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L^p NORMS OF THE BOREL TRANSFORM AND THE DECOMPOSITION OF MEASURES

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ABSTRACT. We relate the decomposition over $[a, b]$ of a measure $d\mu$ (on $\mathbb{R}$) into absolutely continuous, pure point, and singular continuous pieces to the behavior of integrals $\int_a^b (\operatorname{Im} F(x + i\epsilon))^p \, dx$ as $\epsilon \downarrow 0$. Here F is the Borel transform of $d\mu$, that is, $F(z) = \int (x - z)^{-1} \, d\mu(x)$.

1. INTRODUCTION

Given any positive measure μ on $\mathbb{R}$ with

$$\int \frac{d\mu(x)}{1 + |x|} < \infty, \tag{1.1}$$

one can define its Borel transform by

$$F(z) = \int \frac{d\mu(x)}{x - z}. \tag{1.2}$$

We have two goals in this note. One is to discuss the relation of the decomposition of μ into components ($d\mu = d\mu_{\text{ac}} + d\mu_{\text{pp}} + d\mu_{\text{sc}}$ with $d\mu_{\text{ac}}(x) = g(x)\, dx$, $d\mu_{\text{pp}}$ a pure point measure, and $d\mu_{\text{sc}}$ a singular continuous measure) to integrals of powers of $\operatorname{Im} F(x + i\epsilon)$. This is straightforward, and global results (e.g., involving $\int_{-\infty}^{\infty} |\operatorname{Im} F(x + i\epsilon)|^2 \, dx$) are well known to harmonic analysts (see, e.g., Koosis [5, pg. 157])—but there seems to be a point in writing down elementary proofs of the local results (e.g., involving $\int_a^b |\operatorname{Im} F(x + i\epsilon)|^2 \, dx$).

Secondly, by proper use of these theorems, we can simplify the proofs in [7] that certain sets of operators are G_δ's in certain metric spaces.

In §2, we will see that $\int_a^b |\operatorname{Im} F(x + i\epsilon)|^p \, dx$ with $p > 1$ is sensitive to singular parts of $d\mu$ and can be used to prove they are absent. In §3, we see the opposite

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results when $p < 1$ and the singular parts are irrelevant, so that integrals can be used for a test of whether $\mu_{\rm ac} = 0$. Finally, in §4, we turn to the aforementioned results on G_δ sets of operators.

Since we only discuss $\operatorname{Im} F(z)$ and

$$\operatorname{Im} F(x+i\epsilon) = \epsilon \int \frac{d\mu(y)}{(x-y)^2+\epsilon^2}, \tag{1.3}$$

our results actually hold if (1.1) is replaced by

$$\int \frac{d\mu(x)}{(1+|x|)^2} < \infty. \tag{1.4}$$

2. p-NORMS FOR $p > 1$

Theorem 2.1. *Fix $p > 1$. Suppose that*

$$\sup_{0<\epsilon<1} \int_a^b |\operatorname{Im} F(x+i\epsilon)|^p \, dx < \infty. \tag{2.1}$$

Then $d\mu$ is purely absolutely continuous on (a, b), $\frac{d\mu_{\rm ac}}{dx} \in L^p(a, b)$; and for any $[c, d] \subset (a, b)$, $\frac{1}{\pi} \operatorname{Im} F(x+i\epsilon)$ converges to $\frac{d\mu_{\rm ac}}{dx}$ in L^p. Conversely, if $[a, b] \subset (e, f)$ with $d\mu$ purely absolutely continuous on (e, f), and if $\frac{d\mu_{\rm ac}}{dx} \in L^p(e, f)$, then (2.1) *holds.*

Remarks. 1. This criterion with $p = 2$ is used by Klein [4], who has a different proof.

2. The $p = 2$ results can be viewed as following from Kato's theory of smooth perturbations [2,6].

3. It is easy to construct measures supported on $\mathbb{R}\backslash(a, b)$ so that (2.1) fails or so that the L^p norm oscillates, for example, suitable point measures $\sum \alpha_n \delta_{x_n}$ with $x_n \uparrow a$. For this reason, we are forced to shrink/expand (a, b) to $(c, d)/(e, f)$.

Proof. Let $d\mu_\epsilon(x) = \pi^{-1} \operatorname{Im} F(x+i\epsilon)\, dx$. Then [8] $d\mu_\epsilon \to d\mu$ weakly, as $\epsilon \downarrow 0$, that is, $\lim_{\epsilon\downarrow 0} \int f(x)\, d\mu_\epsilon(x) = \int f(x)\, d\mu(x)$ for f a continuous function of compact support. Let q be the dual index to p and f a continuous function supported in (a, b). Then

$$\begin{aligned} \left| \int f\, d\mu \right| &= \lim_{\epsilon\downarrow 0} \left| \int f\, d\mu_\epsilon \right| \\ &\le \varlimsup_{\epsilon\downarrow 0} \left[\int_a^b |f(x)|^q \, dx \right]^{1/q} \left[\int_a^b \left(\frac{1}{\pi} \operatorname{Im} F(x+i\epsilon) \right)^p dx \right]^{1/p} \\ &\le C \|f\|_q. \end{aligned}$$

Thus, $f \mapsto \int f\, d\mu$ is a bounded functional on L^q, and thus $\chi_{(a,b)}\, d\mu = g\, dx$ for some $g \in L^p(a, b)$.

We claim that when $\chi_{(a,b)}\, d\mu = g\, dx$ with $g \in L^p(a, b)$, then for any $[c, d] \subset (a, b)$, $\frac{1}{\pi} \operatorname{Im} F(x+i\epsilon) \to g$ in $L^p(c, d)$—this implies the remaining parts of the theorem.

To prove the claim, write $F = F_1 + F_2$ where F_1 comes from $d\mu_1 \equiv \chi_{(a,b)}\, d\mu$ and $d\mu_2 = (1 - \chi_{(a,b)})\, d\mu$. $\frac{1}{\pi} \operatorname{Im} F_1$ is a convolution of $g\,dx$ with an approximate delta function. So, by a standard argument, $\frac{1}{\pi} \operatorname{Im} F_1 \to g$ in L^p. On the other hand, since $\operatorname{dist}([c, d], \mathbb{R}\backslash(a, b)) > 0$, one easily obtains the bound

$$|\operatorname{Im} F_2(x + i\epsilon)| \le C\epsilon \qquad \text{for } x \in [c, d].$$

So $\frac{1}{\pi} \operatorname{Im} F_2 \to 0$ in L^p. □

The following is a local version of Wiener's theorem.

Theorem 2.2.

$$\text{(2.1)} \quad \lim_{\epsilon \downarrow 0} \epsilon \int_a^b |\operatorname{Im} F(x+i\epsilon)|^2\, dx = \frac{\pi}{2}\left(\frac{1}{2}\mu(\{a\})^2 + \frac{1}{2}\mu(\{b\})^2 + \sum_{x \in (a,b)} \mu(\{x\})^2\right).$$

Proof. Using (1.3), we see that

$$\epsilon \int_a^b (\operatorname{Im} F(x + i\epsilon))^2\, dx = \int\int g_\epsilon(x, y)\, d\mu(x)\, d\mu(y),$$

where

$$g_\epsilon(x, y) = \int_a^b \frac{\epsilon^3\, dw}{((w - x)^2 + \epsilon^2)((w - y)^2 + \epsilon^2)}.$$

It is easy to see that for $0 < \epsilon < 1$:

(i) $g_\epsilon(x, y) \le \pi \frac{1}{\operatorname{dist}(x, [a, b])^2 + 1}$,
(ii) $\lim_{\epsilon \downarrow 0} g_\epsilon(x, y) = 0$ if $x \ne y$ or $x \notin [a, b]$,
(iii) $\lim_{\epsilon \downarrow 0} g_\epsilon(x, y) = \frac{\pi}{2}$ if $x = y \in (a, b)$,
(iv) $\lim_{\epsilon \downarrow 0} g_\epsilon(x, y) = \frac{\pi}{4}$ if $x = y$ is a or b.

Thus, the desired result follows from dominated convergence. □

Remarks. 1. It is not hard to extend this to $\epsilon^{p-1} \int_a^b |\operatorname{Im} F(x + i\epsilon)|^p\, dx$ for any $p > 1$. The limit has $\int_{-\infty}^{\infty} (1 + x^2)^{-p/2}\, dx$ in place of π (which can be evaluated exactly in terms of gamma functions) and $\mu(\{x\})^p$ in place of $\mu(\{x\})^2$; for the above proof extends to p an even integer. Interpolation then shows that the continuous part of μ makes no contribution to the limit, and a simple argument restricts the result to a finite sum of point measure where it is easy. (Note: For $1 < p < 2$, one interpolates between boundedness for $p = 1$ and the zero limit if $p = 2$ and μ is continuous.)

2. On the other hand, $\sup_{0<\epsilon<1} \epsilon^\alpha \int_a^b \operatorname{Im} F(x + i\epsilon)^2\, dx$ for $0 < \alpha < 1$ says something about how singular the singular part of $d\mu$ can be. If the sup is finite, then $\mu(A) = 0$ for any subset A of $[a, b]$ with Hausdorff dimension $d < 1 - \alpha$. This will be proven in [1].

Corollary 2.3. *μ has no pure points in $[a, b]$ if and only if*

$$\varlimsup_{k\to\infty} \frac{1}{k} \int_a^b (\operatorname{Im} F(x+ik^{-1}))^2\, dx = 0.$$

(Of course the limit exists, but we'll need this form in §4.)

3. p-NORMS FOR $p < 1$

Theorem 3.1. *Fix $p < 1$. Then*

$$\lim_{\epsilon\downarrow 0} \int_a^b \left|\frac{1}{\pi} \operatorname{Im} F(x+i\epsilon)\right|^p dx = \int_a^b \left(\frac{d\mu_{\mathrm{ac}}}{dx}\right)^p dx.$$

First Proof. Write $d\mu$ as three pieces: $d\mu_1 = (1-\chi_{[a-1, b+1]})\, d\mu$, $d\mu_2 = g\, dx$ with $g \in L^1(a-1, b+1)$, and $d\mu_3$ singular and finite and concentrated on $[a-1, b+1]$ and correspondingly, $F = F_1 + F_2 + F_3$. It is easy to see that $|\operatorname{Im} F_1(x+i\epsilon)| \le C\epsilon$ on $[a, b]$, so its contribution to the limit of the integral is 0. Since $\frac{1}{\pi}\operatorname{Im} F_2(x+i\epsilon)$ is a convolution of g with an approximate delta function, $\frac{1}{\pi}\operatorname{Im} F_2 \to g$ in L^1, and so by Holder's inequality,

$$\int_a^b |\frac{1}{\pi} \operatorname{Im} F_2(x+i\epsilon)|^p\, dx \to \int_a^b g(x)^p\, dx \quad \text{for any } p < 1.$$

It thus suffices to prove that

$$\int_a^b \left|\frac{1}{\pi} \operatorname{Im} F_3(x+i\epsilon)\right|^p dx \to 0. \tag{3.1}$$

Let S be a set with $\mu_3(\mathbb{R}\backslash S) = 0$ and $|S| = 0$. Given δ, by regularity of measures, find $C \subset S \subset \mathscr{O}$ with C compact and $\mathscr{O} \subset (a-2, b+1)$ open, so $\mu(S\backslash C) < \delta$ and $|\mathscr{O}\backslash S| < \delta$, so $\mu(\mathbb{R}\backslash C) < \delta$ and $|\mathscr{O}| < \delta$. Let h be a continuous function which is 1 on $\mathbb{R}\backslash\mathscr{O}$ and 0 on C.

By Holder's inequality (with index $\frac{1}{p}$),

$$\int_A \left(\frac{1}{\pi}\operatorname{Im} F_3\right)^p dx \le |A|^{1-p}\left[\int_A \left(\frac{1}{\pi}\operatorname{Im} F_3\right)\right]^p \tag{3.2}$$

for any set A. Noting that $\int_{\mathbb{R}} (\frac{1}{\pi}\operatorname{Im} F_3)\, dx = \mu_3(\mathbb{R}) < \infty$, we see that

$$\int_{\mathscr{O}} \left(\frac{1}{\pi}\operatorname{Im} F_3\right)^p dx \le \mu_3(\mathbb{R})^p \delta^{1-p}. \tag{3.3}$$

On the other hand,

$$\begin{aligned}\int_{[a,b]\backslash\mathscr{O}} \left(\frac{1}{\pi}\operatorname{Im} F_3\right)^p dx &\le |b-a|^{1-p}\left[\int_{[a,b]\backslash\mathscr{O}} \left(\frac{1}{\pi}\operatorname{Im} F_3\right) dx\right]^p \\ &\le |b-a|^{1-p}\left[\int_a^b h(x)\left(\frac{1}{\pi}\operatorname{Im} F_3\right)(x+i\epsilon)\, dx\right]^p.\end{aligned}$$

The last integral converges to $\int h(x)\,d\mu_3(x) \leq \int_{\mathbb{R}\backslash C} d\mu_3(x) = \mu_3(\mathbb{R}\backslash C) = \delta$. Thus

$$\overline{\lim_{\epsilon\downarrow 0}} \int_a^b \frac{1}{\pi} \operatorname{Im} F_3(x+i\epsilon)^p\,dx \leq \mu_3(\mathbb{R})^p \delta^{1-p} + |b-a|^{1-p}\delta^p.$$

Since δ is arbitrary, the $\overline{\lim}$ is a zero and so the limit is zero. □

Second Proof (suggested to me by T. Wolff). As in the first proof, by writing μ as a sum of a finite measure and a measure obeying (1.1) but supported away from $[a, b]$, we can reduce the result to the case where μ is finite. Let $M_\mu(x)$ be the maximal function of μ:

$$M_\mu(x) = \sup_{t>0} (2t)^{-1} \mu(x-t, x+t).$$

By the standard Hardy-Littlewood argument (see, e.g., Katznelson [3]),

$$|\{x \mid M_\mu(x) > t\}| \leq C\mu(\mathbb{R})/t,$$

which in particular implies

$$\int_a^b M_\mu(x)^p\,dx < \infty$$

for all $p < 1$.

Since $\frac{1}{\pi}\operatorname{Im} F(x+i\epsilon) \leq M_\mu(x)$ for all ϵ and $\frac{1}{\pi}\operatorname{Im} F(\cdot + i\epsilon) \to (\frac{d\mu_{\text{ac}}}{dx})(x)$ a.e. in x, the desired result follows by the dominated convergence theorem. □

Remark. The reader will note that the first proof is similar to the proof in [7] that the measures with no a.c. part are a G_δ. In a sense, this part of our discussion in §4 is a transform for the proof of [7] to this proof instead!

Corollary 3.2. *A measure μ has no absolutely continuous part on (a, b) if and only if*

$$\underline{\lim}_{k\to\infty} \int_a^b \operatorname{Im} F(x + ik^{-1})^{1/2}\,dx = 0.$$

4. G_δ PROPERTIES OF SETS OF MEASURES AND OPERATORS

Lemma 4.1. *Let X be a topological space and $f_n : X \to \mathbb{R}$ a sequence of non-negative continuous functions. Then $\{x \mid \underline{\lim}_{n\to\infty} F_n(x) = 0\}$ is a G_δ.*

Proof.

$$\left\{x \mid \underline{\lim}_{n\to\infty} F_n(x) = 0\right\} = \left\{x \mid \forall k\, \forall N\, \exists n \geq N\ F_n(x) < \frac{1}{k}\right\}$$
$$= \bigcap_{k=1}^{\infty} \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \left\{x \mid F_n(x) < \frac{1}{k}\right\}$$

is a G_δ. □

As a corollary of this and Corollaries 2.3 and 3.2, we obtain a proof of the result of [9].

Theorem 4.1. *Let M be the set of probability measures on $[a, b]$ in the topology of weak convergence (this is a complete metric space). Then* $\{\mu \mid \mu$ is purely singular continuous$\}$ *is a dense* G_δ.

Proof. By Corollary 3.2

$$\{\mu \mid \mu_{\text{ac}} = 0\} = \left\{\mu \mid \varliminf_{k\to\infty} \int_a^b (\operatorname{Im} F_\mu(x + ik^{-1})^{1/2}\, dx = 0\right\},$$

and by Corollary 2.3

$$\{\mu \mid \mu_{\text{pp}} = 0\} = \left\{\mu \mid \varliminf_{k\to\infty} k^{-1} \int_a^b \operatorname{Im} F_\mu(x + ik^{-1})^2\, dx = 0\right\},$$

so by Lemma 4.1, each is a G_δ. Here we use the fact that $\mu \mapsto F_\mu(x + i\epsilon)$ is weakly continuous for each x, $\epsilon > 0$ and dominated above for each $\epsilon > 0$, so the integrals are weakly continuous. By the convergence of the Riemann-Stieltjes integrals, the point measures are dense in M, so $\{\mu \mid \mu_{\text{ac}} = 0\}$ is dense. On the other hand, the fact that $\frac{1}{\pi} \operatorname{Im} F_\mu(x + i\epsilon)\, dx$ converge in M to $d\mu$ shows that the a.c. measures are dense in M, so $\{\mu \mid \mu_{\text{pp}} = 0\}$ is dense. Thus, by the Baire category theorem, $\{\mu \mid \mu_{\text{pp}} = 0\} \cap \{\mu \mid \mu_{\text{ac}} = 0\}$ is a dense G_δ! □

Finally, we recover our results in [7]. We call a metric space X of selfadjoint operators on a Hilbert space $\mathscr{H}$ regular if and only if $A_n \to A$ in the metric topology implies that $A_n \to A$ in strong resolvent sense. (Strong resolvent convergence of selfadjoint operators means $(A_n - z)^{-1}\varphi \xrightarrow{\|\,\|} (A - z)^{-1}\varphi$ for all φ and all z with $\operatorname{Im} z \neq 0$. Notice this implies that for any a, b, p and $\epsilon > 0$ and any $\varphi \in \mathscr{H}$, $A \mapsto \int_a^b \operatorname{Im}(\varphi, (A - x - i\epsilon)^{-1}\varphi)^p\, dx \equiv F_{a,b,p,\epsilon,\varphi}(A)$ is a continuous function in the metric topology.

Theorem 4.3. *For any open set $\mathscr{O} \subset \mathbb{R}$ and any regular metric space of operators,* $\{A \mid A$ has no a.c. spectrum in $\mathscr{O}\}$ *is a* G_δ.

Proof. Any $\mathscr{O}$ is a countable union of intervals, so it suffices to consider the case $\mathscr{O} = (a, b)$. Let φ_n be an orthonormal basis for $\mathscr{H}$. Then

$$\{A \mid A \text{ has no a.c. spectrum in } (a, b)\} = \bigcap_n \left\{A \mid \varliminf_{k\to\infty} F_{a,b,1/2,1/k,\varphi_n}(A)\right\}$$

is a G_δ by Lemma 4.1 and Corollary 3.2. □

Similarly, using Corollary 2.3, we obtain

Theorem 4.4. *For any interval $[a, b]$ and any regular metric space of operators,* $\{A \mid A$ has no point spectrum in $[a, b]\}$ *is a* G_δ.

Note. This is slightly weaker than the result in [7] but suffices for most applications. One can recover the full result of [7], namely Theorem 4.4 with $[a, b]$ replaced by an arbitrary closed set K, by first noting that any closed set is a union of compacts, so it suffices to consider compact K. For each K, let

$K_\epsilon = \{x \mid \text{dist}(x, K) < \epsilon\}$. Then one can show that if $d\mu$ has no pure points in K, then

$$\lim_{\epsilon \downarrow 0} \epsilon \int_{K_\epsilon} (\text{Im}\, F_\epsilon(x + i\epsilon))^2\, dx = 0;$$

and if it does have pure points in K, then

$$\varliminf_{k \to \infty} k^{-1} \int_{K_\epsilon} |\text{Im}\, F(x + ik^{-1})|^2\, dx > 0$$

and Theorem 4.4 extends.

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