Essential Self-Adjointness of Schrödinger Operators with Positive Potentials

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§ 1. Introduction

It is an amusing mathematical game to study questions of the essential self-adjointness of $-A + V$ despite the fact that for most of the Schrödinger operators of physical interest, the necessary result has been known for twenty years [12] and for the others for almost ten years [31, 7].

Let us first consider a one body system in an external potential where $V$ is bounded at $\infty$ in the sense that $V \in L^p + L^\infty$ for some $p$. The best general self-adjointness result for such potentials seems to be:

**Theorem** (Nelson-Faris). Let

(a) $V \in L^p(\mathbb{R}^m) + L^\infty(\mathbb{R}^m)$, $m \leq 3$,
(b) $V \in L^p(\mathbb{R}^d) + L^\infty(\mathbb{R}^d)$, $p > 2$,
(c) $V \in L^{m/2}(\mathbb{R}^m) + L^\infty(\mathbb{R}^m)$, $m \geq 5$.

Then $-A + V$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^m)$ as an operator on $L^2(\mathbb{R}^m)$.

**Remarks.** 1. (a) is the famous result of Kato [12]; (b) and (c) for $L^p$ with $p > m/2$ are due to Nelson [18] although in a slightly different form they appear in Stummel [31] and Brownell [1].

2. The $L^{m/2}$ result for $m \geq 5$ seems to have first been noted by Faris [5]; see also Jörgens [10] and Müller-Pfeiffer [16].

If we allow $V$'s of arbitrary sign, the conditions of the Nelson-Faris theorem cannot be much improved (in case $m \geq 4$) for $-A - cr^{-2}$ is not essentially self-adjoint on $C_0^\infty(\mathbb{R}^m)$ for $c$ large, and $r^{-2} \in L^p(\mathbb{R}^m) + L^\infty(\mathbb{R}^m)$ for any $p < m/2$.

On the other hand, if $V$ is positive we expect that we may be able to do better. First, we know that if $V \in L^1 + L^\infty$, $V \geq 0$, then $-A + V$ can be defined as a form sum on $C_0^\infty$ which is closable and so associated with a self-adjoint operator. (Form techniques are described in [20, 21].) Second, in case $V$ is non locally $L^p$ ($p > n/2$) on a small set and positive, one can sometime prove self-adjointness results, for example:

**Example.** ([9]; see also [32, 11, 23]).

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Let $V = V_1 + V_2$ where $V_2 \in L^p(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$; $p > m/2$, $m \geq 4$. Let $V_1 \in L^2(\mathbb{R}^n)$, with $V_1$ locally $L^p (p > m/2)$ away from 0. Suppose $V_1 \geq 0$. Then $-A + V$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$.

One thus expects that one should be able to handle a fairly large class of positive $L^2$ potentials, in fact:

**Theorem 1.** Let $V \in L^2(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$ be bounded from below. Then $-A + V$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$.

This is the main result of this paper. It has an $n$-body analog:

**Theorem 2.** Let $\{V_i\}_{i=1}^n$ and $\{V_{ij}\}_{1 < j \leq n}$ be in $L^2(\mathbb{R}^n) + L^\infty(\mathbb{R}^n)$ and bounded from below. Then $- \sum_{i=1}^n A_i + \sum_{i \neq j} V_{ij} (r_i - r_j)$ as an operator on $L^2(\mathbb{R}^n)$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^n)$.

**Remark.** If the $v_i$, $V_i \in L^2 + L^p$, $p < n$, one can use Cook's method to construct wave operators [3].

The techniques we use to prove Theorems 1 and 2 were originally developed to study two dimensional self-coupled Bose field models [6]. It is something of a joke that methods invented to treat relativistic models which are unphysical because space has too small a dimension turn out to be useful to treat non-relativistic models which are unphysical because space has too big a dimension.

§ 2. The Tools

There are four main tools we need to prove Theorems 1 and 2:

(i) The Abstract Theory of Hypercontractive Semigroups. This theory is an abstraction of some techniques in constructive quantum field theory [6] originating with Nelson [19]. The abstract self-adjointness theorem (see Lemma A.4) is due to Segal [24] with refinements by Simon-Hoegh Krohn [27]. An operator $H_0$ on $L^2(M, d\mu)$ where $M$ is a measure space of total mass 1 is said to generate a hypercontractive semigroup if $H_0$ is self-adjoint and $e^{-tH_0}$ obeys estimates $\|e^{-tH_0}v\|_p \leq \|v\|_p$ all $t > 0$, all $1 \leq p \leq \infty$ and all $v \in L^p \cap L^2$; $\|e^{-tH_0}v\|_4 \leq \|v\|_2$ all $t > T$ for some large $T$. Under various conditions on $V$, e.g. if $V \in L^2(M, d\mu)$ and $V$ is bounded from below, $H_0 + V$ has been proven essentially self-adjoint on $D(H_0) \cap D(V)$. In an appendix we extend this theorem to show essential self-adjointness on $C_0^\infty(H_0) \cap D(V)$ (Theorem A.1).

(ii) Hermite Operators. $(\mathbb{R}^n, \pi^{-m/2} e^{-\xi^2} d^n x)$ is a measure space of total mass 1. Consider the map $U : L^2(\mathbb{R}^n, d^n x) \to L^2(\mathbb{R}^n, d\mu)$ where $d\mu$ is the above Gaussian measure; $(Uf)(x) = \pi^{m/4} e^{\xi^2/2} f(x)$. $U$ has the following properties:

(a) $U$ is unitary,
(b) $UVU^{-1} = V$ when $V$ is any multiplication operator,
(c) $U(-A + x^2 - 1)U^{-1} = -A + 2x \cdot V$. 

This latter operator \(-\Delta + 2x \cdot V\) has Hermite polynomials as eigenfunctions and is sometimes called a Hermite operator. It has one important property:

**Theorem 3**  Nelson [20]). \(-\Delta + 2x \cdot V\) as an operator on \(L^2(\mathbb{R}^n, d\mu)\) is the generator of a hypercontractive semigroup.

Proofs of this fact are also discussed in several other references, e.g. [24, 27].

(iii) Konrad’s Trick and the Kato-Wüst Theorem. Konrad [14] has introduced a simple trick that sometimes works in proving \(A + B\) essentially self-adjoint on some set. First choose a third operator \(C\) so that \(A + C\) is self-adjoint and so that \(A + C + B\) is self-adjoint (e.g. if \(B\) is a Kato small perturbation of \(A^2\), one might take \(C = A^2\)). Then one tries to prove

\[
\|C\psi\|^2 \leq \|(A + C + B)\psi\|^2 + b^2\|\psi\|^2
\]

for some \(b\) and all \(\psi\) in a core for \(A + C + B\). It then follows that \(A + B = A + C + B - C\) is essentially self-adjoint on any core for \(A + C + B\) for one has the following theorem of Kato [13] (recently strengthened by Wüst [33], although we don’t need the strengthened version):

**Theorem 4.** If \(X\) is self-adjoint and \(Y\) is symmetric, and for some \(b\) and all \(\psi\) in a core for \(X\):

\[
\|Y\psi\|^2 \leq \|X\psi\|^2 + b^2\|\psi\|^2
\]

then \(X + Y\) is essentially self-adjoint on any core for \(X\).

(iv) The Double Commutator Trick. To apply Theorem 4 in the case where \(X = A + C + B\) and \(Y = -C\), one needs an operator inequality

\[
C^2 \leq (A + B + C)^2 + b^2 \text{ or equivalently } 0 \leq (A + B)^2 + C(A + B) + (A + B) C + b^2.
\]

If \(C\) and \(A + B\) are positive, Jaffe [8] has noted that the identity

\[
C(A + B) + (A + B)C = 2C(A + B)C + [C, [C, (A + B)]]
\]

is often useful.

§3. Proof of Theorems 1 and 2

We first note that it is easy to show that \(\mathcal{S} \subset D(\mathcal{C} \circ \mathcal{C}^\mathcal{D})\) so we need only prove \(-\Delta + V\) is essentially self-adjoint on \(\mathcal{S}\). Let \(H_0 = -\Delta + x^2 - 1\). Then \(C^\mathcal{D}(H_0) = \mathcal{S}\) (see e.g. [25]). Thus \(C^\mathcal{D}(H_0) \subset D(V)\) so it is sufficient to prove \(-\Delta + V\) essentially self-adjoint on \(C^\mathcal{D}(H_0) \cap D(V) = C^\mathcal{D}(H_0)\). Without loss, we may suppose \(V \geq 0\). Let us first consider \(H_0 + V\). Let \(U : L^2(\mathbb{R}^n, dx) \to L^2(\mathbb{R}^n, \pi^{-m/2} e^{-x^2} dx)\) as in §2. Let \(H_0 = U H_0 U^{-1}\). By Theorem 3, \(H_0\) generates a hypercontractive semigroup. Moreover, \(V = U V U^{-1}\) is in \(L^2(\mathbb{R}^n, \pi^{-m/2} e^{-x^2} dx)\) and if
\[ V = \sum_i V_i + \sum_{i,j} V_{ij}, \quad V \in L^2(\mathbb{R}^m, \pi^{-m/2} e^{-x^2} \, dx) \], so by Theorem A.1, \( H_0 + V \) is essentially self-adjoint on \( C^\infty(\mathbb{R}^m) \cap D(V) \). Thus \( H_0 + V + 1 \) is essentially self-adjoint on \( C^\infty(\mathbb{R}^m) \cap D(V) = \mathcal{S} \).

Next, let us prove that

\[ x^2 \leq (H_0 + V + 1)^2 + 2m. \]

First note, that (as forms on \( \mathcal{S} \times \mathcal{S} \)):

\[
(1 + H_0 + V)^2 = (-A + V)^2 + x^4 + x^2(-A + V) + (-A + V)x^2
\]

\[ = (-A + V)^2 + x^4 + 2 \sum_{i=1}^m x_i(-A + V)x_i + 2 \sum_{i=1}^m [x_i, x_i, (-A + V)] \cdot x^4. \]

Since \((-A + V)^2\) and \(-A + V\) are positive and \([x_i, x_i, (-A + V)]\) is the commutator of \(-A + V\),

Thus, for any \( \psi \in \mathcal{S} \),

\[ \|x^2 \psi\|^2 \leq \| (H_0 + V + 1) \psi \|^2 + 2m \| \psi \|^2. \]

Since \( \mathcal{S} \) is a core for \( H_0 + V \), Theorem 4 tells us that \( \mathcal{S} \) is a core for \(-A + V = H_0 + V - x^2 + 1\).

\section*{4. Some Conjectures}

It is clearly not necessary for \( V \) to be bounded at \( \infty \) for our methods to imply \(-A + V\) is essentially self-adjoint. It is enough that \( V \) be bounded by some \( e^{\epsilon x^2} \) in the sense that \( \int V(x)^2 e^{-2x^2} \, dx < \infty \). This, of course, suggests:

**Conjecture 1.** If \( V > 0 \) and \( V \in L^2(\mathbb{R}^m)_{\text{loc}} \), then \(-A + V\) is essentially self-adjoint on \( C_0^\infty(\mathbb{R}^m) \).

By working hard, our methods could probably handle \(-A + V\) where \( V = V_+ - V_- \), \( V_+ , V_- \geq 0 \) with \( V_+ \in L^p + L^\infty \), \( V_- \in (L^q)_{\text{comp}} \) for some \( p > 2 \) and \( q > m/2 \) (here \((L^q)_{\text{comp}}\) means the \( L^q \) functions of compact support). However, we suspect:

**Conjecture 2.** If \( V = V_+ - V_- ; V_+ , V_- \geq 0 \); \( V_+ \in L^2(\mathbb{R}^m)_{\text{loc}} \) and \( V_- \in L^p(\mathbb{R}^m) + L^q(\mathbb{R}^m) \), \( p > m/2 \), then \(-A + V\) is essentially self-adjoint on \( C_0^\infty(\mathbb{R}^m) \).

The strongest conjecture suggested by this line of reasoning is:

**Conjecture 3.** If \( V \in L^2(\mathbb{R}^m)_{\text{loc}} \) and \(-A + V\) is bounded from below on \( C_0^\infty(\mathbb{R}^m) \), then \(-A + V\) is essentially self-adjoint on \( C_0^\infty(\mathbb{R}^m) \).

In case, \( m \leq 3 \), Conjecture 3 is a result of Stetkaer-Hansen [29]; however if \( m \geq 5 \) Conjecture 3 is false as we show by counterexample in Appendix 2.
Unfortunately the tricks we have used to prove Theorems 1 and 2 will not prove Conjecture 2. Perhaps their most interesting “application” is that they suggest Conjecture 2 is true and that they will therefore motivate someone to prove it!

Appendix 1

Some New Results in the Theory of Hypercontractive Semigroups

Segal’s theorem [24] says that under certain circumstances \( H_0 + V \) is essentially self-adjoint on \( D(H_0) \cap D(V) \). We wish to show here that under those circumstances, \( H_0 + V \) is essentially self-adjoint on \( C^\omega(H_0) \cap D(V) \).

We recall that a hypercontractive semigroup [27] is a family \( e^{-tH_0} \) generated by a self-adjoint operator, \( H_0 \), on \( L^2(M, d\mu) \) for some probability measure space \( (M, d\mu) \), so that

(i) \( e^{-tH_0} \) is a contraction on each \( L^p \) (\( 1 \leq p \leq \infty \); \( 0 \leq t \)).

(ii) \( e^{-tH_0} \) is bounded from \( L^2 \) to \( L^q \) for some \( T > 0 \).

We recall that interpolation theorems imply [24, 27]:

(ii') For any \( 1 < r, s < \infty \), \( e^{-tH_0} \) is bounded from \( L^r \) to \( L^s \) for \( t \) sufficiently large.

The first step in our improvement is to note (all lemmas and the theorem suppose \( e^{-tH_0} \) is hypercontractive):

**Lemma A.1.** Let \( 1 \leq p \leq 2 \). Then \( \{e^{-tH_0}\}_{t \geq 0} \) is strongly continuous on \( [0, \infty) \) and if \( 2 \leq p > 1 \), \( \text{Ker} (e^{-rH_0} \uparrow L^p) = \{0\} \).

**Proof.** Since the maps \( e^{-tH_0} \uparrow L^p \) are uniformly bounded, it is sufficient to prove \( e^{-tH_0} \psi \) is strongly continuous for a dense set of \( \psi \). Since \( p < 2 \), \( L^2 \) is dense in \( L^p \) and continuity of \( e^{-tH_0} \psi \) in \( L^2 \) implies \( L^p \) continuity. Fix \( 2 > p > 1 \). By the Stein interpolation theorem (interpolation in arg \( z \) [28], \( e^{-zH_0} \) is bounded from \( L^p \) to \( L^q \) for \( z \) in a sectorial neighborhood of \( (0, \infty) \). Since \( e^{-zH_0} \psi \) is \( L^2 \) and hence \( L^p \) analytic when \( \psi \in L^2 \), \( e^{-zH_0} \psi \) is \( L^p \)-analytic in a neighborhood of \( (0, \infty) \) for any \( \psi \in L^p \). Now suppose \( e^{-tH_0} \psi = 0 \) for some \( \psi \in L^p \) and some \( t \in (0, \infty) \). By the semigroup property, \( e^{-zH_0} \psi = 0 \) for \( s \geq t \). By the analyticity, \( e^{-zH_0} \psi = 0 \) for all \( s \in (0, \infty) \). Finally, by strong continuity, \( \psi = s \lim_{z \to 0} e^{-zH_0} \psi = 0 \). ☐

**Lemma A.2.** Let \( 2 \leq p < \infty \). Then \( \{e^{-tH_0}\}_{t \geq 0} \) is strongly continuous on \( [0, \infty) \).

**Remarks.** 1. Even without hypothesis (ii), one can prove strong continuity on each \( L^p \) by using the analyticity technique used in Lemma A.1.

2. \( e^{-tH_0} \) may not be strongly continuous on \( L^\infty \). For example, if \( H_0 = -d^2/dx + 2x d/dx \) on \( (\mathbb{R}, \pi^{-\frac{1}{2}} e^{-x^2} d\mu) \), \( e^{-tH_0} [L^\infty] \subset C^\omega(H_0) \)
\[ \{ e^{x^2/2} f | f \in \mathcal{S} \} \] contains only continuous functions, so \( e^{-itH_0} \psi \) does not converge in \( L^\infty \) if \( \psi \) is \( L^\infty \) but discontinuous.

**Proof.** By uniform boundedness, we need only prove \( e^{-itH_0} \psi \) is \( L^p \)-continuous for a dense subset of \( \psi \). Pick \( T \) so that \( e^{-iT H_0} \) is bounded from \( L^2 \) to \( L^p \). By Lemma A.1, \( \text{Ker}(e^{-iT H_0} \uparrow L^p) = \{ 0 \} \) where \( q^{-1} + p^{-1} = 1 \). As a consequence, \( \text{Ran}(e^{-iT H_0} \uparrow L^p) \) is dense in \( L^p \). A fortiori, \( e^{-iT H_0}[L^2] \) is dense in \( L^p \). But if \( \psi \in L^2 \), \( e^{-iT H_0} \psi \) is \( L^2 \)-continuous so \( e^{-iT H_0}(e^{-iT H_0} \psi) = e^{-iT H_0}(e^{-iT H_0} \psi) \) is \( L^p \)-continuous. \( \square \)

In case \( p = \infty \), \( e^{-itH_0} \) may not be strongly continuous but we have the weaker:

**Lemma A.3.** If \( \psi \in L^\infty \) and \( V \in L^p \) \((p < \infty)\) then

\[ V e^{-itH_0} \psi \xrightarrow{L^p} V \psi \quad \text{as} \quad t \to 0. \]

**Proof.** First suppose \( V \in L^\infty \). Then the result follows from Lemma A.2. If \( V \in L^2 \), choose \( V_n \in L^2 \) with \( V_n \to V \) in \( L^p \). Since \( \{ e^{-iT H_0} \psi \} \) is \( L^2 \) bounded a simple argument proves that \( \forall \psi \) bounded \( \forall \psi \to 0 \) in \( L^p \).

Finally we need the essential self-adjointness of \( H_0 + V \) in the following form proven in [24] and [27]:

**Lemma A.4.** (a) Let \( V \in L^p \) for some \( p > 2 \), \( e^{-iV} \in L^1 \) for all \( t > 0 \). Then \( H_0 + V \) is essentially self-adjoint on \( L^2 \cap D(H_0) \cap D(V) \) for any \( q < \infty \).

(b) Let \( V \in L^2 \); \( V \) bounded from below. Then \( H_0 + V \) is essentially self-adjoint on \( L^2 \cap D(H_0) \cap D(V) \).

**Remark.** The symbols \( D(H_0) \), \( D(V) \), \( C^\infty(H_0) \) always refer to \( L^2 \)-domain.

We thus conclude:

**Theorem A.1.** Under either hypothesis (a) or (b) of Lemma A.4, \( H_0 + V \) is essentially self-adjoint on \( C^\infty(H_0) \cap D(V) \).

**Remarks.** 1. In both the field theory case and the case we discuss here \( C^\infty(H_0) \subset D(V) \) but this need not be so \((\text{e.g. } H_0 = -\frac{d^2}{dx^2} + 2x \frac{d}{dx}\) on \((\mathbb{R}, \pi^{-\frac{1}{2}} e^{-x^2} dx)\); \( V = e^{ix} \)).

2. Actually, we will prove essential self-adjointness on \( \mathfrak{A}(H_0) \cap D(V) \) where \( \mathfrak{A}(H_0) \) is the set of analytic vectors for \( H_0 \) [17],

3. In the \( D(\phi_2) \) field theory, this essential self-adjointness is a result of Rosen [22], who used "higher order estimates". It is perhaps a little surprising that these H.O.E. are not needed for the result.

**Proof.** Since \( H_0 + V \) is symmetric on \( C^\infty(H_0) \cap D(V) \), we need only find a core inside \( C^\infty(H_0) \cap D(V) \). We consider case (b); case (a) is proved similarly using Lemma A.2 in place of Lemma A.3. By Lemma A.4, we need only find for any \( \psi \in L^p \cap D(H_0) \cap D(V) \), a sequence \( \psi_n \in C^\infty(H_0) \cap D(V) \)
with \( \psi_n \xrightarrow{L^2} \psi \), \( V \psi_n \xrightarrow{L^2} V \psi \) and \( H_0 \psi_n \xrightarrow{L^2} H_0 \psi \). Let \( \psi_n = e^{-H_0/\alpha} \psi \in C^0(H_0) \cap L^p \subseteq C^0(H_0) \cap D(V) \). That \( \psi_n \xrightarrow{L^2} \psi \) and \( H_0 \psi_n \xrightarrow{L^2} H_0 \psi \) follow from strong continuity on \( L^2 \). That \( V \psi_n \xrightarrow{L^2} V \psi \) follows from Lemma A.3. \( \square \)

Appendix 2

A Counterexample

Let \( m \geq 5 \) and let \( H_0 = -\Delta \) on \( L^2(\mathbb{R}^m, dx) \). Let \( V = -r^{-2} \). Then:

(a) \( V \in (L^2)_{\text{loc}} \).

(b) \( V \) is \( H_0 \) - bounded.

(c) \( H_0 + \alpha V \) is bounded from below on \( D(H_0) \) if and only if \( \alpha \leq \frac{1}{2}(m - 1)(m - 3) + \frac{1}{2} \).

(d) \( H_0 + \alpha V \) is essentially self-adjoint on \( C_0^\infty(\mathbb{R}^m) \) if and only if \( \alpha \leq \frac{1}{2}(m - 1)(m - 3) - \frac{1}{2} \).

(a) is trivial. Since \( V \) is in weak \( L^{\infty} \) (i.e. \( v \{x \| V(x) > t \} \leq C t^{-m/2} \)) where \( v \) is Lebesgue measure, (b) follows from a result of Strichartz [30]. Alternatively, (b) can be proven from an inequality of Rellich discussed in [23]. To prove (c) and (d), notice first that, by (b), if \( D \) is any core for \( H_0 \), then the closure of \( H_0 + V \) is the same as the closure of \( H_0 + V \) (defined on \( D(H_0) \)). Moreover since \( m \geq 5 \), \( C_0^\infty(\mathbb{R}^m \setminus \{0\}) \) is easily seen to be a core for \( H_0 \) (this fact remains true even if \( m = 4 \), but it is much harder to prove in that case). Thus we need only prove (c) [respectively (d)] when \( D(H_0) \) [respectively \( C_0^\infty(\mathbb{R}^m) \)] is replaced by \( C_0^\infty(\mathbb{R}^m \setminus \{0\}) \).

Let \( f_{\mathfrak{m}l} \) be a complete set of eigenfunctions for the Laplace-Beltrami operator, \( A \), on the \( m-1 \) sphere, so \( Af_{\mathfrak{m}l} = a_{\mathfrak{m}l} f_{\mathfrak{m}l} \). Here \( a_{m0} = 0 \) and \( a_{ml} \geq 0 \) for all \( l \). Let \( \mathfrak{H}_{\mathfrak{m}l} = \{f(r) e^{-1/2} f_{\mathfrak{m}l} \mid f \in L^2(0, \infty) \} \). Then \( H_0 \) and \( V \) leave \( \mathfrak{H}_{\mathfrak{m}l} \) invariant and under the natural equivalence of \( \mathfrak{H}_{\mathfrak{m}l} \) and \( L^2(0, \infty), H_0 \) acts as

\[
\tilde{H}_0; \; ml = -\frac{d^2}{dr^2} + a_{\mathfrak{m}l} r^{-2}
\]

where \( a_{\mathfrak{m}l} = \frac{1}{2}(m - 1)(m - 3) + a_{ml} \). Thus, to prove (c) and (d) we must only prove:

(c') On \( L^2(0, \infty), -\frac{d^2}{dr^2} - \alpha r^{-2} \) is bounded from below if and only if \( \alpha \leq \frac{1}{2} \).

(d') On \( L^2(0, \infty), -\frac{d^2}{dr^2} - \alpha r^{-2} \) is essentially self-adjoint if and only if \( \alpha \leq -\frac{1}{4} \).
(c') is well known (see [4], pp. 446–447). (d') follows from a simple application of the Weyl limit point limit circle method (see [2]). (a)–(d) have a number of negative consequences:

1. "Conjecture" 3 in §4 is false if \( m \geq 5 \) for take \( \alpha = \frac{1}{4} (m - 1)(m - 3) \).

2. It is possible to have an analytic family of type (B) (in the sense of [13]) in a region \( R \) (in this case \( |x| < \frac{1}{4} (m - 1)(m - 3) + \frac{1}{2} \)) which is of type (A) in a strict subregion of \( R \) (in this case excluding \( \frac{1}{4} (m - 1)(m - 3) - \frac{1}{2} < |x| < \frac{1}{4} (m - 1)(m - 3) + \frac{1}{2} \)).

3. One cannot prove essential self-adjointness of \( H_0 + W \) when \( W \) is \( H_0 \)–bounded of relative bound larger than 1, even if one knows that \( aH_0 + V \) is positive for some \( a < 1 \).

Since it may be surprising to some that essential self-adjointness breaks down before semiboundedness, let us make a few remarks about the phenomena. First, let us explain semi-heuristically the mathematics behind the phenomena: Let \( u \) solve the ordinary differential equation, \( u'' = \alpha r^{-2} u \). Then \( u_\pm = r^\beta \pm \) where \( \beta_\pm \) solves \( \beta (\beta - 1) = \alpha \), i.e.

\[
\beta_\pm = \frac{1}{2} (-1 \pm \sqrt{1 + 4 \alpha})
\]

Boundedness from below is related to \( \beta_\pm \) being real (see [15], pp. 120–121 for a heuristic explanation of this) while by the Weyl criterion, essential self-adjointness requires one of \( u_\pm \) to be non-\( L^2 \) at \( r = 0 \), i.e. \( \beta_\pm \leq -\frac{1}{2} \), i.e. \( \alpha \leq \frac{1}{4} \).

From another point of view, this is just the difference between quantum mechanics from a form point of view [26] and from an operator point of view. Essential self-adjointness on \( D(H_0) \cap D(V) \) is a useful technical result but the crucial physical requirement is that \( H_0 + V \) as a sum of forms be closable on \( Q(H_0) \cap Q(V) \) so that the sum of forms defines a self-adjoint operator. Semi-boundedness and this closability break down at the same point. In fact on the more familiar \( \mathbb{R}^d \), this break-down of essential self-adjointness before semiboundedness occurs also; \( -A - r^{-2} \) is self-adjoint on \( D(-A) \) only if \( \beta < \frac{d}{2} \) but it is semibounded if \( \beta < 2 \) and in the range \( \frac{d}{4} < \beta < 2 \), all the usual quantum mechanics can be developed [26].

It is a pleasure to thank P. Chernoff, D. Masson, E. Nelson and L. Rosen for useful conversation or correspondence related to the material of Appendix 2.


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(Received January 24, 1972; in revised form April 20, 1972)