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Essential Self-Adjointness of Schrödinger Operators with Positive Potentials

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§ 1. Introduction

It is an amusing mathematical game to study questions of the essential self-adjointness of $-\Delta + V$ despite the fact that for most of the Schrödinger operators of physical interest, the necessary result has been known for twenty years [12] and for the others for almost ten years [31, 7].

Let us first consider a one body system in an external potential where V is bounded at ∞ in the sense that $V \in L^p + L^\infty$ for some p. The best general self-adjointness result for such potentials seems to be:

Theorem (Nelson-Faris). *Let*
(a) $V \in L^2(\mathbb{R}^m) + L^\infty(\mathbb{R}^m)$, $m \leqq 3$,
(b) $V \in L^p(\mathbb{R}^4) + L^\infty(\mathbb{R}^4)$, $p > 2$,
(c) $V \in L^{m/2}(\mathbb{R}^m) + L^\infty(\mathbb{R}^m)$, $m \geqq 5$.
Then $-\Delta + V$ *is essentially self-adjoint on* $C_0^\infty(\mathbb{R}^m)$ *as an operator on* $L^2(\mathbb{R}^m)$.

Remarks. 1. (a) is the famous result of Kato [12]; (b) and (c) for L^p with $p > m/2$ are due to Nelson [18] although in a slightly different form they appear in Stummel [31] and Brownell [1].

2. The $L^{m/2}$ result for $m \geqq 5$ seems to have first been noted by Faris [5]; see also Jörgens [10] and Müller-Pfeiffer [16].

If we allow V's of arbitrary sign, the conditions of the Nelson-Faris theorem cannot be much improved (in case $m \geqq 4$) for $-\Delta - cr^{-2}$ is not essentially self-adjoint on $C_0^\infty(\mathbb{R}^m)$ for c large, and $r^{-2} \in L^p(\mathbb{R}^m) + L^\infty(\mathbb{R}^m)$ for any $p < m/2$.

On the other hand, if V is positive we expect that we may be able to do better. First, we know that if $V \in L^1 + L^\infty$, $V \geqq 0$, then $-\Delta + V$ can be defined as a form sum on C_0^∞ which is closable and so associated with a self-adjoint operator. (Form techniques are described in [20, 21].) Second, in case V is non locally L^p ($p > n/2$) on a small set and positive, one can sometime prove self-adjointness results, for example:

Example. ([9]; see also [32, 11, 23]).

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Let $V = V_1 + V_2$ where $V_2 \in L^p(\mathbb{R}^m) + L^\infty(\mathbb{R}^m)$; $p > m/2$, $m \geqq 4$. Let $V_1 \in L^2(\mathbb{R}^m)$, with V_1 locally L^p $(p > m/2)$ away from 0. Suppose $V_1 \geqq c r^{-2}$ $(c > 1)$. Then $-\Delta + V$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^m)$.

One thus expects that one should be able to handle a fairly large class of positive L^2 potentials, in fact:

Theorem 1. *Let $V \in L^2(\mathbb{R}^m) + L^\infty(\mathbb{R}^m)$ be bounded from below. Then $-\Delta + V$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^m)$.*

This is the main result of this paper. It has an n-body analog:

Theorem 2. *Let $\{V_i\}_{i=1}^n$ and $\{V_{ij}\}_{i<j=1}^n$ be in $L^2(\mathbb{R}^m) + L^\infty(\mathbb{R}^m)$ and bounded from below. Then $-\sum_{i=1}^n \Delta_i + \sum_i V_i(\mathbf{r}_i) + \sum_{i<j} V_{ij}(\mathbf{r}_i - \mathbf{r}_j)$ as an operator on $L^2(\mathbb{R}^{mn})$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^m)$.*

Remark. If the V_{ij}, $V_i \in L^2 + L^p$, $p < n$, one can use Cook's method to construct wave operators [3].

The techniques we use to prove Theorems 1 and 2 were originally developed to study two dimensional self-coupled Bose field models [6]. It is something of a joke that methods invented to treat relativistic models which are unphysical because space has too small a dimension turn out to be useful to treat non-relativistic models which are unphysical because space has too big a dimension.

§ 2. The Tools

There are four main tools we need to prove Theorems 1 and 2: *(i) The Abstract Theory of Hypercontractive Semigroups.* This theory is an abstraction of some techniques in constructive quantum field theory [6] originating with Nelson [19]. The abstract self-adjointness theorem (see Lemma A.4) is due to Segal [24] with refinements by Simon-Hoegh Krohn [27]. An operator H_0 on $L^2(M, d\mu)$ where M is a measure space of total mass 1 is said to generate a hypercontractive semigroup if H_0 is self-adjoint and e^{-tH_0} obeys estimates $\|e^{-tH_0}\psi\|p \leqq \|\psi\|_p$ all $t > 0$, all $1 \leqq p \leqq \infty$ and all $\psi \in L^p \cap L^2$; $\|e^{-tH_0}\psi\|_4 \leqq \|\psi\|_2$ all $t > T$ for some large T. Under various conditions on V, e.g. if $V \in L^2(M, d\mu)$ and V is bounded from below, $H_0 + V$ has been proven essentially self-adjoint on $D(H_0) \cap D(V)$. In an appendix we extend this theorem to show essential self-adjointness on $C^\infty(H_0) \cap D(V)$ (Theorem A.1).

(ii) Hermite Operators. $(\mathbb{R}^m, \pi^{-m/2} e^{-x^2} d^m x)$ is a measure space of total mass 1. Consider the map $U: L^2(\mathbb{R}^m, d^m x) \to L^2(\mathbb{R}^m, d\mu)$ where $d\mu$ is the above Gaussian measure; $(Uf)(x) = \pi^{m/4} e^{+x^2/2} f(x)$. U has the following properties:

(a) U is unitary,

(b) $UVU^{-1} = V$ when V is any multiplication operator,

(c) $U(-\Delta + x^2 - 1)U^{-1} = -\Delta + 2\mathbf{x} \cdot \boldsymbol{\nabla}$.

This latter operator $-\Delta + 2\boldsymbol{x}\cdot\boldsymbol{V}$ has Hermite polynomials as eigenfunctions and is sometimes called a Hermite operator. It has one important property:

Theorem 3 Nelson [20]). $-\Delta + 2x\cdot\boldsymbol{V}$ *as an operator on* $L^2(\mathbb{R}^m, d\mu)$ *is the generator of a hypercontractive semigroup.*

Proofs of this fact are also discussed in several other references, e.g. [24, 27].

(iii) Konrady's Trick and the Kato-Wüst Theorem · Konrady [14] has introduced a simple trick that sometimes works in proving $A+B$ essentially self-adjoint on some set. First choose a third operator C so that $A+C$ is self-adjoint and so that $A+C+B$ is self-adjoint (e.g. if B is a Kato small perturbation of A^2, one might take $C=A^2$). Then one tries to prove

$$\|C\psi\|^2 \leqq \|(A+C+B)\psi\|^2 + b^2\|\psi\|^2$$

for some b and all ψ in a core for $A+C+B$. It then follows that $A+B=A+C+B-C$ is essentially self-adjoint on any core for $A+C+B$ for one has the following theorem of Kato [13] (recently strengthened by Wüst [33], although we don't need the strengthened version):

Theorem 4. *If X is self-adjoint and Y is symmetric, and for some b and all ψ in a core for X:*

$$\|Y\psi\|^2 \leqq \|X\psi\|^2 + b^2\|\psi\|^2$$

then $X+Y$ is essentially self-adjoint on any core for X.

(iv) The Double Commutator Trick. To apply Theorem 4 in the case where $X=A+C+B$ and $Y=-C$, one needs an operator inequality $C^2 \leqq (A+B+C)^2 + b^2$ or equivalently $0 \leqq (A+B)^2 + C(A+B) + (A+B)C + b^2$. If C and $A+B$ are positive, Jaffe [8] has noted that the identity

$$C(A+B)+(A+B)C = 2C^{\frac{1}{2}}(A+B)C^{\frac{1}{2}} + [C^{\frac{1}{2}}, [C^{\frac{1}{2}}, (A+B)]]$$

is often useful.

§3. Proof of Theorems 1 and 2

We first note that it is easy to show that $\mathscr{S} \subset D(\overline{-\Delta + V \restriction C_0^\infty})$ so we need only prove $-\Delta + V$ is essentially self-adjoint on $\mathscr{S}$. Let $H_0 = -\Delta + x^2 - 1$. Then $C^\infty(H_0) = \mathscr{S}$ (see e.g. [25]). Thus $C^\infty(H_0) \subset D(V)$ so it is sufficient to prove $-\Delta + V$ essentially self-adjoint on $C^\infty(H_0) \cap D(V) = C^\infty(H_0)$. Without loss, we may suppose $V \geqq 0$. Let us first consider $H_0 + V$. Let $U: L^2(\mathbb{R}^m, dx) \to L^2(\mathbb{R}^m, \pi^{-m/2} e^{-x^2} dx)$ as in §2. Let $H_0' = U H_0 U^{-1}$. By Theorem 3, H_0' generates a hypercontractive semigroup. Moreover, $V = U V U^{-1}$ is in $L^2(\mathbb{R}^m, \pi^{-m/2} e^{-x^2} dx)$ and if

$V=\sum_i V_i+\sum_{i,j} V_{ij}$, $V\in L^2(\mathbb{R}^{mn}, \pi^{-mn/2}e^{-x^2}dx)$, so by Theorem A.1, $H_0'+V$ is essentially self-adjoint on $C^\infty(H_0')\cap D(V)$. Thus H_0+V+1 is essentially selfadjoint on $C^\infty(H_0)\cap D(V)=\mathscr{S}$.

Next, let us prove that

$$x^4\leqq (H_0+V+1)^2+2m\,.$$

First note, that (as forms on $\mathscr{S}\times\mathscr{S}$):

$$(1+H_0+V)^2=(-\Delta+V)^2+x^4+x^2(-\Delta+V)+(-\Delta+V)x^2$$
$$=(-\Delta+V)^2+x^4+2\sum_{i=1}^m x_i(-\Delta+V)x_i+\sum_{i=1}^m [x_i,[x_i,(-\Delta+V)]]\,.$$

Since $(-\Delta+V)^2$ and $-\Delta+V$ are positive and $[x_i,[x_i,(-\Delta+V)]]=-2$, $(H_0+1+V)^2+2m\geqq x^4$.

Thus, for any $\psi\in\mathscr{S}$,

$$\|x^2\psi\|^2\leqq\|(H_0+V+1)\psi\|^2+2m\|\psi\|^2\,.$$

Since $\mathscr{S}$ is a core for H_0+V, Theorem 4 tells us that $\mathscr{S}$ is a core for $-\Delta+V=H_0+V-x^2+1$.

§4. Some Conjectures

It is clearly not necessary for V to be bounded at ∞ for our methods to imply $-\Delta+V$ is essentially self-adjoint. It is enough that V be bounded by some $e^{+\alpha x^2}$ in the sense that $\int|V(x)|^2e^{-2\alpha x^2}dx<\infty$. This, of course, suggests:

Conjecture 1. If $V>0$ and $V\in L^2(\mathbb{R}^m)_{\text{loc}}$, then $-\Delta+V$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^m)$.

By working hard, our methods could probably handle $-\Delta+V$ where $V=V_+-V_-(V_+,V_-\geqq 0)$ with $V_+\in L^p+L^\infty$, $V_-\in(L^q)_{\text{comp}}$ for some $p>2$ and $q>m/2$ (here $(L^q)_{\text{comp}}$ means the L^q functions of compact support). However, we suspect:

Conjecture 2. If $V=V_+-V_-$; $V_+,V_-\geqq 0$; $V_+\in L^2(\mathbb{R}^m)_{\text{loc}}$ and $V_-\in L^p(\mathbb{R}^m)+L^\infty(\mathbb{R}^m)$, $p>m/2$, then $-\Delta+V$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^m)$.

The strongest conjecture suggested by this line of reasoning is:

Conjecture 3. If $V\in L^2(\mathbb{R}^m)_{\text{loc}}$ and $-\Delta+V$ is bounded from below on $C_0^\infty(\mathbb{R}^m)$, then $-\Delta+V$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^m)$.

In case, $m\leqq 3$, Conjecture 3 is a result of Stetkaer-Hansen [29]; however *if $m\geqq 5$ Conjecture 3 is false* as we show by counterexample in Appendix 2.

Unfortunately the tricks we have used to prove Theorems 1 and 2 will not prove Conjecture 2. Perhaps their most interesting "application" is that they suggest Conjecture 2 is true and that they will therefore motivate someone to prove it!

Appendix 1

Some New Results in the Theory of Hypercontractive Semigroups

Segal's theorem [24] says that under certain circumstances $H_0 + V$ is essentially self-adjoint on $D(H_0) \cap D(V)$. We wish to show here that under those circumstances, $H_0 + V$ is essentially self-adjoint on $C^\infty(H_0) \cap D(V)$.

We recall that a hypercontractive semigroup [27] is a family e^{-tH_0} generated by a self-adjoint operator, H_0, on $L^2(M, d\mu)$ for some probability measure space $(M, d\mu)$, so that

(i) e^{-tH_0} is a contraction on each L^p $(1 \leqq p \leqq \infty; 0 \leqq t)$.

(ii) e^{-tH_0} is bounded from L^2 to L^4 for some $T > 0$.

We recall that interpolation theorems imply [24, 27]:

(ii′) For any $1 < r, s < \infty$, e^{-tH_0} is bounded from L^r to L^s for t sufficiently large.

The first step in our improvement is to note (all lemmas and the theorem suppose e^{-tH_0} is hypercontractive):

Lemma A.1. *Let* $1 \leqq p \leqq 2$. *Then* $\{e^{-tH_0}\}_{t \geqq 0}$ *is strongly continuous on* $[0, \infty)$ *and if* $2 \geqq p > 1$, $\operatorname{Ker}(e^{-tH_0} \upharpoonright L^p) = \{0\}$.

Proof. Since the maps $e^{-tH_0} \upharpoonright L^p$ are uniformly bounded, it is sufficient to prove $e^{-tH_0}\psi$ is strongly continuous for a dense set of ψ. Since $p < 2$, L^2 is dense in L^p and continuity of $e^{-tH_0}\psi$ in L^2 implies L^p continuity. Fix $2 > p > 1$. By the Stein interpolation theorem (interpolation in arg z) [28], e^{-zH_0} is bounded from L^p to L^p for z in a sectorial neighborhood of $(0, \infty)$. Since $e^{-zH_0}\psi$ is L^2 and hence L^p analytic when $\psi \in L^2$, $e^{-zH_0}\psi$ is L^p-analytic in a neighborhood of $(0, \infty)$ for any $\psi \in L^p$. Now suppose $e^{-tH_0}\psi = 0$ for some $\psi \in L^p$ and some $t \in (0, \infty)$. By the semigroup property, $e^{-sH_0}\psi = 0$ for $s \geqq t$. By the analyticity, $e^{-sH_0}\psi = 0$ for all $s \in (0, \infty)$. Finally, by strong continuity, $\psi = s\text{-}\lim_{s \to 0} e^{-sH_0}\psi = 0$. □

Lemma A.2. *Let* $2 \leqq p < \infty$. *Then* $\{e^{-tH_0}\}_{t \geqq 0}$ *is strongly continuous on* $[0, \infty)$.

Remarks. 1. Even without hypothesis (ii), one can prove strong continuity on each L^p by using the analyticity techniques used in Lemma A.1.

2. e^{-tH_0} may not be strongly continuous on L^∞. For example, if $H_0 = -d^2/dx + 2x\, d/dx$ on $(\mathbb{R}, \pi^{-\frac{1}{2}} e^{-x^2} dx)$, $e^{-tH_0}[L^\infty] \subset C^\infty(H_0)$

$\equiv \{e^{+x^2/2} f \mid f \in \mathscr{S}\}$ contains only continuous functions, so $e^{-tH_0}\psi$ does not converge in L^∞ if ψ is L^∞ but discontinuous.

Proof. By uniform boundedness, we need only prove $e^{-tH_0}\psi$ is L^p-continuous for a dense subset of ψ. Pick T so that e^{-TH_0} is bounded from L^2 to L^p. By Lemma A.1, $\mathrm{Ker}(e^{-TH_0} \upharpoonright L^q) = \{0\}$ where $q^{-1} + p^{-1} = 1$. As a consequence, $\mathrm{Ran}(e^{-TH_0} \upharpoonright L^p)$ is dense in L^p. A fortiori, $e^{-TH_0}[L^2]$ is dense in L^p. But if $\psi \in L^2$, $e^{-tH_0}\psi$ is L^2-continuous so $e^{-tH_0}(e^{-TH_0}\psi) = e^{-TH_0}(e^{-tH_0}\psi)$ is L^p-continuous. □

In case $p = \infty$, e^{-tH_0} may not be strongly continuous but we have the weaker:

Lemma A.3. *If $\psi \in L^\infty$ and $V \in L^p$ ($p < \infty$) then*

$$Ve^{-tH_0}\psi \xrightarrow{L^p} V\psi \quad \text{as} \quad t \to 0 .$$

Proof. First suppose $V \in L^\infty$. Then the result follows from Lemma A.2. If $V \in L^\infty$, choose $V_n \in L^\infty$ with $V_n \to V$ in L^p. Since $\{e^{-tH_0}\psi\}$ is L^∞ bounded a simple argument proves that $Ve^{-tH_0}\psi - V\psi \to 0$ in L^p. □

Finally we need the essential self-adjointness of $H_0 + V$ in the following form proven in [24] and [27]:

Lemma A.4. (a) *Let $V \in L^p$ for some $p > 2$, $e^{-tV} \in L^1$ for all $t > 0$. Then $H_0 + V$ is essentially self-adjoint on $L^q \cap D(H_0) \cap D(V)$ for any $q < \infty$.*

(b) *Let $V \in L^2$; V bounded from below. Then $H_0 + V$ is essentially self-adjoint on $L^\infty \cap D(H_0) \cap D(V)$.*

Remark. The symbols $D(H_0)$, $D(V)$, $C^\infty(H_0)$ always refer to L^2-domain. We thus conclude:

Theorem A.1. *Under either hypothesis* (a) *or* (b) *of Lemma* A.4, *$H_0 + V$ is essentially self-adjoint on $C^\infty(H_0) \cap D(V)$.*

Remarks. 1. In both the field theory case and the case we discuss here $C^\infty(H_0) \subset D(V)$ but this need not be so $\left(\text{e.g. } H_0 = -\frac{d^2}{dx^2} + 2x\frac{d}{dx}\right.$ on $(\mathbb{R}, \pi^{-\frac{1}{2}} e^{-x^2} dx)$; $V = e^{|x|}$).

2. Actually, we will prove essential self-adjointness on $\mathfrak{A}(H_0) \cap D(V)$ where $\mathfrak{A}(H_0)$ is the set of analytic vectors for H_0 [17],

3. In the $P(\Phi)_2$ field theory, this essential self-adjointness is a result of Rosen [22], who used "higher order estimates". It is perhaps a little surprising that these *H.O.E.* are not needed for the result.

Proof. Since $H_0 + V$ is symmetric on $C^\infty(H_0) \cap D(V)$, we need only find a core inside $C^\infty(H_0) \cap D(V)$. We consider case (b); case (a) is proved similarly using Lemma A.2 in place of Lemma A.3. By Lemma A.4, we need only find for any $\psi \in L^\infty \cap D(H_0) \cap D(V)$, a sequence $\psi_n \in C^\infty(H_0) \cap D(V)$

with $\psi_n \xrightarrow{L^2} \psi$, $V\psi_n \xrightarrow{L^2} V\psi$ and $H_0\psi_n \xrightarrow{L^2} H_0\psi$. Let $\psi_n = e^{-H_0/n}\psi \in C^\infty(H_0) \cap L^\infty \subset C^\infty(H_0) \cap D(V)$. That $\psi_n \xrightarrow{L^2} \psi$ and $H_0\psi_n \longrightarrow H_0\psi$ follow from strong continuity on L^2. That $V\psi_n \xrightarrow{L^2} V\psi$ follows from Lemma A.3. □

Appendix 2

A Counterexample

Let $m \geqq 5$ and let H_0 be $-\Delta$ on $L^2(\mathbb{R}^m, dx)$. Let $V = -r^{-2}$. Then:

(a) $V \in (L^2)_{\text{loc}}$.

(b) V is H_0- bounded.

(c) $H_0 + \alpha V$ is bounded from below on $D(H_0)$ if and only if $\alpha \leqq \frac{1}{4}(m-1)(m-3) + \frac{1}{4}$.

(d) $H_0 + \alpha V$ is essentially self-adjoint on $C_0^\infty(\mathbb{R}^m)$ if and only if $\alpha \leqq \frac{1}{4}(m-1)(m-3) - \frac{3}{4}$.

(a) is trivial. Since V is in weak $L^{m/2}$ (i.e. $\nu\{x \| V(x)| > t\} \leqq C t^{-m/2}$ where ν is Lebesque measure), (b) follows from a result of Strichartz [30]. Alternatively, (b) can be proven from an inequality of Rellich discussed in [23]. To prove (c) and (d), notice first that, by (b), if D is any core for H_0, then the closure of $H_0 + V \upharpoonright D$ is the same as the closure of $H_0 + V$ (defined on $D(H_0)$). Moreover since $m \geqq 5$, $C_0^\infty(\mathbb{R}^m \backslash \{0\})$ is easily seen to be a core for H_0 (this fact remains true even if $m = 4$, but it is much harder to prove in that case). Thus we need only prove (c) [respectively (d)] when $D(H_0)$ [respectively $C_0^\infty(\mathbb{R}^m)$] is replaced by $C_0^\infty(\mathbb{R}^m \backslash \{0\})$.

Let f_{ml} be a complete set of eigenfunctions for the Laplace-Beltrami operator, A, on the $m-1$ sphere, so $A f_{ml} = a_{ml} f_{ml}$. Here $a_{m0} = 0$ and $a_{ml} \geqq 0$ for all l. Let $\mathscr{H}_{ml} = \{g(r) r^{(n-1)/2} f_{ml} | g \in L^2(0, \infty)\}$. Then H_0 and V leave $\mathscr{H}_{ml}$ invariant and under the natural equivalence of $\mathscr{H}_{ml}$ and $L^2(0, \infty)$, H_0 acts as

$$\tilde{H}_{0;\, ml} = -\frac{d^2}{dr^2} + \alpha_{ml} r^{-2}$$

where $\alpha_{ml} = \frac{1}{4}(m-1)(m-3) + a_{ml}$. Thus, to prove (c) and (d) we must only prove:

(c′) On $L^2(0, \infty)$, $-\dfrac{d^2}{dr^2} - \alpha r^{-2}$ is bounded from below if and only if $\alpha \leqq \frac{1}{4}$.

(d′) On $L^2(0, \infty)$, $-\dfrac{d^2}{dr^2} - \alpha r^{-2}$ is essentially self-adjoint if and only if $\alpha \leqq -\frac{3}{4}$.

(c′) is well known (see [4], pp. 446 – 447). (d′) follows from a simple application of the Weyl limit point limit circle method (see [2]). (a) – (d) have a number of negative consequences:

1. "Conjecture" 3 in §4 is false if $m \geqq 5$ for take $\alpha = \frac{1}{4}(m-1)(m-3)$.

2. It is possible to have an analytic family of type (B) (in the sense of [13]) in a region R (in this case $|\alpha| < \frac{1}{4}(m-1)(m-3) + \frac{1}{4}$) which is of type ($A$) in a strict subregion of R (in this case excluding $\frac{1}{4}(m-1)(m-3) - \frac{3}{4} < \alpha < \frac{1}{4}(m-1)(m-3) + \frac{1}{4}$).

3. One cannot prove essential self-adjointness of $H_0 + W$ when W is H_0 – bounded of relative bound larger than 1, even if one knows that $aH_0 + V$ is positive for some $a < 1$.

Since it may be surprising to some that essential self-adjointness breaks down before semiboundedness, let us make a few remarks about the phenomena. First, let us explain semi-heuristically the mathematics behind the phenomena: Let u solve the ordinary differential equation, $u'' = \alpha r^{-2} u$. Then $u_\pm = r^{\beta} \pm$ where $\beta_\pm$ solves $\beta(\beta - 1) = \alpha$, i.e.

$$\beta_\pm = \tfrac{1}{2}(-1 \pm \sqrt{1 + 4\alpha}) .$$

Boundedness from below is related to $\beta_\pm$ being real (see [15], pp. 120 – 121 for a heuristic explanation of this) while by the Weyl criterion, essential self-adjointness requires one of $u_\pm$ to be non-L^2 at $r = 0$, i.e. $\beta_- \leqq -\frac{1}{2}$, i.e. $\alpha \geqq \frac{3}{4}$.

From another point of view, this is just the difference between quantum mechanics from a form point of view [26] and from an operator point of view. Essential self-adjointness on $D(H_0) \cap D(V)$ is a useful technical result but the crucial physical requirement is that $H_0 + V$ as a sum of forms be closable on $Q(H_0) \cap Q(V)$ so that the sum of forms defines a self-adjoint operator. Semi-boundedness and this closability break down at the same point. In fact on the more familiar $\mathbb{R}^3$, this breakdown of essential self-adjointness before semiboundedness occurs also; $-\Delta - r^{-\beta}$ is self-adjoint on $D(-\Delta)$ only if $\beta < \frac{3}{2}$ but it is semibounded if $\beta < 2$ and in the range $\frac{3}{2} < \beta < 2$, all the usual quantum mechanics can be developed [26].

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Note Added in Proof: Conjectures 1 and 2 have been proven by T. Kato, Proc. Jerusalem Conf. Func. Anal., 1972 (to appear).

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