CYCLIC VECTORS IN THE ANDERSON MODEL

To Elliott Lieb on his 60th birthday

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We prove that for Anderson models in the localized regime, the vectors δ_n are cyclic with probability one and, in particular, the spectrum is simple.

1. Introduction

In this paper, we'll discuss the Anderson model, by which we mean a family of operators h_ω on $\ell^2(\mathbb{Z}^\nu)$ given by

$$(h_\omega u)(j) = \sum_{|n|=1} u(j+n) + V_\omega(j)u(j) ,$$

where $|n| = \sum_{i=1}^{\nu} |n_i|$ and $V_\omega(j)$ are independent, identically distributed random variables with distribution on $d\lambda(E) = F(E)\,dE$.

Let δ_n be the vector in $\ell^2(\mathbb{Z}^\nu)$ with $\delta_n(j) = 0$ if $j \neq n$ and 1 if $j = n$.

Our goal is to prove:

Theorem 1. *Suppose that* $[a,b]$ *is an interval in* $\text{spec}(h_\omega)$ *for a.e.* ω *and that the spectrum is pure point there (with probability one). Then for a.e.* ω*, each* δ_n *is a cyclic vector for* h_ω *on this interval.*

Such a result is often technically useful, which is our interest in it. It implies that h_ω has simple spectrum. The most interesting question is whether this result holds without assuming that states are localized.

If $\nu = 1$, we know that the localization hypothesis holds for all F and all a, b [2, 3, 6, 5]. For $\nu > 1$, we know that for suitable F there are localized states for all $[a,b]$ and for other F at some $[a,b]$ [4, 1].

This paper is a birthday present for Elliott Lieb. We've written over a dozen papers together, many among my significant ones. Working with Elliott is always a stimulating and rewarding experience and I thank him for the pleasure of those collaborations.

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2. Proof of the Main Theorem

Lemma 2.1. *For any Jacobi matrix, h, the vectors $\{h^j\delta_n\}_{j=0;n \text{ fixed}}^{\infty}$ are linearly independent.*

Proof. Suppose not. Let $\sum_{j=0}^{J} \alpha_j h^j \delta_n = 0$ with J minimal. Let e be a root of $\sum \alpha_j X^j$ so $\sum \alpha_j h^j = (h-e)Q(h)$. Since J is minimal, $\psi = Q(h)\delta_n \neq 0$. But $(h-e)\psi = 0$ so h has an eigenfunction of compact support. Let $C = \{m \mid \psi(m) \neq 0\}$. Let $M = \max\{m_1 \mid m \in C\}$. Let $\tilde{m}$ be a point with $\tilde{m}_1 = M$. Then $(h\psi)(\tilde{m}+\delta_1) \neq 0$ but $\psi(\tilde{m}+\delta_1) = 0$ so $(h-e)\psi \neq 0$; that is, h has no eigenfunction of compact support. Thus, we have the required contradiction. □

Given h and n, define $\eta_m(h,n)$ to be the orthonormal set given by Gram-Schmidt from $h^j\delta_n$, that is

$$\eta_m(h,n) = \left[h^m\delta_n - \sum_{j=0}^{m=1} (\eta_j(h,n), h^m\delta_n)\eta_j\right] / N ,$$

where $N = \left\| h^m\delta_n - \sum_{j=0}^{m-1} \cdots \right\|$. To prove the theorem, we need only show that for each m, k and a.e. ω:

$$\sum_{j=0}^{\infty} |(P_{(a,b)}(h_\omega)\delta_m, \eta_j(h_\omega, k))|^2 = 1$$

(this is clearly a measurable set).

Fix $\tilde{\omega}$ so that $h_{\tilde{\omega}}$ has point spectrum on (a,b) and so that the same is true for $\omega^\lambda = \tilde{\omega}_0 + \lambda\delta_m$ for a.e. λ with $F(\tilde{\omega}_m + \lambda) \neq 0$. Let $d\mu_\lambda$ be the spectral measure of the vector δ_m and operator h_{ω^λ}.

Let $G_{ij}(e) = (\delta_i, (h_{\tilde{\omega}} - e)^{-1}\delta_j)$. As $e \to -\infty$,

$$G_{ij}(e) = (-e)^{|i-j|}(1 + 0(|e|^{-1})) ,$$

so G is not identically zero. G is the Stieltjes transform of a signed measure so it has boundary values for a.e. $e \in \mathbb{R}$, $G_{ij}(e+i0)$. Moreover, since G is not identically zero,

$$A_{ij} = \{e \mid G_{ij}(e+i0) = 0 \text{ or } \lim G_{ij}(e+i\epsilon) = \infty \text{ or the limit does not exist}\}$$

has Lebesgue measure zero.

By the argument of Simon-Wolff [6], $\int (1+\lambda^2)^{-1}\, d\mu_\lambda(E)\, d\lambda \equiv d\eta(E)$ is absolutely continuous w.r.t. dE. Thus, for a.e. λ:

$$d\mu_\lambda(A_{m,k} \cup B) = 0 \qquad (1)$$

where

$$B = \{\text{eigenvalues of } h_{\tilde{\omega}}\} .$$

If we also restrict to those λ with $d\mu_\lambda$ pure point on (a,b), $d\mu_\lambda(B) = 0$ implies each eigenvalue e of h_{ω^λ} with $(\delta_m, P_e\delta_m) \neq 0$ has $\dim(P_e) = 1$; so if we also prove that $(\delta_k, P_e\delta_k) \neq 0$, then $\operatorname{Ran} P_e$ is in the cyclic subspace generated by h_{ω^λ} on δ_k and thus $P_{(a,b)}\delta_m$ is in that cyclic subspace also, as was to be proven.

Now,

$$(\delta_k, (h_{\omega^\lambda} - z)^{-1}\delta_m) = (\delta_k, (h_\omega - z)^{-1}\delta_m) - \lambda(\delta_k, (h_\omega - z)^{-1}\delta_m)(\delta_m(h_{\omega^\lambda} - z)^{-1}\delta_k)\ . \tag{2}$$

Let e be any pure point of $d\mu_\lambda$ on (a,b). By (1), $(\delta_k, (h_\omega - z)^{-1}\delta_m)$ with $z = e + i\epsilon$ has a non-zero finite limit as $\epsilon \downarrow 0$. Multiplying (2) by $i\epsilon$ and taking $\epsilon \downarrow 0$,

$$(\delta_k, P_e\delta_m) = -\lambda G_{km}(e + i0)(\delta_m, P_e\delta_m)\ .$$

It follows that $(\delta_k, P_e\delta_m) \neq 0$ so $P_e\delta_k \neq 0$, as was to be proven. □

References

[1] M. Aizenman and S. Molchanov, "Localization at large disorders and at extreme energies: An elementary derivation", *Commun. Math. Phys.* **157** (1993) 245–278.

[2] F. Delyon, H. Kunz and B. Souillard, "One-dimensional wave equations in disordered media", *J. Phys.* **A16** (1983) 25–42.

[3] F. Delyon, Y. Lévy and B. Souillard, "Anderson localization for multi-dimensional systems at large disorder or large energy", *Commun. Math. Phys.* **100** (1985) 463–470.

[4] J. Fröhlich and T. Spencer, "Absence of diffusion in the Anderson tight binding model for large disorder or low energy", *Commun. Math. Phys.* **88** (1983) 151–184.

[5] S. Kotani and B. Simon, "Localization in general one-dimensional random systems, II. Continuum Schrödinger operators", *Commun. Math. Phys.* **112** (1987) 103–119.

[6] B. Simon and T. Wolff, "Singular continuous spectrum under rank one perturbations and localization for random Hamiltonians", *Commun. Pure Appl. Math.* **39** (1986) 75–90.