \in Z^d \backslash X) |m - n|_+ = 1\} . \tag{2.9}$$

For X, $Y \in Z^d$, let

$$d(X, Y) = \min\{|n - m|_+ : n \in X, m \in Y\} .$$

The matrix elements of the resolvent of $H = H_0 + V$ we denote by

$$R(n, m; z) = (\delta_n, (H - z)^{-1} \delta_m) .$$

The following (deterministic) result, which we will prove in the Appendix, has been announced in [16].

Theorem 2.2. *Suppose that we have ascending sequence of connected sets in Z^d*

$$\Gamma_1^- \subset \Gamma_1^+ \subset \Gamma_2^- \subset \Gamma_2^+ \subset \ldots \subset \Gamma_n^- \subset \Gamma_n^+ \subset \ldots$$

such that

$$\# \Gamma_n^\pm = O(D_n^d) \quad \# \partial \Gamma_n^\pm = O(D_n^{d-1}), \quad d_n = d(\Gamma_n^-, \Gamma_n^+) , \tag{2.10}$$

where $D_n \to \infty$ is a sequence of integers satisfying $D_n \leqq c^n$, for some c. Let V be a potential satisfying

$$V(m) \geqq C(D_n^{d-1} n^{1+\delta})^{1/d_n} \quad \text{if } m \in \Gamma_n^+ \backslash \Gamma_n^- , \tag{2.11}$$

for some $C, \delta > 0$. *Then, for every* $n \in Z^d$ *and for* a.e. $E \in R$ (*with respect to the Lebesgue measure*)

$$\limsup_{\varepsilon \to 0} \sum_{m \in Z^d} |R(n, m; E + i\varepsilon)|^2 < \infty \ . \tag{2.12}$$

The above theorem is a by-product of some recent results [14–16] on the localization in one dimension. On the intuitive level [15, 19], high-potential walls in $\Gamma_n^+ \backslash \Gamma_n^-$ are under conditions (2.10), (2.11) effective in stopping tunneling at large distances. An immediate consequence of (2.12) is that $\sigma_{\mathrm{ac}}(H) = \emptyset$.

To see how model (1.1) fits in the above picture we proceed as follows. Let C_k, C_k' be boxes of the sides $2^k - [2^{k\gamma - 1}]$, $2^k + [2^{k\gamma - 1}]$ where γ is chosen as in (2.8). It is easy to show (using Lemma 2.1) that for a.e. ω there exists connected sets $\Gamma_k^-(\omega)$, $\Gamma_k^+(\omega)$, such that for $k > k_0(\omega)$

$$C_k \subset \Gamma_k^-(\omega) \subset \Gamma_k^+(\omega) \subset C_k', \quad d_k = d(\Gamma_k^+(\omega), \Gamma_k^-(\omega)) \geqq C_1 2^{k\gamma} \ ,$$

and that $V_\omega(m) > C_2 2^{k\gamma}$ for $m \in \Gamma_k^+(\omega) \backslash \Gamma_k^-(\omega)$. Thus, (2.10), (2.11) are obviously satisfied, and we have that for a.e. ω and a.e. E and for every $n \in Z^d$

$$\limsup_{\varepsilon \to 0} \sum_{m \in Z^d} |R_\omega(n, m; E + i\varepsilon)|^2 < \infty \ . \tag{2.13}$$

Equation (2.13) and the Simon–Wolff theorem [2, 20] yields that $\sigma(H_\omega) = \sigma_{\mathrm{pp}}(H_\omega)$ for a.e. ω.

Using Theorem 2.2 we get an easy proof that for the non-stationary model (1.1) (and for many others with unbounded potential) we have pure point spectrum. On the other hand, by their very nature, the theorems of the above type [16] cannot be used to prove exponential decay of eigenfunctions or to analyse the discrete spectrum. For that, one has to rely more heavily on the particular structure of our potential, and we will do it in Chapter 4.

3. Taming the Animals

This chapter is devoted to the proof of Theorem 1.3. We start by introducing the continuous analog of (1.3). If Ω is a region in R^d, the Dirichlet Laplacian $-\Delta_D^\Omega$ is the unique self-adjoint operator whose quadratic form is given by the closure of

$$D(u) = \int_\Omega |\nabla u|^2 dx$$

on $C_0^\infty(\Omega)$. In the sequel we will suppose that all regions under consideration have a piecewise smooth boundary. If $|\Omega|$, the Lebesgue measure of Ω, is finite, $-\Delta_D^\Omega$ has a compact resolvent and its smallest eigenvalue is given by

$$\Lambda(\Omega) = \inf_{\substack{u \in C_0^\infty(\Omega) \\ \|u\| = 1}} D(u) \ . \tag{3.1}$$

The corresponding eigenfunction u is a $C^\infty(\Omega)$ function satisfying

$$-\Delta u(x) = \Lambda(\Omega) u(x) \quad \text{for } x \in \Omega, \quad u(x) = 0, \text{ if } x \in \partial\Omega \ .$$

It is also known that $u(x) > 0$ if $x \in \Omega$. For positive r let

$$\lambda_r = \inf\{\Lambda(\Omega) : |\Omega| = r\} . \tag{3.2}$$

The infimum in (3.2) is achieved when Ω is a ball of volume r [4], $\lambda_1 = \lambda_D$ and $\lambda_r = \lambda_D / r^{2/d}$. For notational simplicity, in the sequel we will suppose that $d = 2$. We will prove

Lemma 3.1.

$$a_k \leqq \frac{\lambda_D}{k} \cdot (1 + O(1/\sqrt{k})) .$$

Lemma 3.2.

$$a_k \geqq \frac{\lambda_D}{k} \cdot \frac{1 + O(1/\sqrt{k})}{1 + 3 \cdot \# \partial A_{T,k}/k} .$$

Lemma 3.3. If $A_{T,k}$, $k \geqq 1$ is a sequence of tamed k-animals, we have

$$\lim_{k \to \infty} \frac{\# \partial A_{T,k}}{k} = 0 .$$

Theorem 1.3 is an immediate consequence of the above three lemmas. Their proofs are somewhat subtle and we devote to each a separate section.

3.1. Proof of Lemma 3.1. To avoid confusion, throughout the chapter we use Latin letters to denote elements of $L^2(R^d)$, and Greek letters for elements of $l^2(Z^d)$. By $\|\cdot\|_\infty$ we denote the norm on R^2 given by $\|(\varepsilon_1, \varepsilon_2)\|_\infty = \max\{|\varepsilon_1|, |\varepsilon_2|\}$. To any subset X of Z^2 we correspond a region X_a in R^2 whose closure is given by

$$\bar{X}_a = \left\{ x \in R^2 : \sup_{y \in X} \|x - y\|_\infty \leqq a \right\} . \tag{3.3}$$

If A_k is an arbitrary animal, $A_{k,1/2}$ the region given by (3.3) with $a = 1/2$. For $\Lambda(A_{k,1/2})$ given by (3.1), we denote the corresponding eigenfunction by u and set $u(x) = 0$ if $x \notin A_{k,1/2}$. To every $x \in F = \{x \in R^2 : \|x\|_\infty < 1/2\}$ we associate a function ϕ_x on Z^2 as $\phi_x(n) = u(n + x)$, $n \in Z^2$. Obviously, $\phi_x(n) \neq 0$ iff $n \notin A_k$. Furthermore,

$$\int_F \|\phi_x\|^2 dx = \int_F \sum_n u(n + x)^2 dx = \sum_n \int_F u(n + x)^2 dx = \|u\|^2 . \tag{3.4}$$

Let $e_1 = (1, 0)$, $e_2 = (0, 1)$. We have for $i = 1, 2$,

$$\begin{aligned}
\int_F \sum_n (\phi_x(n + e_i) - \phi_x(n))^2 dx &= \sum_n \int_F (u(n + x + e_i) - u(n + x))^2 dx \\
&= \sum_n \int_F \left(\int_0^1 \frac{\partial u(n + x + te_i)}{\partial x_i} dt \right)^2 dx \\
&\leqq \int_0^1 \sum_n \int_F \left(\frac{\partial u(n + x + te_i)}{\partial x_i} \right)^2 dx\, dt \\
&= \int_{R^2} \left(\frac{\partial u(x)}{\partial x_i} \right)^2 dx .
\end{aligned}$$

Summing over i we obtain

$$\int_F \mathscr{D}(\phi_x)dx \leqq D(u)\,, \tag{3.5}$$

and consequently there exists at least one $x_0 \in F$ such that

$$\mathscr{D}(\phi_{x_0}) \leqq D(u) = \Lambda(A_{k,1/2})\,\|u\|^2 = \Lambda(A_{k,1/2})\,\|\phi_{x_0}\|^2\,.$$

It is immediate that

$$a_k \leqq \Lambda(A_k) \leqq \Lambda(A_{k,1/2})\,.$$

Let $a > 1/\sqrt{2}$ be arbitrary, and denote by $B(r)$ the ball of radius r centered at 0. For k large enough let $S = B(\sqrt{k/\pi} - a) \cap Z^2$. We have

$$\#S = |S_{1/2}| \leqq |B(\sqrt{k/\pi} - a + 1/\sqrt{2})| < |B(\sqrt{k/\pi})| = k\,, \tag{3.6}$$

and so there exists a k-animal A_k that contains S. We have

$$A_{k,1/2} \supset S_{1/2} \supset B(\sqrt{k/\pi} - a - 1/\sqrt{2})\,,$$

and consequently

$$\begin{aligned} a_k \leqq \Lambda(A_{k,1/2}) &\leqq \Lambda(B(\sqrt{k/\pi} - a - 1/\sqrt{2}) \\ &= \frac{\lambda_D}{\pi(\sqrt{k/\pi} - a - 1/\sqrt{2})^2} \\ &= \frac{\lambda_D}{k}(1 + O(1/\sqrt{k}))\,. \end{aligned}$$

The lemma follows.

3.2. Proof of Lemma 3.2. Let $A_k \subset Z^2$ be an arbitrary animal, and let ϕ be the normalized eigenfunction corresponding to $\Lambda(A_k)$. We extend ϕ to a continuous function u: $R^2 \to R$ in the following way: $u(k) = \phi(k)$ if $k \in Z^2$, and on the each quadrant $Q_{ij} = \{(x_1, x_2): i \leqq x_1 \leqq i+1, j \leqq x_2 \leqq j+1\}$, u is of the form

$$u(x_1, x_2) = a_{ij} + b_{ij}x_1 + c_{ij}x_2 + d_{ij}x_1x_2\,.$$

Such extension exists and is unique. We divide the rest of the argument into three steps.

Step 1. $D(u) \leqq \mathscr{D}(\phi)$. The change of variable $\varepsilon_1 = x_1 - i - 1/2$, $\varepsilon_2 = x_2 - j - 1/2$, transforms Q_{ij} into the quadrant $Q = \{(\varepsilon_1, \varepsilon_2): |\varepsilon_1| \leqq 1/2, |\varepsilon_2| \leqq 1/2\}$, and $u(x)$ into the function

$$U(\varepsilon) = h + a\varepsilon_1 + b\varepsilon_2 + c\varepsilon_1\varepsilon_2\,,$$

for suitable i, j-dependent constants h, a, b, c. We have

$$\int_{Q_{ij}} \left(\frac{\partial u}{\partial x_1}\right)^2 dx = \int_Q \left(\frac{\partial U}{\partial \varepsilon_1}\right) d\varepsilon = \int_Q (a + c\varepsilon_2)^2 d\varepsilon = a^2 + c^2/12\,, \tag{3.7}$$

and

$$\frac{1}{2}((u(i+1,j)-u(i,j))^2+(u(i+1,j+1)-u(i,\ j+1))^2)$$

$$=\frac{1}{2}((U(1/2,\ -1/2)-U(-1/2,\ -1/2))^2$$

$$+(U(1/2,1/2)-U(-1/2,1/2))^2)$$

$$=\frac{1}{2}((a-c/2)^2+(a+c/2)^2)=a^2+c^2/4\,. \quad (3.8)$$

From (3.7), (3.8) we obtain, summing over i, j,

$$\int_{R^2}\left(\frac{\partial u}{\partial x_1}\right)^2 dx \leqq \sum_{i,j}(u(i+1,j)-u(i,j))^2\,, \quad (3.9)$$

and similarly

$$\int_{R^2}\left(\frac{\partial u}{\partial x_2}\right)^2 dx \leqq \sum_{i,j}(u(i,j+1)-u(i,j))^2\,. \quad (3.10)$$

The result follows by adding (3.9), (3.10).

Step 2. $\|u\|^2 \geqq 1-3\Lambda(A_k)/4$. First, note that

$$\int_{Q_{i,j}} u(x)^2 dx = \int_Q (h+a\varepsilon_1+b\varepsilon_2+c\varepsilon_1\varepsilon_2)^2 d\varepsilon_1 d\varepsilon_2$$

$$= h^2+(a^2+b^2)/12+c^2/124$$

$$\geqq h^2 = u(i+1/2, j+1/2)^2\,.$$

Consequently,

$$\|u\|^2 \geqq \sum_{i,j} u(i+1/2, j+1/2)^2\,. \quad (3.11)$$

Let $\hat{Z}^2$ be a dual lattice of Z^2, obtained by translating Z^2 by the vector (1/2, 1/2). Let T: $l^2(Z^2)\to l^2(\hat{Z}^2)$ be the averaging operator defined as

$$T\psi(i+1/2, j+1/2)=\frac{1}{4}\sum_{\pm,\pm}\psi(i+1/2\pm 1/2, j+1/2\pm 1/2)\,.$$

From its very definition, $u|_{\hat{Z}^2}=T\phi$, and

$$\|u\| \geqq \|u\|_{\hat{Z}^2}\| = \|T\phi\|\,. \quad (3.12)$$

The adjoint T^*: $l^2(\hat{Z}^2)\to l^2(Z^2)$ is given by

$$T^*\eta(i,j)=\frac{1}{4}\sum_{\pm,\pm}\eta(i\pm 1/2, j\pm 1/2)\,,$$

and we have $\|T\|=\|T^*\|$,

$$\|u\| \geqq \|T^*T\phi\|\,. \quad (3.13)$$

Let $B_n(a)$ be a ball of radius a centered at n (in the euclidean metric), and denote by $B(a)$ the n-independent number of integer points contained in $B_n(a)$. Let $K_a\colon l^2(Z^2) \to l^2(Z^2)$ be defined as

$$K_a u(n) = \frac{1}{B(a)} \sum_{m \in B_n(a)} u(m) .$$

For example, $H_0 = 4(K_1 - 1)$. For any k we have $\Lambda(A_k) \leqq 4$, $\phi \geqq 0$ on A_k, and consequently

$$K_1 \phi(n) \geqq (1 - \Lambda(A_k)/4)\phi(n), \quad K_1^2 \phi(n) \geqq (1 - \Lambda(A_k)/4)^2 \phi(n) .$$

A direct calculation yields

$$T^*T = \frac{1}{4}\mathbf{1} + \frac{1}{2}K_1 + \frac{1}{4}K_{\sqrt{2}} ,$$

$$K_{\sqrt{2}} = 2K_1^2 - \frac{1}{2}K_2 - \frac{1}{2}\mathbf{1} .$$

Thus, using (3.12) we obtain

$$\|u\| \geqq \|T^*T\phi\| \geqq (T^*T\phi, \phi) \geqq \frac{1}{2}(1 - \Lambda(A_k)/4) + \frac{1}{2}(1 - \Lambda(A_k/4))^2 , \tag{3.14}$$

and consequently,

$$\|u\|^2 \geqq 1 - 3\Lambda(A_k)/4 .$$

Step 3

$$a_k \geqq \frac{\lambda_D}{k} \cdot \frac{1 + O(1/\sqrt{k})}{1 + 3 \cdot \# \partial A_{T,k}/k} .$$

From the above two steps we obtain, if $\Lambda(A_k) \leqq 4/3$,

$$\frac{D(u)}{\|u\|} \leqq \frac{\mathscr{D}(\phi)}{1 - 3\Lambda(A_k)/4} = \frac{\Lambda(A_k)}{1 - 3\Lambda(A_k)/4} . \tag{3.15}$$

Let $A_{k,1}$ be given by (3.3) with $a = 1$. u belongs to the quadratic form domain of the Dirichlet Laplacian in the region $A_{k,1}$, and from (3.15) we get

$$\Lambda(A_{k,1}) \leqq \frac{\Lambda(A_k)}{1 - 3\Lambda(A_k)/4} .$$

Consequently,

$$\begin{aligned} \Lambda(A_k) \geqq \frac{\Lambda(A_{k,1})}{(1 + 3\Lambda(A_{k,1})/4)} &\geqq \frac{\lambda_D |A_{k,1}|^{-1}}{1 + 3\lambda_D |A_{k,1}|^{-1}/4} \\ &= \lambda_D |A_{k,1}|^{-1}(1 + O(1/k)) . \end{aligned} \tag{3.16}$$

Equation (3.16) is trivially satisfied if $\Lambda(A_k) > 4/3$, and so is true for all k. We also have $|A_{k,1}| \leqq \# A_k + 3 \cdot \# \partial A_k$ and consequently

$$\Lambda(A_k) \geqq \frac{\lambda_D}{k + 3 \cdot \# \partial A_k} \cdot (1 + O(1/k)) \geqq \frac{\lambda_D}{k} \cdot \frac{1 + O(1/\sqrt{k})}{k + 3 \cdot \# \partial A_k} . \tag{3.17}$$

Equation (3.17) is true for any k-animal, and in particular for any tamed k-animal $A_{T,k}$. The lemma follows.

3.3. Proof of Lemma 3.3. For $X \subset Z$ denote

$$\partial_n X = \{m \in X\text{: there exists a path of length } \leqq n \text{ starting at } m \text{ and ending at } m' \in Z^2 \backslash X\}\,.$$

Obviously, $\partial_1 X = \partial X$. If $A_{T,k}$, $k > 0$, is a chosen sequence of tamed animals, we will prove

$$\lim_{k \to +\infty} \frac{\# \partial_n A_{T,k}}{k} = 0\,. \tag{3.18}$$

The lemma follows by setting $n = 1$ in (3.18). As before, ϕ denotes the normalized eigenfunction corresponded to $a_k = \Lambda(A_{T,k})$. For $X \subset Z^d$ let

$$\|\phi\|_X^2 = \sum_{n \in X} |\phi(n)|^2\,.$$

Again, we split the proof into three steps.

Step 1. $\|\phi\|^2_{\partial_n A_{T,k}} = O(a_k) = O(1/k)$.

If $m \in \partial_n A_{T,k}$, there exists a path of length $l \leqq n$, $m = m_1, m_2, \ldots, m_l$, $|m_{j-1} - m_j|_+ = 1$, $m_l \notin A_{T,k}$. We have

$$|\phi(m)| \leqq \sum_{j=1}^{l} |\phi(m_{j-1}) - \phi(m_j)|^2\,,$$

$$|\phi(m)|^2 \leqq n \sum_{j=1}^{l} |\phi(m_{j-1}) - \phi(m_j)|^2\,.$$

Any couple (m, m'), $|m - m'|_+ = 1$ can belong only to finitely many paths of length. n, the number depending only on n, and consequently there exists a uniform constant C_n so that

$$\sum_{m \in \partial_n A_{T,k}} \phi(k)^2 \leqq C_n \sum_{\substack{\langle m', m\rangle \\ |m' - m|_+ = 1}} |\phi(m') - \phi(m)|^2 = C_n \cdot a_k = O(1/k)\,.$$

Step 2. Denote $O_k^n = A_{T,k} \backslash \partial_n A_{T,k}$, and let $\psi(m) = \max\{0, \phi(m) - b/\sqrt{k}\}$ for $0 < b < 1$. Then

(a) $\mathscr{D}(\psi) \leqq \mathscr{D}(\phi)$.
(b) $\|\psi\| \geqq 1 - b$.
(c) If $Z_k^n = O_k^n \cup \{m \in A_{T,k} : \psi(m) > 0\}$ then $\#(Z_k^n \backslash O_k^n) = O(1)$ as $k \to \infty$.

The part (a) is trivial. To prove (b), let $\hat{\phi}(m) = \max\{\phi(m), b/\sqrt{k}\}$. Then

$$\|\hat{\phi}\|^2_{A_{T,k}} = \sum_{m \in A_{T,k}} |\hat{\phi}(m)|^2 \leqq \|\phi\|^2 = 1, \quad \psi(m) = \hat{\phi}(m) - b/\sqrt{k}\,,$$

and the Minkowski inequality yields

$$\|\psi\| \geqq \|\hat{\phi}\|_{A_{T,k}} - \left(\sum_{m \in A_{T,k}} b^2/k\right)^{1/2} \geqq 1 - b\,.$$

To prove (c), note that from Step 1 we have

$$\#\{m \in \partial_n A_{T,k}\colon \phi(m) > b/\sqrt{k}\} = O(1)\,.$$

On the other hand, $\psi(m) > 0$ iff $\phi(m) > b/\sqrt{k}$ and (c) follows.

Step 3. If $\alpha_n = \limsup_{k\to\infty}(\#\partial_n A_{T,k})/k$ then $\lim_{n\to\infty}\alpha_n = 0$.

We first remark that $0 \leqq \alpha_1 \leqq \alpha_2 \leqq \ldots \leqq 1$, and consequently $\lim_{n\to\infty}\alpha_n = \alpha$ exists. Let $\theta = 1 - \alpha$ and suppose that $\theta < 1$. Let $\rho \in (\theta, 1)$ be arbitrary, and denote

$$\theta_n = \liminf_{k\to\infty} \frac{\# Z_k^n}{k}\,.$$

From Step 2 we have that

$$\frac{\# Z_k^n}{k} = \frac{O(1)}{k} + 1 - \frac{\#\partial_n A_{T,k}}{k}\,, \tag{3.19}$$

and consequently $\theta_n = 1 - \alpha_n \leqq \rho$ for n large enough. Step 2 also implies that for any $b > 0$ and any $k_0 \in \mathfrak{N}$, there exist k, m, $k > k_0$, $m < \rho k$, so that $\# A_k^n = m$ and

$$\Lambda(Z_k^n) \leqq a_k/(1-b)^2\,. \tag{3.20}$$

First, (3.20) implies that $\theta \geqq 1/4$. Otherwise, we can choose $\rho < 1/4$ and derive from (3.17), (3.20)

$$\begin{aligned}\frac{\lambda_D}{4m}\cdot(1 + O(1/m)) &\leqq a_m \leqq \Lambda(Z_k^n)\\ &\leqq \frac{a_k}{(1-b)^2} \leqq \frac{\lambda_D}{k(1-b)^2}\cdot(1 + O(1/\sqrt{k}))\,,\end{aligned} \tag{3.21}$$

$m/k < \rho$, and so when $k \to \infty$, $m \to \infty$ as well, and we obtain $1/4 \leqq \rho(1-b)^2$ for all $b > 0$, which contradicts the choice $\rho < 1/4$.

Let $0 < \varepsilon < 1$ be chosen in such a way that $\rho/\theta < 1/(1-\varepsilon)$. For n large enough, $\theta_n < \rho$ and consequently $\theta/\theta_n \geqq 1 - \varepsilon$. There exists a sequence $k_j \to \infty$ so that

$$\lim_{j\to\infty} \frac{\#\partial_n A_{T,k_j}}{k_j} = \alpha_n\,.$$

On the other hand, from (3.19) we derive

$$\limsup_{j\to\infty} \frac{\#\partial_{n+1} A_{T,k_j}}{k_j} \leqq \alpha_{n+1} < 1\,,$$

$$\lim_{j\to\infty} \frac{\# O_n^{k_j}}{k_j} = \theta_n, \quad \liminf_{j\to\infty} \frac{\# O_{n+1}^{k_j}}{k_j} \geqq \theta_{n+1} \geqq \theta\,.$$

Because $\partial O_k^n = O_k^n \backslash O_{k+1}^n$, we have

$$\limsup_{j\to\infty} \frac{\#\partial O_{k_j}^n}{\# O_{k_j}^n} \leqq \frac{\theta_n - \theta}{\theta_n} < \varepsilon\,.$$

Again, Step 2 (c) yields

$$\limsup_{j\to\infty} \frac{\#\partial Z^n_{k_j}}{\# Z^n_{k_j}} < \varepsilon .$$

For j large enough, to any k_j we correspond m_j with $m_j < \rho k_j$ so that $\# Z^n_{k_j} = m_j$ and (3.20) is true. It follows from (3.17) that for any $\delta > 0$ and large enough j

$$\Lambda(Z^n_{k_j}) \geqq \frac{\lambda_D(1-\delta)}{m_j(1 + 3\cdot \#\partial Z^n_{k_j}/m_j)} \geqq \frac{\lambda_D(1-\delta)}{\rho k_j(1+3\varepsilon)} .$$

On the other hand, from (12.1) and Lemma 12.1 we obtain

$$\Lambda(Z^n_{k_j}) \leqq \frac{a_{k_j}}{(1-b)^2} \leqq \frac{\lambda_D}{k_j(1-b)^2}\cdot(1 + O(1/\sqrt{k_j})) ,$$

and consequently for a large j we have

$$\rho \geqq (1-b)^2 \frac{1-\delta}{(1+3\varepsilon)(1+\delta)} .$$

The numbers b, ε, δ can be chosen to be arbitrarily small and so $\rho \geqq 1$, which contradicts the choice of ρ, namely that $\theta < \rho < 1$. So, $\theta = 1$, and for all n, $\alpha_n = 0$. The lemma follows.

4. On the Pure Point and the Discrete Spectrum

The following simple consequences of Lemma 2.1 will be of essential importance below. Let $d/k \geqq \alpha > d/(k+1)$, and for suitable γ let

$$A_\omega = \{n : V_\omega(n) \leqq |n|^\gamma\} .$$

According to Lemma 2.1, we have a decomposition

$$A_\omega = \bigcup_{n\geqq 0} A(n,\omega), \quad A(n,\omega)\cap A(m,\omega) = \emptyset , \tag{4.1}$$

for a.e. ω. (In the sequel we drop the ω dependence whenever it is clear within the context.) $A(0)$ is a finite region chosen in such a way that for $n \notin A(0)$ no events B_n take place (see (2.8)), and $A(n)$ are animals of size $\leqq k$. Let $\sim$ be a relation of equivalence on $\{A(n) : n > 0\}$ defined as

$$A(n) \sim A(m) \Leftrightarrow \exists n_1 = n, n_2, \ldots, n_l = m \text{ such that for } 1 \leqq i \leqq l-1 ,$$

$$d(A(n_i), A(n_{i+1})) \leqq C \max\{|s|^\gamma, |s'|^\gamma : s \in A(n_i), s' \in A(n_{i+1})\} .$$

We order classes of equivalence by their distance from the origin (viewing classes as subsets of Z^d), ordering arbitrarily the equidistant ones. Let $G^{(i)}$ be a union of the elements of the ith class. By taking C small enough, it is easy to show that $\# G^{(i)} \leqq k$ (otherwise some event B_n happens for $n \notin A(0)$) and that

$$d(G^{(i)}, G^{(k)}) > C \max\{|s|^\gamma, |s'|^\gamma : s \in G^{(i)}, s' \in G^{(k)}\} .$$

We will refer to $G^{(i)}$ as a *connected group of animals.*

The point of the above is the following: if we think about $G^{(i)}$ as potential wells, then, for typical ω and outside a finite (ω-dependent) set, these wells have no more than k points and are separated by high and long potential barriers. The probabilistic framework differs now from the one in Sect. 2.3, where we were interested only in potential wells encircling each other successively and thus preventing tunneling. By taking into account the structure of $G^{(i)}$, one should be able to obtain better control of the resolvent matrix elements, and that will be our main concern below. We will prove

Theorem 4.1. *Let $I = (0, E_0)$. Then for every $E_0 > 0$ there exists $a > 0$ such that for every $n \in Z^d$ and for a.e. $\omega \in \Omega$ and a.e. $E \in I$ (with respect to the Lebesgue measure)*

$$\sup_{0<\varepsilon<1} |R_\omega(n, m; E + i\varepsilon)| \leqq C_{n,\omega,E} \exp(-a|m|) . \tag{4.2}$$

The above theorem is certainly expected on intuitive grounds. (Actually, more is expected, namely that $|m|$ can be replaced with $|m|^\beta$ for some $\beta > 1$, although we will not try to prove that.) Simple modification of our argument can improve decay in (4.2) by a factor $\ln |m|$.) Once proven, Theorem 4.1 yields part (i) of Theorem 1.2 by the well-known argument of Simon–Wolff [20] or Delyon–Levy–Souillard [6] (see also [2]).

The rest of this chapter is devoted to the proof of Theorem 4.1. Part (iii) of Theorem 1.2 is a simple by-product of the analysis below, and necessary modifications in the argument are outlined in Sect. 4.2.

The following decomposition of a probability space $\Omega = \bigotimes_{n \in Z^d} I_n$, $I_n = [0, 1]$, will be useful. If $\gamma > 0$ is the constant from Lemma 2.1, writing

$$I_n = I_n^0 \cup I_n^1, \quad I_n^0 = [0, |n|^\gamma/(1 + |n|^\alpha)], \quad I_n^1 = [|n|^\gamma/(1 + |n|^\alpha), 1] ,$$

we obtain the decomposition

$$\Omega = \bigcup_c \Omega_c$$

indexed by all possible sequences of 0 and 1's. One simple consequence of discussion in the beginning of this section is that one can pick countably many $\{\Omega_i\}_{i \geqq 1}$ out of $\{\Omega_c\}$ such that $\bigcup_i \Omega_i$ has a full P measure, and that every Ω_i has the following form: for n outside finite set F_i (which can be chosen once for all $\omega \in \Omega_i$), I_n^0's are exactly corresponded to the n's which belong to the connected groups of animals, or more precisely, at site n we have I_n^0 iff $n \in G^{(i)}$ for some i. Furthermore, $G^{(i)}$'s are independent of the choice $\omega \in \Omega_i$, and thus our decomposition fixes the position of "potential wells," although potential within them can vary (at site n, between 0 and $|n|^\gamma$). We will prove below that (4.2) is valid for a.e. $\omega \in \Omega_1$ and a.e. $E \in I$. Since our argument does not depend on the choice of Ω_1 (it also applies to any Ω_k, $k > 1$, all what we use is the fixed position of "wells"), it yields the statement. In the sequel any ω, unless otherwise stated, will refer to an event in Ω_1.

Let $L > 1$ be a positive constant and define a new potential ($C_L = \{n\colon \max |n_i| \leqq L\}$)

$$V_{\omega,L} = \begin{cases} V_\omega(n), & \text{if } n \notin C_L ; \\ L, & \text{if } n \in C_L . \end{cases}$$

The right choice of L will circumvent technical problems with convergence of the resolvent cluster expansion constructed below. First, by making L large enough, we can suppose that $C_L \supset F_1$. If

$$H_{\omega,L} = H_0 + V_{\omega,L}$$

we have that H_ω is a finite rank, L-dependent perturbation of $H_{\omega,L}$. Denote by

$$R_{\omega,L}(n, m; z) = (\delta_n, (H_{\omega,L} - z)^{-1}\delta_m)$$

the matrix elements of the resolvent of $H_{\omega,L}$. The main technical ingredient in our proof is the following

Lemma 4.2. *To every L large enough we can correspond $\Omega_L \subset \Omega_1$, $\Omega_L \to \Omega_1$ as $L \to \infty$, such that for* a.e. *$\omega \in \Omega_L$, $i \in C_L$, $m \in Z^d$, and for every fixed $E \in I$, we have for some fixed $a > 0$,*

$$\sup_{0<\varepsilon<1} |R_{\omega,L}(i, m; E + i\varepsilon)| \leqq C_{\omega,L} \exp(-a|m|) . \tag{4.3}$$

Lemma 4.1 yields (4.2) as follows. Since

$$H_\omega - H_{\omega,L} = \sum_{i \in C_L} \lambda_i(\cdot, \delta_i)\delta_i ,$$

where $\lambda_i \in I_i^0 - L$ or $I_i^1 - L$, we have that for $\omega \in \Omega_L$ and L large enough

$$R_\omega(n, m; z) = R_{\omega,L}(n, m; z) - \sum_{i \in C_L} \lambda_i R_{\omega,L}(i, m; z) R_\omega(n, i; z) . \tag{4.4}$$

Since $R_\omega(n, i; z)$ is a difference of two Herglotz functions in the upper half plane, we have that for a.e. $\omega \in \Omega_L$ and for a.e. $E \in I$,

$$\max_{i \in C_L} \sup_{0<\varepsilon<1} |R_\omega(n, i; E + i\varepsilon)| \leqq D_{n,\omega,E} ,$$

where $D_{n,\omega,E}$ is a finite constant (whose dependence on L is comprised within ω). Thus, (4.2) is valid for a.e. $\omega \in \Omega_L$ and a.e. $E \in I$. Since $\Omega_L \to \Omega_1$ as $L \to \infty$, (4.2) follows.

It remains to prove Lemma 4.2.

4.2. Proof of Lemma 4.2. For arbitrary $X \subset Z^d$ denote

$$\partial_{\text{ext}} X = \{m \in Z^d \backslash X : (\exists n \in X) |m - n|_+ = 1\} ,$$

$$\partial_{\text{int}} X = \{m \in X : (\exists n \in Z^d \backslash X) |m - n|_+ = 1\} ,$$

$$\partial X = \{(m, n) : m \in \partial_{\text{ext}} X,\ n \in \partial_{\text{int}} X,\ |m - n|_+ = 1\} .$$

If $H = H_0 + V$, we say that H_D is the operator H with Dirichlet boundary condition along ∂X if

$$H_D - H = \sum_{(m,n) \in \partial X} (\cdot, \delta_m)\delta_n + (\cdot, \delta_n)\delta_m . \tag{4.5}$$

H_D is a direct sum of two operators, one on $l^2(X)$ and one on $l^2(Z^d \backslash X)$. The part which acts on $l^2(X)$ will be referred to as the *restriction* of H to X.

For $i \geqq i(L)$ all connected groups $G^{(i)}$ have empty intersection with C_L. Denote by $H^{(i)}_{\omega,L}$ the restriction of $H_{\omega,L}$ to $G^{(i)}$ for $i \geqq i(L)$, and by $R^{(i)}_{\omega,L}(n, m; z)$ the matrix element of the $(H^{(i)}_{\omega,L} - z)^{-1}$. Let

$$G^{(0)} = Z^d \Big\backslash \left(\bigcup_{i \geqq i(L)} G^{(i)} \right).$$

The size of L will be fixed by requirements below (recall that the statement of lemma is valid for L large enough). We denote by $H^{(0)}_{\omega,L}$ and $R^{(0)}_{\omega,L}$ the restrictions of $H_{\omega,L}$, $R_{\omega,L}$ to $G^{(0)}$. From now on, let $n \in C_L$, $m \in Z^d$ be two fixed sites. For definiteness we will suppose that $m \in G^{(j)}$ for some $j \geqq i(L)$, the discussion of the case when $m \in G^{(0)}$ is analogous.

For $\operatorname{Im}(z) > 2d$ we have expansion

$$R_{\omega,L}(n, m; z) = -\sum_{\tau} \prod_{s \in \tau} \frac{1}{V_{\omega,L}(s) + 2d - z}, \tag{4.6}$$

where series on the right converges uniformly, and sum is over all paths connecting n and m. By regrouping the terms in the series, we can write

$$\begin{aligned} R_{\omega,L}(n, m; z) = -\sum_{k \geqq 0} \sum_{\substack{(i_1,\ldots,i_k) \\ i_l \geqq i(L)}} \sum_{\substack{(s_l, t_l) \in \partial G^{(i_l)} \\ (s'_l, t'_l) \in \partial G^{(i_l)}}} \sum_{(s,t) \in \partial G^{(j)}} R^{(0)}_{\omega,L}(n, s_1; z) \cdot R^{(i_1)}_{\omega,L}(t_1, t'_1; z) \,. \\ \cdot R^{(0)}_{\omega,L}(s'_1, s_2; z) \cdot R^{(i_2)}_{\omega,L}(t_2, t'_2; z) \cdot R^{(0)}_{\omega,L}(s'_2, s_3; z) \cdot \ldots \\ \ldots \cdot R^{(i_k)}_{\omega,L}(t_k, t'_k; z) \cdot R^{(0)}_{\omega,L}(s'_k, s; z) \cdot R^{(j)}_{\omega,L}(t, m; z) \,. \end{aligned} \tag{4.7}$$

The above formula clearly follows from (4.6), separating the contribution of the paths within different $G^{(i)}$'s, and using the formula analogous to the (4.6) for each $R^{(i)}_{\omega,L}$. For $i > 0, j > 0$ we set

$$d(i, j) = \max\{|n - m|; n \in G^{(i)}, m \in G^{(j)}\}, \quad d_n(i) = \max_{m \in G^{(i)}} |n - m| \,.$$

The following lemma gives the basic estimates needed to control (4.7).

Lemma 4.3. *For $E \in I$, $0 < \varepsilon < 1$ and L suffciently large:*

(i) *There exist random variables $A^{(i)}_E(\omega)$ for every $i \geqq i(L)$ such that for $t \in \partial_{\text{int}} G^{(i)}$, $t' \in G^{(i)}$,*

$$|R^{(i)}_{\omega,L}(t, t'; E + i\varepsilon)| < A^{(i)}_E(\omega) \,.$$

(ii)

$$P\{\omega \in \Omega_1 : A^{(i)}_E(\omega) > M\} \leqq C/M$$

for a uniform constant C independent of i.

(iii) *If $s \in \partial_{\text{ext}} G^{(i)}$ and $s' \in \partial_{\text{ext}} G^{(j)}$ we have*

$$R^{(0)}_{\omega,L}(s', s; E + i\varepsilon) \leqq C_L \exp(-a_L d(i, j)) \,,$$

$$R^{(0)}_{\omega,L}(n, s'; E + i\varepsilon) \leqq C_L \exp(-a_L d_n(j)) \,.$$

Constants C_L, a_L depend only on L and furthermore $C_L \to 0$, $a_L \to \infty$ as $L \to \infty$.

Proof. Let $H = H_0 + V$ be a discrete Schrödinger operator, and let H_D be its restriction to a set $X \subset Z^d$. If $\delta = \operatorname{dist}(z, \sigma(H_D))$, it is a standard result (see e.g. [2], Exercise III.6.8) that for $m, n \in X$ we have

$$|R_D(m, n; z)| \leqq \frac{2}{\delta} \exp(-\ln(1 + \delta/4d)|m - n|) . \tag{4.8}$$

Defining

$$A_E^{(i)}(\omega) = 2/\operatorname{dist}(E, \sigma(H_{\omega,L}^{(i)})) ,$$

we observe that part (i) follows from (4.8). Part (ii) is an immediate consequence of Propositions VIII.1.8 and VIII.4.8 in [2]. Part (iii) follows from (4.8) and observation that for L large enough

$$\operatorname{dist}(E, \sigma(H_{\omega,L}^0)) \geqq L^\gamma - E_0 - 4d . \quad \blacksquare$$

The sets Ω_L are defined as

$$\Omega_L = \{\omega \in \Omega_1 ; A_E^{(i)}(\omega) \leqq |n_i|^{d+1} \text{ for } i \geqq i(L)\} .$$

Part (ii) of the above lemma yields

$$P(\Omega_L) \geqq P(\Omega_1) - \sum_{i \geqq i(L)} C/|n_i|^{d+1} = P(\Omega_1) - O(1/i(L)) .$$

Since $i(L) \to \infty$ as $L \to \infty$ sets Ω_L are as required by Lemma 4.1. For L large enough, $\omega \in \Omega_L$, $E \in I$ and for $1 > \varepsilon > 0$ we have

$$|R_{\omega,L}(n, m; E + i\varepsilon)| < C_L \exp[-(a_L/2) \cdot |n - m|] \sum_{n \geqq 0} (\alpha_L)^n , \tag{4.9}$$

where

$$\alpha_L = \sup_{i \geqq i(L)} \sup_{s \in \partial_{\text{ext}} G^{(i)}} |n_i|^{d+1} \sum_{j \geqq i(L)} \sup_{\substack{(s_1, t_1) \in \partial G^{(j)} \\ t_2 \in \partial_{\text{int}} G^{(j)}}} |R_{\omega,L}^{(0)}(s, s_1; E + i\varepsilon)$$
$$\cdot R_{\omega,L}^{(j)}(t_1, t_2; E + i\varepsilon)|^{1/2} .$$

In (4.9) $C_L \to 0$ as $L \to \infty$. One can easily estimate

$$\alpha_L \leqq \sup_{i \geqq i(L)} C|n_i|^{d+1} \exp(-D|n_i|) , \tag{4.10}$$

where constants C and D are uniform. For L large enough, $\alpha_L < 1$, and statement follows from (4.10).

4.2. Discrete Spectrum is Finite. To prove that for a.e. ω the number of eigenvalues lying in $[0, a_k)$ is finite, in the notation of Lemma 4.1, it suffices to show that to every L large enough we can correspond $\Omega_L \subset \Omega_1$, $\bigcup \Omega_L = \Omega_1$, such that for $\omega \in \Omega_L$ $H_{\omega,L}$ has no spectrum in $[0, a_k)$ (since $H_{\omega,L} - H_\omega$ is an operator of rank $(2L + 1)^d$, we then have $\sigma_{\text{disc}}(H_\omega) \leqq (2L + 1)^d$).

$H_{\omega,L}$ has no eigenvalues in $[0, a_k)$ if for all $n, m \in Z^d$,

$$\sup_{E < a_k} \sup_{\varepsilon > 0} |R_{\omega,L}(n, m; E + i\varepsilon)| < \infty . \tag{4.11}$$

We write the cluster expansion (4.7) of $R_{\omega,L}$, and estimate for $E < a_k$

$$|R^{(i)}_{\omega,L}(t, t'; E + i\varepsilon)| \leqq 2/\mathrm{dist}(a_k, \sigma(H^{(i)}_{\omega,L})) = A^{(i)}_{a_k}(\omega) .$$

The rest of the argument follows line by line that of the previous section.

5. Proof of Theorem 1.4

We will prove Theorem 1.4 in the case $d = 1$, the argument being essentially the same for $d > 1$. For a purpose of this section it is convenient to redefine H_D (operator with Dirichlet boundary condition) by subtracting $(\cdot, \delta_n)\delta_n + (\cdot, \delta_m)\delta_m$ from (4.5), ensuring that $H \leqq H_D$. For L a positive integer, let C_L be an interval of size $2L$ centered at the origin, and let $H_{\omega,L}$ be the restriction of H_ω to C_L. We suppose that $1/k > \alpha > 1/(k+1)$, and that $E \in (a_j, a_{j-1})$ $(a_0 = \infty)$ for some $1 \leqq j \leqq k$. The case $\alpha = 1/k$, we treat separately.

Let $0 < \delta < 1 - j\alpha$, and for fixed ω suppose that $C_{L,\delta} = C_{[L^\delta]+1} \supset A(0)$ (see (4.1), $[\cdot]$ stands for the greatest integer part). Define $H^\delta_{\omega,L}$ from $H_{\omega,L}$ by setting the potential to be of the size L within $C_{L,\delta}$. In the sequel $N_E(H)$ stands for the number of eigenvalues of an operator H which are less than E. Since $|N_E(H_{\omega,L}) - N_E(H^\delta_{\omega,L})| = O(L^\delta)$, it suffices to prove the statement for $N_E(H^\delta_{\omega,L})$. We numerate the connected group of animals by their distance from the origin, and set

$$Z^{(i)}_L(\omega) = G^{(i)}(\omega) \cap (C_L \setminus C_{L,\delta}), \quad Z_L(\omega) = \bigcup Z^{(i)}_L(\omega), \quad Z^{(0)}_L(\omega) = C_L \setminus Z_L(\omega) .$$

Let $\hat{H}_{L,\omega}$ be the operator obtained from $H_{\omega,L}$ by putting the Dirichlet boundary condition along $\partial Z(\omega)$.

Lemma 5.1. *There exist* $\varepsilon > 0$ *such that for* a.e. ω,

$$\liminf_{L\to\infty} \frac{N_E(H^\delta_{\omega,L})}{L^{1-j\alpha}} \geqq \liminf_{L\to\infty} \frac{N_{(E-1/L^\varepsilon)}(\hat{H}_{\omega,L})}{L^{1-j\alpha}} , \tag{5.1}$$

$$\limsup_{L\to\infty} \frac{N_{(E+1/L^\varepsilon)}(\hat{H}_{\omega,L})}{L^{1-j\alpha}} \geqq \limsup_{L\to\infty} \frac{N_E(H^\delta_{\omega,L})}{L^{1-j\alpha}} . \tag{5.2}$$

Proof. (5.1) is immediate since $\hat{H}_{\omega,L} \geqq H^\delta_{\omega,L}$. To prove (5.2) we proceed as follows. Let $R(n, m)$, $\hat{R}(n, m)$ be matrix elements of $R = (H_{\omega,L} + 1)^{-1}$, $\hat{R} = (\hat{H}_{\omega,L} + 1)^{-1}$. First, we have that

$$|R(n, m) - \hat{R}(n, m)| \leqq \frac{C}{L^{\gamma\delta}} \exp(-a|n - m|) , \tag{5.3}$$

for uniform constants C and a (γ is as in (2.8)). We will prove (5.2) in the case $n \in Z^{(i)}$, $m \in Z^{(i)}$, the other cases having similar proofs. We have

$$R(n, m) = \hat{R}(n, m) + \sum_{(s,t)\in\partial Z^{(i)}} \sum_{(s',t')\in\partial Z^{(i)}} \hat{R}(n, s)\hat{R}(t, s')R(t', m) .$$

Using (4.8) we estimate

$$|\hat{R}(t,s')| \leqq \frac{C}{L^{\gamma\delta}} \exp(-a|t-s'|)\,,$$

$$|\hat{R}(n,s)| \leqq C\exp(-a|n-s|), \quad |R(t',m)| \leqq C\exp(-a|t'-m|)\,,$$

where C and a are uniform constants, and (5.3) follows.

$\hat{R}$ splits into direct sum of operators (after eventually renumerating $Z^{(i)}$'s)

$$\hat{R} = \bigoplus_{i=1}^{l} \hat{R}^{(i)} \bigoplus \hat{R}^{(0)} \text{ acting on } \bigoplus_{i=1}^{l} l^2(Z^{(i)}) \bigoplus l^2(Z^{(0)})\,.$$

By permuting the basis of $l^2(C_L)$ we can write $\hat{R}$ in the matrix form

$$\hat{R} = \begin{pmatrix} \hat{R}^{(1)} & 0 & \dots & 0 \\ 0 & \hat{R}^{(2)} & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \hat{R}^{(0)} \end{pmatrix}.$$

For $1 \leqq i \leqq l$, $\hat{R}^{(i)}$ is a matrix of dimension $k_i \times k_i$ for some $k_i \leqq k$. Let $\tilde{R}$ be the operator R in permuted basis, and let ($k_0 = 0$)

$$\rho_i = \max_{k_{i-1} \leqq n \leqq k_i} \sum_m |\hat{R}^{(i)}(n,m) - \tilde{R}(n,m)|\,.$$

It follows from (5.3) that $\rho_i \leqq C/L^{\gamma\delta}$ for uniform constant C.

Let $1/(\lambda+1)$ be an eigenvalue of R and let ϕ be corresponding eigenvector. If $\max_{n \in C_L} |\phi(n)|$ is achieved for some $n \in Z^{(i)}$, $i > 0$, then proceeding as in the proof of Gershorgin's theorem [21], one can show that there exist eigenvalue $E_j^{(i)}$ of $\hat{R}^{(i)}$ such that

$$|E_j^{(i)} - \lambda| \leqq C/L^{\gamma\delta}\,. \tag{5.4}$$

If $\max|\phi(n)|$ is achieved for $n \in Z^{(0)}$, it is elementary to show that $\lambda \geqq cL^{\gamma}$ for some $c > 0$. Thus, for L large enough, the eigenvalues of $H^{\delta}_{\omega,L}$ which are less than E are on the distance $C/L^{\gamma\delta}$ from the eigenvalues of $\hat{H}_{\omega,L}$ which are less than E, and statement follows.

The probabilistic part of the proof goes as follows. Let $H^{(i)}$ be the restriction of $\hat{H}_{\omega,L}$ to $Z^{(i)}$ and let χ_n be random variables defined on the following way:

$$\chi_n(\omega) = \begin{cases} N_E(H^{(i)}), & \text{if } n \in Z^{(i)} \text{ and } d(Z^{(i)},0) = |n|\,; \\ 0, & \text{otherwise}\,. \end{cases}$$

χ_n are independent random variables for L large enough.

Lemma 5.2. *If $n \in C_{[L^{\delta}]+j}$ or $||n| - L| < j$, $\mathbf{E}(\chi_n(\omega)) = 0$. Otherwise, there exist constant $\varepsilon(L)$, $\varepsilon(L) \to 0$ as $L \to \infty$, such that*

$$(1-\varepsilon(L))\,\frac{N_j(E)}{|n|^{j\alpha}} \leqq \mathbf{E}(\chi_n) \leqq (1+\varepsilon(L))\,\frac{N_j(E)}{|n|^{j\alpha}}\,, \tag{5.5}$$

where $N_j(E)$ is a continuous function on (a_j, a_{j-1}). Furthermore, $N_j(E) \sim D_j(E-a_j)^j$ as $E \to a_j$ for a suitable constant D_j.

Proof. To show that $\mathbf{E}(\chi_n) = 0$ under the conditions $|n| \leqq [L^\delta] + j$ or $||n| - L| < j$ is trivial. To prove (5.5), note that $N_E(H^{(i)}) = 0$ unless $Z^{(i)}$ contains an animal of the size j or bigger. Let A be the operator on l-dimensional space ($l \geqq j$) defined by

$$A(\lambda_1, \lambda_2, \ldots, \lambda_l) = H_0 + \sum_{i=1}^{l} \lambda_i(\cdot, \delta_i)\delta_i ,$$

where H_0 is a finite dimensional Laplacian (see (1.3)). Ordering eigenvalues of A in the increasing order,

$$E_1(\lambda_1, \lambda_2, \ldots, \lambda_l) < E_2(\lambda_1, \lambda_2, \ldots, \lambda_l) \leqq \ldots \leqq E_l(\lambda_1, \lambda_2, \ldots, \lambda_l) ,$$

we observe that E_i are continuous functions of $(\lambda_1, \lambda_2, \ldots, \lambda_l)$, and if any λ_i is large enough, $E_1 > E$ [12]. If $A_p = \{(\lambda_1, \ldots, \lambda_l): N_E(A) = p\}$, then A_p, $1 \leqq p \leqq l$, are disjoint bounded measurable sets. The estimates (5.5) are now obvious (we remark that $\varepsilon(L)$ part contains contribution from the animals within $Z^{(i)}$ of the size strictly bigger than j, if any). The continuity of function $N_j(E)$ is a consequence of Proposition VIII.4.10 in [2].

Finally, let ϕ be the ground state corresponded to the eigenvalue a_j. The regular perturbation theory [17] yields that

$$E_1(\lambda_1, \lambda_2, \ldots, \lambda_j) = a_j + \sum \lambda_i |\phi(n)|^2 + O(\sum |\lambda_i|^2) .$$

Furthermore, for $E - a_j$ small enough we have

$$N_j(E) = \mu\{(\lambda_1, \lambda_2, \ldots, \lambda_j): \sum |\phi(i)|^2 \lambda_i + O(\sum |\lambda_i|^2) < E - a_j\} ,$$

where μ stands for a Labesgue measure. It is now elementary to show that as $E \to a_j$,

$$N_j(E) \sim D_j(E - a_j)^j, \quad D_j = \frac{1}{j! \prod |\phi(i)|^2} .$$

Remark. For $d > 1$, the constants D_j have a more complicated form, and are intrinsically of combinatorial nature.

Since $N_E(\hat{H}_{\omega,L}) = \sum_{n \in C_L} \chi_n(\omega)$, we have

$$\begin{aligned} \lim_{L \to \infty} \frac{\mathbf{E}(N_E(\hat{H}_{\omega,L}))}{L^{1-j\alpha}} &= \lim_{L \to \infty} \sum_{n \in C_L} \frac{\mathbf{E}(\chi_n(\omega))}{L^{1-j\alpha}} \\ &= 2 \int_0^1 \frac{N_j(E)}{|x|^{j\alpha}} dx \\ &= \frac{2}{1 - j\alpha} N_j(E) . \end{aligned}$$

It remains to show that

$$\lim_{L \to \infty} \frac{\mathbf{E}(N_E(\hat{H}_{\omega,L})) - N_E(\hat{H}_{\omega,L})}{L^{1-j\alpha}} = 0 \quad \text{for a.e. } \omega . \tag{5.6}$$

Let $\eta_n(\omega) = \chi_n(\omega) - \mathbf{E}(\chi_n(\omega))$. Then, if p is a positive integer, we get as in (2.4)

$$\mathbf{E}\left(\sum_{n\in C_L} \eta_n(\omega)\right)^{2p} \leqq C\mathbf{E}\left(\sum_{n\in C_L} \chi_n(\omega)\right)^p$$
$$\leqq CL^{p(1-j\alpha)}, \tag{5.7}$$

where constant C does not depend on L. Defining

$$A_L(\omega) = |\mathbf{E}(N_E(\hat{H}_{\omega,L})) - N_E(H_{\omega,L})|, \tag{5.8}$$

we have as in Sect. 2.1,

$$P\{\omega\colon (2L)^{j\alpha-1}A_L(\omega) \geqq \varepsilon\} \leqq \frac{C}{\varepsilon^{2p}} L^{2p(j\alpha-1)} \cdot L^{p(1-j\alpha)}$$
$$\leqq \frac{C}{\varepsilon^{2p}} L^{-p(1-j\alpha)}.$$

For p large enough, $p(1-j\alpha) > 1$, and the Borel–Cantelli Lemma yields (5.6).

When $\alpha = 1/k$ and $E \in (a_j, a_{j-1})$, $j > k$, the above argument applies without changes. The case $\alpha = 1/k$, $E \in (a_k, a_{k-1})$, is special, and to prove (1.9) we argue as follows. In the construction in the beginning of section interval $C_{[L^\delta]+1}$ is replaced by an interval $C_{[(\ln L)^\delta]+1}$ for δ small enough. Lemma 5.1 is valid after replacement of L^ε with $(\ln L)^\varepsilon$, and Lemma 5.2 is valid as it stands. Thus

$$\lim_{L\to\infty} \frac{\mathbf{E}(N_E(\hat{H}_{\omega,L}))}{\ln L} = 2N_k(E).$$

It remains to show that for a.e. ω

$$\lim_{L\to\infty} \frac{A_L(\omega)}{\ln L} = 0$$

with $A_L(\omega)$ given by (5.6). First, one can show that

$$\mathbf{E}((A_L(\omega))^{2p}) \leqq C^p(\ln L)^p,$$

where C is a uniform constant *independent* of p and L. Thus

$$\mathbf{E}(\exp(A_L(\omega)^2)) \leqq \sum_k \frac{\mathbf{E}(A_L(\omega)^{2k})}{k!} \leqq L^{C^2}.$$

We have

$$P\{\omega\colon (\ln L)^{-1}A_L(\omega) \geqq \varepsilon\} \leqq P\{\omega\colon \exp(A_L(\omega)/\varepsilon)^2 \geqq \exp((\ln L)^2)\}$$
$$\leqq \frac{1}{\exp((\ln L)^2)} \mathbf{E}(\exp(A_L(\omega)/\varepsilon^2))$$
$$\leqq \frac{L^{(C/\varepsilon)^2}}{L^{\ln L}},$$

and the Borel–Cantelli Lemma again yields the statement.

Appendix

Proof of Theorem 2.2. For simplicity of notation we set $n_0 = 0$. It suffices to prove that under the conditions of theorem, (2.12) is valid for a.e. $E \in (a, b)$ where (a, b) is a fixed interval of finite size. We introduce boundary blocks B_k and main blocks M_k as $B_0 = \emptyset$, $M_0 = \emptyset$,

$$B_k = \Gamma_k^+ \backslash \Gamma_k^- , \quad M_k = (\Gamma_k^+ \backslash M_{k-1}) \cup B_{k-1} .$$

The restrictions of H to M_k, B_k we denote by H_{M_k}, H_{B_k}, and by $R_{M_k}(n, m; z)$, $R_{B_k}(n, m; z)$ the kernels of $(H_{M_k} - z)^{-1}$, $(H_{B_k} - z)^{-1}$. For $m \in Z^d$, let M_k be the main block such that $m \in M_k$, $m \notin M_{k+1}$. Let's suppose that $m \in M_k \backslash B_{k-1}$, the treatment of the case $m \in B_{k-1}$ is analogous. Our first goal is to obtain a suitable cluster expansion with respect to the blocks M_k, B_k. For $\mathrm{Im}(z) > 2d$ we have

$$R(0, m; z) = -\sum_{\tau} \prod_{s \in \tau} \frac{1}{V(s) + 2d - z} ,$$

where summation is over all paths τ connecting 0 and m. We will divide set of all such paths into disjoint classes on the following way. Let Γ be graph whose vertices are positive integers and whose bonds connect only successive integers. To every path τ in Z^d connecting 0 and m we correspond a unique path γ on Γ connecting 0 and k on the following way: γ starts at 0 and whenever τ leaves M_i and crosses to $M_{i+1} \backslash M_i$ or $M_{i-1} \backslash M_i$, γ jumps from vertex i to respectively vertex $i + 1$ or $i - 1$. Thus, the set of all paths connecting 0 and m can be divided into disjoint classes, associated to the paths on Γ connecting 0 and k. We denote by γ the class corresponding to γ.

For any j, the boundary ∂B_j can be divided into two parts, the inner boundary

$$\partial_{\mathrm{in}} B_j = \{(n, m) : (n, m) \in \partial B_j, m \in \Gamma_j^- \} ,$$

and outer boundary $\partial_{\mathrm{out}} B_j = \partial B_j \backslash \partial_{\mathrm{in}} B_j$. If path γ is given by $(1, i_1, i_2, \ldots, i_l, k)$ (in fact, one has $i_1 = 2$ unless the path is of length 1), we define a new sequence $(1, j_1, j_2, \ldots, j_l)$ such that

$$j_k = \begin{cases} i_s, & \text{if } i_{s+1} > i_s ; \\ i_{s-1}, & \text{otherwise} , \end{cases}$$

and a sequence of boundaries $L_1 = \partial_{\mathrm{in}} B_1$, $L_1' = \partial_{\mathrm{out}} B_1$,

$$L_{j_s} = \begin{cases} \partial_{\mathrm{in}} B_{i_s}, & \text{if } i_{s+1} > i_s; \\ \partial_{\mathrm{out}} B_{i_{s-1}}, & \text{otherwise} , \end{cases}$$

$$L'_{j_s} = \begin{cases} \partial_{\mathrm{out}} B_{i_s}, & \text{if } i_{s+1} > i_s; \\ \partial_{\mathrm{int}} B_{i_{s-1}} & \text{otherwise} . \end{cases}$$

If

$$R^{(\gamma)}(0, m; z) = -\sum_{\tau \in (\gamma)} \prod_{s \in \tau} \frac{1}{V(s) + 2d - z} ,$$

we can write

$$R^{(\gamma)}(0, m; z) = \sum_{\substack{(t_1, s_1)\in\partial L_1 \\ (t'_1, s'_1)\in\partial L'_1}} \sum_{\substack{(t_{i_j}, s_{i_j})\in\partial L_{i_j} \\ (t'_{i_j}, s'_{i_j})\in\partial L'_{i_j} \\ 1\leqq j\leqq l}} R_{M_1}(0, t_1; z)\cdot R_{B_1}(s_1, s'_1, z) R_{M_{i_1}}(t'_1, t_{i_1}; z) \,.$$

$$\cdot R_{B_{i_1}}(s_{i_1}, s'_{i_1}; z)\cdot \ldots \cdot R_{M_{i_l}}(t'_{i_{l-1}}, t_{i_l}) R_{B_{i_l}}(s_{i_l}, s'_{i_l}) R_{M_k}(t'_{i_l}, m) \,.$$

(If $m = 0$ the term $1/(V(0) + 2d - z)$) should be added.) The strategy of controlling $R^{(\gamma)}$ is now clear: the resolvents of boundary blocks are small and will compensate for the size of resolvents of main blocks. The following lemma, proven in [15], is the main technical ingredient in establishing (2.12). We give proof for readers convenience.

Lemma A.1. *Let M_k be arbitrary main block.*

(i) *For any $L > 0$ and any $n, m \in M_k$,*

$$\mu\{E\in(a, b); |R_{M_k}(n, m; E)| > L\} \leqq C/L \,,$$

where μ is a Lebesgue measure and C a uniform constant.

(ii) *For any $n \in M_k$,*

$$\mu\{E\in(a, b): \sum_{m\in M_k} |R_{M_k}(n, m; E)|^2 > L\} \leqq 4(\# M_k/L)^{1/2} \,.$$

Proof. (i) Let $L_k = \# M_k$. Then

$$R_{M_k}(n, m; E) = \sum_{i=1}^{L_k} \frac{\phi_i(n)\phi_i(m)}{E - E_i} \,,$$

where ϕ_i are eigenfunctions and E_i eigenvalues of H_{M_k}. We can write

$$R_{M_k}(n, m; E) = \int_R \frac{d\nu(\lambda)}{E - \lambda} \,,$$

where measure ν has atoms $\phi_i(n)\phi_i(m)$ at points E_i. Since

$$\sum_{i=1}^{L_k} |\phi_n|\cdot|\phi_m| \leqq \left(\sum_{i=1}^{L_k} |\phi_i(n)|^2\right)^{1/2} \cdot \left(\sum_{i=1}^{L_k} |\phi_i(m)|^2\right)^{1/2}$$

$$\leqq \|\delta_n\|\cdot\|\delta_n\| = 1 \,,$$

we have $|\mu|_{\mathrm{var}} \leqq 1$, and (i) follows from Kolmogorov's theorem (see [15] for an elementary proof due to T. Wolff).

(ii) Let

$$a(E) = \sum_{m\in M_k} |R_{M_k}(n, m; E)|^2 = \sum_{i=1}^{L_k} \frac{|\phi_i(n)|^2}{|E - E_i|^2} \,,$$

and let

$$\Delta_i = \{E\in(a, b): |E - E_i| \leqq |\phi_i(n)|/L^{1/2}\} \,.$$

First, we have that

$$\sum_{i=1}^{L_k} \mu(\Delta_i) = 2L^{-1/2} \sum_{i=1}^{L_k} |\phi_i(n)| \leqq 2\cdot(L_k/L)^{1/2} .$$

On the other hand

$$\begin{aligned}\int_{R\setminus(\bigcup \Delta_i)} a(E)dE &\leqq \sum_{i=1}^{L_k} \int_{(|\phi_i(n)|/L)^{1/2}} \frac{|\phi_i(n)|^2}{E^2} dE \\ &\leqq \sum_{i=1}^{L_k} 2L^{1/2}|\phi_i(n)| \\ &\leqq 2\cdot(L\cdot L_k)^{1/2} .\end{aligned}$$

Chebyshev's inequality yields

$$\mu\{E\in R\setminus(\bigcup \Delta_i): a(E) > L\} \leqq \frac{1}{L} \int_{R\setminus(\bigcup \Delta_i)} a(E)dE = 2(L_k/L)^{1/2}$$

and result follows.

We now finish the proof of Theorem 2.2 as follows. Equation (2.11) implies that for $E\in(a, b)$ and for j large enough,

$$|R_{B_j}(n, m; E)| \leqq CD_j^{-(d-1)} j^{-(1+\delta)} , \tag{A.1}$$

where C is a uniform constant. Without loss of generality we can assume that (A.1) holds for $j \geqq 1$. The part (i) of Lemma A.1 yields that for $\varepsilon > 0$,

$$\sum_j \mu\{E\in(a, b): \max_{n, m\in M_j} |R_{M_j}(n, m; E)| > \varepsilon j^{1+\delta}\} \leqq (C/\varepsilon) \sum j^{-(1+\delta)} < \infty .$$

Thus, for a.e. $E\in(a, b)$ and for $j \geqq j(E)$,

$$\max_{n, m\in M_j} |R_{M_j}(n, m; E)| \leqq \varepsilon j^{1+\delta} .$$

Without loss of generality we can assume that $j(E) = 1$.

We now have that for any $\varepsilon > 0$

$$|R^{(\gamma)}(0, m; E + i\varepsilon)| \leqq |C\varepsilon|^{|\gamma|} \sum_{(t,s)\in\partial L'_{j_s}} |R_{M_k}(t, m; E)| ,$$

where C is a uniform constant and $|\gamma|$ is the length of path γ. Thus,

$$|R(0, m; E + i\varepsilon)|^2 \leqq \sum_\gamma \#\partial B_k |C\varepsilon|^{|\gamma|} \sum_{(t,s)\in\partial L'_{j_s}} |R_{M_k}(t, m; E)|^2 , \tag{A.2}$$

where summation is over all paths γ on Γ connecting 0 and k. The inequality (A.2) is derived under the condition that $m\in M_k\setminus B_{k-1}$. When $m\in B_{k-1}$, arguing on an analogous way, one can derive (A.2) (some care is required since path can connect 0 and m without leaving M_{k-1}, but with right conventions such ambiguities are easily resolved). If $C\varepsilon < 1/2$ it is elementary to show that

$$\sum_\gamma (C\varepsilon)^{|\gamma|} \leqq C_1(C\varepsilon)^k ,$$

where C_1 is uniform constant. Thus

$$\sum_{m\in M_k} |R(0, m; E + i\varepsilon)|^2 \leqq \#\partial B_k C_1^2 (C\varepsilon)^{2k} \sum_{m\in M_k} \sum_{(t,s)\in\partial L'_{j_s}} R_{M_k}(t, m; E)$$
$$\leqq (\#\partial B_k)^2 C_1^2 (C\varepsilon)^{2k} \max_t \sum_{m\in M_k} |R_{M_k}(t, m; E)|^2 .$$

Since $\# M_j \leqq CD_j^d$ for uniform C, part (ii) of Lemma A.1 yields

$$\mu\{E\in(a, b)\colon \max_t \sum_{m\in M_j} |R_{M_j}(t, m; E)|^2 > D_j^d j^{2(1+\delta)}\} \leqq 4/j^{1+\delta} ,$$

and thus for a.e. $E\in(a, b)$ and for $j \geqq j(E)$,

$$\max_{t\in M_j} \sum_{m\in M_j} |R_{M_j}(t, m; E)|^2 \leqq D_j^d j^{2(1+\delta)} .$$

Again, without loss of generality we can assume $j(E) = 1$. Then, using that $D_k \leqq c^k$, we can derive that for a.e. $E\in(a, b)$ and ε small enough

$$\sum_{m\in M_k} |R(0, m; E + i\varepsilon)|^2 \leqq C_1^2 (C_2\varepsilon)^{2k} ,$$

where C_1, C_2 are uniform constants. Choosing ε small enough, we obtain that for a.e. $E\in(a, b)$ there exists α, $0 < \alpha < 1$, such that

$$\sum_{m\in M_k} |R(0, m; E + i\varepsilon)|^2 \leqq \alpha^{2k} .$$

Consequently, for ε small enough

$$\sum_{m\in Z^d} |R(0, m; E + i\varepsilon)|^2 \leqq \sum_k \sum_{m\in M_k} |R(0, m; E + i\varepsilon)|^2 \leqq \sum_k \alpha^{2k} < \infty ,$$

and result follows.

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