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Rank One Perturbations with Infinitesimal Coupling

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We consider a positive self-adjoint operator A and formal rank one perturbations

$$B = A + \alpha(\varphi, \cdot)\varphi,$$

where $\varphi \in \mathscr{H}_{-2}(A)$ but $\varphi \notin \mathscr{H}_{-1}(A)$, with $\mathscr{H}_s(A)$ the usual scale of spaces. We show that B can be defined for such φ and what are essentially negative infinitesimal values of α. In a sense we will make precise, every rank one perturbation is one of three forms: (i) $\varphi \in \mathscr{H}_{-1}(A)$, $\alpha \in \mathbb{R}$; (ii) $\varphi \in \mathscr{H}_{-1}$, $\alpha = \infty$; or (iii) the new type we consider here.

1. INTRODUCTION

There has recently been considerable interest in the study of rank one perturbations of positive self-adjoint operators (see [11, and Refs. therein]). Let $A \geqslant 0$ on a Hilbert space $\mathscr{H}$ and consider

$$B = A + \alpha(\varphi, \cdot)\varphi. \tag{1.1}$$

Simon–Wolff [12] first pointed out that a natural framework for this was to consider $\varphi \in \mathscr{H}_{-1}(A)$, where $\mathscr{H}_s(A)$ is the usual scale of spaces associated to A; that is, if $s \geqslant 0$, $\mathscr{H}_s(A) = D(|A|^{s/2})$ with the norm $\|\cdot\|$ given by

$$\|\varphi\|_s^2 = \langle \varphi, (A+1)^s \varphi \rangle,$$

and if $s < 0$, $\mathscr{H}_s(A)$ is the completion of $\mathscr{H}$ in the $\|\cdot\|_s$ norm. $\mathscr{H}_s \subset \mathscr{H}_t$ if $s > t$ and one can define $\mathscr{H}_\infty(A) = \bigcap_s \mathscr{H}_s(A)$ and $\mathscr{H}_{-\infty}(A) = \bigcup_s \mathscr{H}_s(A)$. $\mathscr{H}_s^* = \mathscr{H}_{-s}$ in a natural way.

When $\varphi \in \mathscr{H}_{-1}(A)$, $\psi \mapsto |(\psi, \varphi)|^2$ defines a quadratic form on $Q(A) = \mathscr{H}_{+1}(A)$, which is A-bounded with relative bound zero. So the standard form perturbation theory [7, 10] lets one define (1.1) for any $\alpha \in \mathbb{R}$.

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Define

$$F_\alpha(z) = (\varphi, (A_\alpha - z)^{-1} \varphi) \tag{1.2a}$$

$$F(z) = F_{\alpha=0}(z). \tag{1.2b}$$

One easily proves the formulae (going back to Krein and Aronszajn),

$$F_\alpha(z) = F(z)/1 + \alpha F(z) \tag{1.3}$$

$$(A_\alpha - z)^{-1} \varphi = (1 + \alpha F(z))^{-1} (A - z)^{-1} \varphi \tag{1.4a}$$

$$(A_\alpha - z)^{-1} = (A - z)^{-1} - \alpha(1 + \alpha F(z))^{-1} ((A - \bar{z})^{-1} \varphi, \cdot)(A - z)^{-1} \varphi. \tag{1.4b}$$

From (1.4) one sees $s\text{-lim}_{\alpha \to \infty}(A_\alpha - z)^{-1}$ exists. If $\varphi \notin \mathscr{H}_0(A) = \mathscr{H}$, it defines an operator A_∞ on $\mathscr{H}$. This is studied in [5].

Our primary goal here is two-fold:

(a) To construct a family of rank one perturbations $A + \alpha(\varphi, \cdot)\varphi$ where $\varphi \notin \mathscr{H}_{-1}(A)$ but only in $\mathscr{H}_{-2}(A)$. Here α is infinitesimal.

(b) Every pair of semibounded operators with $(A + i)^{-1} - (B + i)^{-1}$ rank one can be written using the $\alpha(\varphi, \cdot)\varphi$ construction with $\varphi \in \mathscr{H}_{-1}$ and α finite or infinite.

These two apparently paradoxical statements are not paradoxical because in (b) we did not specify if B is a perturbation of A or vice-versa. In fact, one can always label them so that $A \leqslant B$. Then we will show that $B = A + \alpha(\varphi, \cdot)\varphi$ with $\varphi \in \mathscr{H}_{-1}(A)$ with $\alpha \in [0, \infty]$. If $\alpha < \infty$, then A can be obtained from B by a rank one perturbation with $\varphi \in \mathscr{H}_{-1}(B)$. But if $\alpha = \infty$, it is necessary to use the $\mathscr{H}_{-2}(B)$ construction to recover A from B.

At first, it is comforting that infinitesimal coupling is needed to undo infinite coupling, but that feeling is unfounded. For multiplicative perturbations, infinitesimal should undo infinite, but these perturbations are additive. In fact, $(\eta, \cdot)\eta$ with $\eta \in \mathscr{H}_{-2}(B)/\mathscr{H}_{-1}(B)$ is so infinite we need infinitesimal coupling to undo $\infty(\varphi, \cdot)\varphi$ with $\varphi \in \mathscr{H}_{-1}(A)$.

A theme that we will explore in this paper is that if A, B have resolvents that differ by a rank one, then there exists a symmetric operator C with deficiency indices $(1, 1)$ so that A and B are both self-adjoint extensions of C. To say that B is $A + \alpha(\varphi, \cdot)\varphi$ with $\alpha = \infty$ and $\varphi \in \mathscr{H}_{-1}(A)$ (equivalently that A is $B + \alpha(\varphi, \cdot)\varphi$ with $\varphi \in \mathscr{H}_{-2}(B)/\mathscr{H}_{-1}(A)$ and α infinitesimal) is equivalent to saying that B is the Friedrich's extension. From this point of view, our assertion (b) above is a special case of the Birman–Krein–Vishik theory of quadratic forms of positive self-adjoint extensions [3, 8, 13, 6, 2].

In Section 2, we present the construction of rank one perturbations with $\varphi \in \mathscr{H}_{-2}$. In Section 3, we use resolvent ordering to prove assertion (b). In

Section 4, we explain the relation of infinite and infinitesimal coupling. In Section 5, we consider fairly general situations $A_n = A + \alpha_n(\varphi_n, \cdot)\varphi_n$ with φ_n a cutoff of $\varphi \in \mathscr{H}_{-\infty}(A)$ and show that as $n \to \infty$, A_n converges to A in strong resolvent sense unless $\varphi \in \mathscr{H}_{-1}(A)$ or $\varphi \in \mathscr{H}_{-2}(A)$, $\alpha_n < 0$ and $\alpha_n \to 0$ at a suitable rate. This provides another view of the fact the only rank one perturbations are the $\mathscr{H}_{-1}(A)$ and $\mathscr{H}_{-2}(A)$ constructions. In Section 6, we discuss the connection to the theory of self-adjoint extensions of deficiency indices (1, 1). Finally, Section 7 presents some simple examples.

2. The Basic $\mathscr{H}_{-2}(A)$ Construction

Let $\varphi \in \mathscr{H}_{-2}(A)$ so $(A-z)^{-1}\varphi$ makes sense for any $z \notin \operatorname{spec}(A)$ and in particular, for $\operatorname{Im} z \neq 0$. Motivated by (1.4), we try to construct a self-adjoint operator whose resolvent $R(z)$ obeys

$$R(z) = (A-z)^{-1} - \sigma(z)\, K(z), \tag{2.1a}$$

where

$$K(z) = ((A-\bar{z})^{-1}\varphi, \cdot)(A-z)^{-1}\varphi. \tag{2.1b}$$

The idea is to define $R(z)$ by (2.1) and then to pick the unknown function $\sigma(z)$ in order that R obey the equation obeyed by any resolvent,

$$\frac{dR}{dz} = R(z)^2. \tag{2.2}$$

Since $dK/dz = (A-z)^{-1}K + K(A-z)^{-1}$ and $(d/dz)(A-z)^{-1} \equiv (A-z)^{-2}$, (2.2) is equivalent to

$$\frac{d\sigma}{dz} K(z) = -\sigma(z)^2 K(z)^2. \tag{2.3}$$

But $K(z)^2 = K(z)(\varphi, (A-z)^{-2}\varphi)$. Thus (2.2) is equivalent to

$$\frac{d}{dz}\sigma^{-1}(z) = (\varphi, (A-z)^{-2}\varphi). \tag{2.4}$$

Supposing that $A \geqslant 0$, we note that (2.4) shows that σ^{-1}, originally defined for $\operatorname{Im} z \neq 0$, can be continued through $(-\infty, 0)$. Self-adjointness for R, that is, $R^*(z) = R(\bar{z})$ requires σ^{-1} be real there; and thus the solutions can be written

$$\sigma^{-1}(z) = \beta + (\varphi, [(A-z)^{-1} - (A+1)^{-1}]\varphi) \tag{2.5}$$

with β real and equal to $\sigma^{-1}(-1)$. This motivates:

THEOREM 2.1. *Fix $\beta \in \mathbb{R}$. Suppose $A \geqslant 0$ and $\varphi \in \mathscr{H}_{-2}(A)$. For* $\operatorname{Im} z \neq 0$, *define $R_\beta(z)$ by* (2.1) *with $\sigma(z)$ given by* (2.5). *Then there is a self-adjoint operator $\tilde{A}_\beta$ with $R_\beta(z) = (\tilde{A}_\beta - z)^{-1}$.*

Proof. Let

$$G(z) \equiv (\varphi, [(A-z)^{-1} - (A+1)^{-1}]\varphi). \tag{2.6}$$

Then for $y \in (-\infty, \theta)$, $dG/dy = (\varphi, (A-y)^{-2}\varphi) > 0$. Thus, there is at most one $y < 0$, call it y_0, so $\sigma(y)^{-1} = 0$. Therefore, $R_\beta(z)$ extends to $\mathbb{C}\backslash[0, \infty) \cup \{y_0\}$ with $R_\beta(y)$ self-adjoint if $y \in \mathbb{R}\backslash[0, \infty) \cup \{y_0\}$. Fix any $y_1 < 0$ with $y_1 \neq y_0$ and define $A_\beta \equiv R_\beta(y_1)^{-1} - y_1$. Then $R_\beta(z)$ and $(A_\beta - z)^{-1}$ obey the same differential equation (1.2) and same initial conditions at $y = y_1$, and so they are equal on $\operatorname{Im} z \neq 0$. ∎

Remark. One can think of (2.1) in the form

$$\begin{aligned} (\tilde{A}_\beta - z)^{-1} &= (A-z)^{-1} - \sigma_\beta(z)\, K(z) \\ \sigma_\beta(z)^{-1} &= \beta + (\varphi, ((A-z)^{-1} - (A+1)^{-1}\varphi) \end{aligned} \tag{2.1c}$$

as a renormalized form of (1.4), which can be written

$$\begin{aligned} (A_\alpha - z)^{-1} &= (A-z)^{-1} - \hat{\sigma}_\alpha(z)\, K(z) \\ \hat{\sigma}_\alpha(z)^{-1} &= \alpha^{-1} + (\varphi, (A-z)^{-1}\varphi). \end{aligned}$$

If $\varphi \in \mathscr{H}_{-1}(A)$, then $\tilde{A}_\beta = A_\alpha$, where β and α are related by

$$\beta = \alpha^{-1} + (\varphi, (A+1)^{-1}\varphi). \tag{2.7}$$

If $\varphi \notin \mathscr{H}_{-1}$, in essence we need to take $\alpha^{-1} = -\infty$ to undo the divergence of $(\varphi, (A+1)^{-1}\varphi)$, and α is infinitesimal and negative. The condition $\varphi \in \mathscr{H}_{-2}(A)$ is required for the single renormalization to work.

THEOREM 2.2. *If $\varphi \notin \mathscr{H}_{-1}(A)$, then each operator A_β defined in Theorem* 2.1 *obeys $\tilde{A}_\beta \leqslant A$ with $\tilde{A}_\beta \neq A$. If $\varphi \in \mathscr{H}_{-1}(A)$, there exist $\tilde{A}_\beta$'s with $\tilde{A}_\beta \geqslant A$ with $\tilde{A}_\beta \neq A$.*

Remark. Recall [7] that we say A, B obey $A \geqslant B$ if and only if there is $a \in \mathbb{R}$ with $A \geqslant a1$, $B \geqslant a1$; and for $z < a$ real, we have $(B-z)^{-1} \geqslant (A-z)^{-1}$ as bounded operators.

Proof. If $\varphi \in \mathscr{H}_{-1}(A)$, we have seen above that $\{\tilde{A}_\beta\}$ is the same as $\{A_\alpha\}$ using (2.7). Since $A_\alpha \geqslant A$ if $\alpha > 0$, that proves the $\mathscr{H}_{-1}$ result.

If $\varphi \notin \mathscr{H}_{-1}$, then $G(-y) \to -\infty$ as $y \to \infty$. Thus, there is some $y_0 \in (-\infty, 0)$, so $G(y) + \beta < 0$ for all $y \leqslant y_0$. By (2.5) and (2.1c), $(\tilde{A}_\beta - y)^{-1} \geqslant (A - y^{-1}) > 0$ for such y, so $\tilde{A}_\beta \geqslant y_0$, $A \geqslant y_0$, and $\tilde{A}_\beta \leqslant A$. ∎

3. Every Rank One Perturbation Is $\mathscr{H}_{-1}(A)$-Bounded

In this section, we want to consider pairs of operators A, B so that $(A+i)^{-1} - (B+i)^{-1}$ is rank one. We start with two results that illuminate the notion:

Proposition 3.1. *Let A, B be self-adjoint operators. Then $Q(z) = (A-z)^{-1} - (B-z)^{-1}$ is rank one for one z with $\operatorname{Im} z \neq 0$ if and only if it is rank one for all such z.*

Proof.

$$(A-z)^{-1} = (1 + (w-z)(A-z)^{-1})(A-w)^{-1}, \tag{3.1}$$

so using the fact that

$$\begin{aligned}&(\varphi, (A-z)^{-1} - (B-z)^{-1}\psi)\\ &\quad= ((A-\bar{z})^{-1}\varphi, B(B-z)^{-1}\psi) - (A(A-\bar{z})^{-1}\varphi, (B-z)^{-1}\psi),\end{aligned}$$

we see that

$$Q(z) = (1 + (w-z)(A-z)^{-1})\, Q(w)(1 + (w-z)(B-z)^{-1})$$

and so Rank $Q(z) \leqslant$ Rank $Q(w)$. ∎

Proposition 3.2. *Suppose that A, B are self-adjoint, $A \geqslant 0$, and $(A+i)^{-1} - (B+i)^{-1}$ is rank one. Then B is bounded from below.*

Proof. By (3.1) for B, $w \in (-\infty, 0)$ is in spec(B) if and only if $1 + (w-i)(B-i)^{-1}$ is not invertible. But

$$\begin{aligned}L(w) &= 1 + (w-i)(B-i)^{-1}\\ &= 1 + (w-i)(A-i)^{-1} + (w-i)((B-i)^{-1} - (A-i)^{-1})\\ &= L_1(w) + L_2(w),\end{aligned}$$

where $L_1 = 1 + (w-i)(A-i)^{-1} = (A-w)(A-i)^{-1}$ is invertible for $w \in (-\infty, 0)$ and $L_2 = (w-i)((B-i)^{-1} - (A+i)^{-1})$ is rank one.

Thus, $L(w)$ is invertible if and only if $1+L_1(w)^{-1}L_2(w)$ is invertible. By (3.1), $w\in\operatorname{spec}(B)$ if and only if $1+L_1(w)^{-1}L_2(w)$ is not invertible. Thus, since L_2 is rank one, $w\in\operatorname{spec}(B)$ if and only if $F(w)\equiv\operatorname{Tr}(L_1(w)^{-1}L_2(w))=-1$. F is an entire analytic function with $F(w)\neq -1$ if $\operatorname{Im} w\neq 0$. We conclude B has isolated point spectrum on $(-\infty,0)$.

Thus, there exist real w_0 with $F(w_0)\neq -1$ and so $(B-w_0)^{-1}-(A-w_0)^{-1}$ is rank one. For rank one perturbations of self-adjoint operators, eigenvalues intertwine. Since A has no eigenvalues in $(-\infty,0)$, B can have only one eigenvalue in $(-\infty,0)$; that is, B is bounded from below. ∎

COROLLARY 3.3. *If $A\geqslant 0$ and $(A+i)^{-1}-(B+i)^{-1}$ is rank one, then either $A\geqslant B$ or $B\geqslant A$.*

Proof. Pick w below $\operatorname{spec}(A)\cup\operatorname{spec}(B)$. Then $(A-w)^{-1}\geqslant 0$, $(B-w)^{-1}\geqslant 0$, and since $(A-w)^{-1}-(B-w)^{-1}$ is rank one and self-adjoint, either $(A-w)^{-1}\geqslant(B-w)^{-1}$ or $(B-w)^{-1}\geqslant(A-w)^{-1}$. It follows that either $A\geqslant B$ or $B\geqslant A$. ∎

THEOREM 3.4. *Let A, B be self-adjoint operators with $B\geqslant A\geqslant 0$. Suppose that $(A+1)^{-1}-(B+1)^{-1}$ is rank one. Then $B=A+\alpha(\varphi,\cdot)\varphi$ with $\varphi\in\mathscr{H}_{-1}(A)$ and $\alpha\in[0,\infty]$ (with $\alpha=\infty$ allowed).*

Proof. Write

$$(A+1)^{-1}=(B+1)^{-1}+(\eta,\cdot)\eta, \tag{3.2}$$

which we can do because $(A+1)^{-1}\geqslant(B+1)^{-1}$.

We claim that $\eta\in\mathscr{H}_{+1}(A)$ with $(\eta,(A+1)\eta)\leqslant 1$; see Lemma 3.5 below. Define $\varphi=(A+1)\eta$ so (3.2) becomes

$$(B+1)^{-1}=(A+1)^{-1}-((A+1)^{-1}\varphi,\cdot)(A+1)^{-1}\varphi,$$

which is just (1.4) if

$$\frac{\alpha}{1+\alpha(\varphi,(A+1)^{-1}\varphi)}=1$$

or

$$\alpha=\frac{1}{1-(\eta,(A+1)\eta)}, \tag{3.3}$$

where $(\eta,(A+1)\eta)=1$ corresponds to $\alpha=\infty$. (1.4) at $z=-1$ implies the general relation for all z. ∎

LEMMA 3.5. *Let $A \geqslant 0$ be self-adjoint. Suppose $\eta \in \mathscr{H}$ with $(\eta, \cdot)\eta \leqslant (A+1)^{-1}$. Then $\eta \in \mathscr{H}_{+1}(A)$ with $(\eta, (A+1)\eta) \leqslant 1$.*

Proof. Let E_k be the spectral projection $E_{[0,k]}(A)$. Let $\varphi_k = (A+1)E_k\eta$. Then, by hypothesis,

$$|(\eta, \varphi_k)|^2 \leqslant (\varphi_k, (A+1)^{-1}\varphi_k). \tag{3.4}$$

Equation (3.4) is equivalent to

$$(\eta, E_k(A+1)\eta)^2 \leqslant (\eta, E_k(A+1)\eta)$$

or

$$(\eta, E_k(A+1)\eta) \leqslant 1.$$

Taking $k \to \infty$, we see $\eta \in \mathscr{H}_{+1}(A)$ and $(\eta, (A+1)\eta) \leqslant 1$. ∎

Remark. It may seem puzzling that the α in (3.3) obeys $1 < \alpha \leqslant \infty$. How about $B = A + \alpha(\varphi, \cdot)\varphi$ with $\alpha < 1$? The resolution is that until we normalize φ in some way, the scale of α is irrelevant. If we demand $\tilde{\varphi}$ obey $(\tilde{\varphi}, (A+1)^{-1}\tilde{\varphi}) = 1$, then we take $\tilde{\varphi} = \varphi/(\eta, (A+1)\eta)^{1/2}$ and $\alpha(\varphi, \cdot)\varphi = \tilde{\alpha}(\tilde{\varphi}, \cdot)\tilde{\varphi}$, where now

$$\tilde{\alpha} = \frac{(\eta, (A+1)\eta)}{1 - (\eta, (A+1)\eta)}.$$

As $(\eta, (A+1)\eta)$ runs from 0 to 1, $\tilde{\alpha}$ runs from 0 to infinity.

As an application of Lemma 3.5, we return to the construction of Section 2:

THEOREM 3.6. *Suppose $A \geqslant 0$, $\varphi \in \mathscr{H}_{-2}(A)$ but $\varphi \notin \mathscr{H}_{-1}(A)$, and that $\tilde{A}_\beta$ is the operator of Theorem 2.1. Then*

(i) $\mathscr{H}_{+1}(\tilde{A}_\beta) \supset \mathscr{H}_{+1}(A)$

(ii) $\mathscr{H}_{+1}(\tilde{A}_\beta) \neq \mathscr{H}_{+1}(A)$.

Remark. We will see later in Section 6 that $\mathscr{H}_{+1}(A)$ has codimension 1 in $\mathscr{H}_{+1}(\tilde{A}_\beta)$.

Proof. By Theorem 2.2, $\tilde{A}_\beta \leqslant A$ which implies (i). To see (ii), note that by the construction in Section 2 for all sufficiently large $c > 0$,

$$(A_\beta + c)^{-1} = (A+c)^{-1} - \sigma(c)((A+c)^{-1}\varphi, \cdot)(A+c)^{-1}\varphi$$

with $\sigma(c) < 0$. Thus by Lemma 3.5, $(A+c)^{-1}\varphi \in \mathscr{H}_{+1}(\tilde{A}_\beta)$. Since $\varphi \notin \mathscr{H}_{-1}(A)$, we have that $(A+c)^{-1}\varphi \notin \mathscr{H}_{+1}(A)$. ∎

4. Relation to Infinite Coupling

Suppose $B = A + \alpha(\varphi, \cdot)\varphi$ with $\varphi \in \mathscr{H}_{-1}(A)$. If $\alpha < \infty$, then $\mathscr{H}_{+1}(B) = \mathscr{H}_{+1}(A)$ and $A = B - \alpha(\varphi, \cdot)\varphi$ so A can be recovered from B by the $\mathscr{H}_{-1}$ construction. Our goal here is to show that when $\alpha = \infty$, A can be recovered from B by the $\mathscr{H}_{-2}(B)$ construction of Section 2, and vice-versa that the $A \to \tilde{A}_\beta$ construction can be undone with infinite coupling.

Recall [5] if $\varphi \in \mathscr{H}_{-1}(A)$ but $\varphi \notin \mathscr{H}$ and $A_\infty = A + \infty(\varphi, \cdot)\varphi$, then there exists a natural $\eta \in \mathscr{H}_{-2}(A_\infty)$ which obeys

$$(A_\infty - z)^{-1}\eta = F(z)^{-1}(A - z)^{-1}\varphi \tag{4.1}$$

with F given by (1.2b).

Proposition 4.1. *Suppose $A \geq 0$, $\varphi \in \mathscr{H}_{-1}(A)$ but $\varphi \notin \mathscr{H}$, and η is given by* (4.1). *Then $\eta \notin \mathscr{H}_{-1}(A_\infty)$.*

Proof. $\eta \in \mathscr{H}_{-1}(A_\infty)$ if and only if $\lim_{c \to \infty}(\eta, (c/(A_\infty + c))(1/(A_\infty + 1))\eta)$ is finite. But by (4.1)

$$\left(\eta, \frac{c}{A_\infty + c}\, \frac{1}{A_\infty + 1}\, \eta\right) = \frac{1}{F(-1)\, F(-c)} \left(\varphi, \frac{c}{A + c}\, \frac{1}{A + 1}\, \varphi\right).$$

The expectation on the right side of this equation has a non-zero limit as $c \to \infty$ since $\varphi \in \mathscr{H}_{-1}(A)$. But $F(-c) \to 0$ as $c \to \infty$ so the limit is infinity; that is, $\eta \notin \mathscr{H}_{-1}(A_\infty)$. ∎

Theorem 4.2. *Suppose $A \geq 0$ and $\varphi \in \mathscr{H}_{-1}(A)$ but $\varphi \notin \mathscr{H}$. Let $B \equiv A_\infty = A + \alpha(\varphi, \cdot)\varphi$. Then for some β and the perturbation η, $\tilde{B}_\beta = A$; that is, A can be recovered from B by the construction of Section 2.*

Proof. By (1.4b) in the limit

$$(B - z)^{-1} = (A - z)^{-1} - F(z)^{-1}((A - \bar{z})^{-1}\varphi, \cdot)(A - z)^{-1}\varphi.$$

By (4.1)

$$(A - z)^{-1} = (B - z)^{-1} + F(z)((B - \bar{z})^{-1}\eta, \cdot)(B - z)^{-1}\eta$$

which shows that $(A + 1)^{-1}$ is a $(\tilde{B}_\beta + 1)^{-1}$. ∎

Remark. By Section 2, the coefficient in front of $((B - \bar{z})^{-1}\varphi, \cdot)(B - z)^{-1}\varphi$ should be $(\beta + G(z))^{-1}$, where $G(z) = (\eta, [(A_\infty - z)^{-1} - (A_\infty + 1)^{-1}]\eta)$. The resulting relation of $\operatorname{Im} F(z)^{-1}$ and $\operatorname{Im}(G(z))$ is exactly what was found in [5].

5. Limits

We have shown in the last two sections that if $(A-z)^{-1}-(B-z)^{-1}$ is rank one (and both are bounded below), then B can be recovered from A via either a $\varphi \in \mathscr{H}_{-1}(A)$ construction with $\alpha \in (-\infty, \infty]$ or else by the $\varphi \in \mathscr{H}_{-2}(A)\backslash\mathscr{H}_{-1}(A)$ construction with α infinitesimal. Thus it should be impossible to define $A+\alpha(\varphi, \cdot)\varphi$ if $\varphi \notin \mathscr{H}_{-2}(A)$. That is what we will prove in this section.

Theorem 5.1. *Let $A \geqslant 0$ and $\varphi \in \mathscr{H}_{-\infty}(A)$. Let $\varphi_n = E_{[0,n]}(A)\varphi$ and*

$$A_n = A + \alpha_n(\varphi_n, \cdot)\varphi_n.$$

Then:

(i) *If $\varphi \notin \mathscr{H}_{-2}(A)$, then for any choice of α_n, $(A_n - z)^{-1}$ converges to $(A-z)^{-1}$ strongly as $n \to \infty$ for any $z \in \mathbb{C}\backslash\mathbb{R}$.*

(ii) *If $\varphi \notin \mathscr{H}_{-1}(A)$ and $\alpha_n \geqslant 0$, then for any choice of α_n (subject to $\alpha_n \geqslant 0$), $(A_n - z)^{-1}$ converges to $(A-z)^{-1}$ strongly as $n \to \infty$ for any $z \in \mathbb{C}\backslash\mathbb{R}$.*

(iii) *If $\varphi \notin \mathscr{H}_{-1}(A)$ and $\alpha_n \to \alpha_\infty \neq 0$, then for any choice of α_n (subject to $\alpha_n \to \alpha_\infty$), $(A_n - z)^{-1}$ strongly to $(A-z)^{-1}$ as $n \to \infty$ for any $z \in \mathbb{C}\backslash\mathbb{R}$.*

Remarks. (1) Thus to obtain a non-trivial limit, we either need $\varphi \in \mathscr{H}_{-1}(A)$ or else $\varphi \in \mathscr{H}_{-2}(A)$ and α_n negative and infinitesimal.

(2) In cases (ii) and (iii), if $\varphi \in \mathscr{H}_{-2}(A)$, our proof shows norm convergence.

Proof. By general principles [9], weak convergence of resolvents implies strong convergence. Since the $\{(A_n - z)^{-1}\}$ are uniformly bounded for fixed $z \in \mathbb{C}\backslash\mathbb{R}$, it suffices to prove convergence of $(\psi_1, (A_n - z)^{-1}\psi_2)$ for $\psi_i \in \mathscr{H}_\infty$.

By (1.4b),

$$(A_n - z)^{-1} = (A-z)^{-1} - [\alpha_n^{-1} + (\varphi_n (A-z)^{-1}\varphi_n)]^{-1} \times ((A-\bar{z})^{-1}\varphi_n, \cdot)(A-z)^{-1}\varphi_n. \tag{5.1}$$

Since $(\psi, (A-z)^{-1}\varphi_n)$ is uniformly bounded if $\psi \in \mathscr{H}_{+\infty}(A)$ (since $\varphi \in \mathscr{H}_{-\infty}(A)$), strong convergence is equivalent to

$$|\gamma_n| \equiv |\alpha_n^{-1} + (\varphi_n, (A-z)^{-1}\varphi_n)| \to \infty.$$

Now

$$\operatorname{Im} \gamma_n = (\operatorname{Im} z)\, \|(A-z)^{-1}\varphi_n\|^2$$

goes to infinity as $n \to \infty$ if $\varphi \notin \mathscr{H}_{-2}$, so (i) is proven.

Suppose now $\varphi \in \mathscr{H}_{-2}$. Since

$$\operatorname{Re} \gamma_n = \alpha_n^{-1} + (\varphi_n, A[(A - \operatorname{Re} z)^2 + (\operatorname{Im} z)^2]^{-1} \varphi_n) - \operatorname{Re} z \, \|(A - z)^{-1} \varphi_n\|^2,$$

we see that if $\alpha_n > 0$ and $\varphi_n \notin \mathscr{H}_{-1}(A)$, then $\operatorname{Re} \gamma_n \to \infty$, and similarly if α_n^{-1} has a finite limit $\operatorname{Re} \gamma_n \to \infty$. ∎

Remark. Friedman [4] has shown that if V_n are functions on $\mathbb{R}^\nu$ with $\operatorname{supp} V_n \subset \{x \mid |x| < n^{-1}\}$ and $H_n = -\Delta + V_n$, then if $\nu \geqslant 2$, $H_n \to H$ in strong resolvent sense if $V_n \geqslant 0$ (irrespective of how big V_n is); and if $\nu \geqslant 4$, $H_n \to H$ with no positivity assumption. Note that $\delta_0 \in \mathscr{H}_{-\alpha}(-\Delta)$ if and only if $2\alpha > \nu$. Thus $\delta_0 \in \mathscr{H}_{-1}$ only if $\nu < 2$ and $\delta_0 \in \mathscr{H}_{-2}$ if and only if $\nu < 4$. We can therefore regard Theorem 5.1 as a kind of analog of Friedman's results.

6. Self-Adjoint Extensions

The punchline of this section is that rank one perturbations of $A \geqslant 0$ is really the same as the theory of self-adjoint extensions of deficiency indices $(1, 1)$ of a positive operator. From this point of view, the $\alpha = \infty$ operator found by Gesztesy–Simon [5] is exactly the Friedrich's extension.

Let $A \geqslant 0$ and $\varphi \in \mathscr{H}_{-2}(A)$. Whatever $A_\alpha = A + \alpha(\varphi, \cdot)\varphi$ is to mean $A_\alpha \psi$ should equal $A\psi$ if $(\varphi, \psi) = 0$. Thus, define

$$D_\varphi = \{\psi \in D(A) \mid (\varphi, \psi) = 0\}.$$

Since $\varphi \in \mathscr{H}_{-2}(A)$, (φ, ψ) is defined for $\psi \in D(A) = \mathscr{H}_{+2}(A)$.

Lemma. *Let $A_0 = A \restriction D_\varphi$ with domain D_φ. Then A_0 has deficiency indices $(1, 1)$.*

Proof. It suffices to prove that $\operatorname{Ran}(A_0 + 1)$ has codimension 1. But by definition, $\psi \in D_\varphi$ if and only if $(A + 1)\psi$ is orthogonal to $(A + 1)^{-1} \varphi$; that is, $\operatorname{Ran}(A_0 + 1) = \{(A + 1)^{-1} \varphi\}^\perp$ has codimension 1. ∎

The rank one perturbations are thus the self-adjoint extensions of A_0. Deficiency one extension of semibounded operators (and generally semibounded extensions of semibounded operators) have been studied extensively [3, 8, 13, 6, 2]. The result of this theory is that these are parametrized by a single parameter γ which runs in $(-\infty, \infty]$ with $+\infty$ allowed. They are best described in terms of quadratic forms. The operator $A^{(\infty)}$ is the Friedrich's extension and has form domain $Q(A^{(\infty)})$. There is a vector ξ defined by $(A_0 + 1)^* \xi = 0$ and for $\gamma \neq \infty$,

$$Q(A^{(\gamma)}) = Q(A^{(\infty)}) \dotplus \{\lambda \xi\}_{\lambda \in \mathbb{C}},$$

where $\dot{+}$ means disjoint sums and

$$((\psi + \lambda\xi), A^{(\gamma)}(\psi + \lambda\xi)) = (\psi, A^{(\infty)}\psi) + \lambda^2\gamma.$$

ξ is easily seen to be $(A+1)^{-1}\varphi$.

The original operator A is some $A^{(\gamma_0)}$. If $A = A^{(\gamma_0)}$ with $\gamma_0 \neq \infty$, then the $A^{(\gamma)}$ are precisely $\{A + c(\gamma - \gamma_0)(\varphi, \cdot)\varphi\}$ for a suitable constant c $(=(\varphi, (A+1)^{-1}\varphi))$. The $\gamma = \infty$ operator is exactly a Friedrich's extension.

If $\gamma_0 = \infty$, we see in this situation, where the other $A^{(\gamma)}$'s are obtained by the construction in Section 2.

7. Examples

Example 1. Take $A = -\Delta$ on $L^2(\mathbb{R}^\nu)$. We want to see what φ can be used for rank one perturbations defined at a single point 0. Since φ is supported at 0, $\varphi \in \mathscr{H}_{-\infty}(A)$ means φ is a distribution, so its Fourier transform is a polynomial P in p. For $\varphi \in \mathscr{H}_{-1}(A)$, we need

$$\int \frac{d^\nu p\, |P(p)|^2}{(p^2+1)} < \infty. \tag{7.1}$$

This can only happen if $\nu = 1$ and P has degree 0, that is, $\varphi = \delta(x)$. For φ to be in $\mathscr{H}_{-2}(A)$, we need the analog of (7.1) with (p^2+1) replaced by $(p^2+1)^2$. This allows P of degree 0 if $\nu = 2, 3$ and degree 1 if $\nu = 1$. Thus, the rank one theory works exactly for $\delta(x)$ in $\nu = 1, 2, 3$, and $\delta'(x)$ in $\nu = 1$. The $\mathscr{H}_{-2}(A)$ construction exactly corresponds to point interactions as discussed extensively (see [1, and Refs. therein]). Of course, our construction specialized to this case is just the standard one for point interactions; so our construction in Section 2 can be viewed as an abstraction of that method. One thing one can look at is undoing the point interaction in dimension 2 and 3. For concreteness, take $\nu = 3$. Then $\mathscr{H}_{+1}(\tilde{A}_\beta)$ is strictly bigger than $\mathscr{H}_{+1}(A)$. The extra functions have a Coulomb singularity at $x = 0$; that is, $\psi \in \mathscr{H}_{+1}(\tilde{A}_\beta)$ has the form

$$\psi(x) = ce^{-\mu|x|}|x|^{-1} + \tilde{\psi}$$

with $\tilde{\psi} \in \mathscr{H}_{+1}(-\Delta)$. μ is a convenient parameter; c is independent of μ. One can think of c as formally given by $\lim_{|x|\to 0} |x|\, \psi(x)$. Since ψ is not bounded, we cannot use that definition but can use

$$c(\psi) = \lim_{r\to 0} r\, \frac{3}{4\pi r^3} \int_{|x|\leq r} \psi(x)\, d^3x.$$

So c defines a vector $\varphi \in \mathscr{H}_{-1}(\tilde{A}_\beta)$ and the various $\tilde{A}_\beta$'s are just $\tilde{A}_{\beta_0} + \alpha(\varphi, \cdot)\varphi$ for $\alpha \in (-\infty, \infty)$. $\alpha = \infty$ recovers the original Laplacian.

EXAMPLE 2. Let A be $-d^2/dx^2$ on $L^2(0, \infty)$ with Neumann boundary condition at zero. Let $\varphi(x) = \delta(x) \in \mathscr{H}_{-1}(A)$. Then $A + \alpha(\varphi, \cdot)\varphi$ precisely corresponds to the boundary conditions

$$\sin(\theta)\, u'(0) + \cos(\theta)\, u(0) = 0,$$

where $\alpha = -\cot(\theta)$. $\alpha = \infty$ corresponds to Dirichlet boundary condition. The corresponding η as discussed in [5] is just $\delta'(x)$; that is, $\delta' \in \mathscr{H}_{-2}(A_\infty)$. The construction in Section 2 tells us how to reconstruct A_θ from A_∞.

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