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SOME SCHRODINGER OPERATORS WITH DENSE POINT SPECTRUM

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ABSTRACT. Given any sequence $\{E_n\}_{n=1}^{\infty}$ of positive energies and any monotone function $g(r)$ on $(0, \infty)$ with $g(0) = 1$, $\lim_{r \to \infty} g(r) = \infty$, we can find a potential $V(x)$ on $(-\infty, \infty)$ such that $\{E_n\}_{n=1}^{\infty}$ are eigenvalues of $-\frac{d^2}{dx^2} + V(x)$ and $|V(x)| \leq (|x| + 1)^{-1} g(|x|)$.

In [7], Naboko proved the following:

Theorem 1. *Let $\{\kappa_n\}_{n=1}^{\infty}$ be a sequence of rationally independent positive reals. Let $g(r)$ be a monotone function on $[0, \infty)$ with $g(0) = 1$, $\lim_{r \to \infty} g(r) = \infty$. Then there exists a potential $V(x)$ on $[0, \infty)$ such that*

(1) *$\{\kappa_n^2\}_{n=1}^{\infty}$ are eigenvalues of $-\frac{d^2}{dx^2} + V(x)$ on $[0, \infty)$ with $u(0) = 0$ boundary conditions.*
(2) $|V(x)| \leq \frac{g(x)}{(|x|+1)}$.

Our goal here is to construct V's that allow the proof of the following theorem:

Theorem 2. *Let $\{\kappa_n\}_{n=1}^{\infty}$ be a sequence of arbitrary distinct positive reals. Let $g(r)$ be a monotone function on $[0, \infty)$ with $g(0) = 1$ and $\lim_{r \to \infty} g(r) = \infty$. Let $\{\theta_n\}_{n=1}^{\infty}$ be a sequence of angles in $[0, \pi)$. Then there exists a potential $V(x)$ on $[0, \infty)$ such that*

(1) *For each n, $(-\frac{d^2}{dx^2} + V(x))u = \kappa_n^2 u$ has a solution which is L^2 at infinity and*

$$\frac{u'(0)}{u(0)} = \cot(\theta_n). \tag{1}$$

(2) $|V(x)| \leq \frac{g(x)}{|x|+1}$.

Remarks. 1. These results are especially interesting because Kiselev [6] has shown that if $|V(x)| \leq C(|x| + 1)^{-\alpha}$ with $\alpha > \frac{3}{4}$, then $(0, \infty)$ is the essential support of $\sigma_{\text{ac}}(-\frac{d^2}{dx^2} + V(x))$, so these examples include ones with dense point spectrum, dense inside absolutely continuous spectrum.

2. For whole line problems, we can take each $\theta_n = 0$ or $\frac{\pi}{2}$ and let $V_\infty(x) = V(|x|)$ and specify even and odd eigenvalues.

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3. For our construction, we'll have $|u_n(x)| \leq C_n(1+|x|)^{-1}$. By the same method, we could also specify $\{m_n\}_{n=1}^{\infty}$ so $|u_n(x)| \leq C_n(1+|x|)^{-m_n}$.

4. By the same method, if $\sum_{n=1}^{\infty} |\kappa_n| < \infty$, we can actually take $|V(x)| \leq C(1+|x|)^{-1}$, providing an answer to an open question of Eastham-Kalf [4, page 95]. If one takes our construction really seriously, one might conjecture that if $V(x) = O(|x|^{-1})$, then zero is the only possible limit point of the eigenvalues E_n and, indeed, even that

$$\sum_{n=1}^{\infty} \sqrt{E_n} < \infty.$$

5. One can probably extend Naboko's method to allow θ's so from a technical point of view, our result goes beyond his in that we show the rational independence condition is an artifact of his proof. The real point is to provide a different construction where the interesting examples of the phenomenon can be found.

Our construction is based on examples of the Wigner-von Neumann type [9]. They found a potential $V(x) = \frac{8\sin(2r)}{r} + O(r^{-2})$ at infinity and such that $-u'' + Vu = u$ has a solution of the form $\frac{\sin(r)}{r^2} + O(r^{-3})$ at infinity. In fact, our potentials will be of the form

$$V(x) = W(x) + \sum_{n=0}^{\infty} 4\kappa_n \chi_n(x) \frac{\sin(2\kappa_n x + \varphi_n)}{x} \tag{2}$$

where $\chi_n(x)$ is the characteristic function of the region $x > R_n$ for suitable large $R_n \to \infty$. Since R_n goes to infinity, the sum in (2) is finite for each x and there is no convergence issue. In (2), W will be a carefully constructed function on $[0,1]$ arranged to make sure that the phases θ_n at $x = 0$ come out right. We'll construct V as a limit of approximations

$$V_m(x) = W_m(x) + \sum_{n=0}^{m} 4\kappa_n \chi_n(x) \frac{\sin(2\kappa_n x + \varphi_n)}{x} \tag{3}$$

where W_m is supported on $[2^{-m}, 1]$ and equals W there. We'll make this construction such that:

(a) For $n \leq m$, $(-\frac{d^2}{dx^2} + V_m(x))u(x) = \kappa_n^2 u(x)$ has a solution $u_n^{(m)}(x)$ obeying $u \in L^2$ and condition (1).

(b)

$$\left| u_n^{(m)}(x) - \frac{\sin(\kappa_n x + \frac{1}{2}\varphi_n)}{1+x} \right| \leq C_n(1+x)^{-2} \tag{4}$$

for C_n uniformly bounded (in m but not in n!). Note in (4), the fact that $1/1+x$ appears (multiplying the sin) rather than, say, $1/(1+x)^2$ comes from the choice of 4 in $4\kappa_n$ in (3) (in general, if $4\kappa_n$ is replaced by γx_n, the decay is $r^{-\gamma/4}$).

Central to our construction is a standard oscillation result that can be easily proven using the method of Harris-Lutz [5] or the Dollard-Friedman method [2, 3] (see [8, problem 98 in Chapter XI]); results of this genre go back to Atkinson [1].

It will be convenient to introduce the norm

$$|||f||| = \|(1+x^2)f\|_\infty + \left\|(1+x^2)\frac{df}{dx}\right\|_\infty$$

for functions on $[0,\infty)$.

Theorem 3. *Fix $x > 0$. Let V_0 be a continuous function on $[0,\infty)$ such that*

$$V_0(x) = 4\kappa \sin(2\kappa x + \varphi_0)/|x|$$

for $x > R_0$ for some R_0. Let V_1, V_2 be two other continuous functions which obey

(i) $|V_i(x)| \leq C_1|x|^{-1}$,

(ii) $V_i(x) = \frac{dW_i}{dx}$ *where* $|W_i(x)| \leq C_2|x|^{-1}$,

(iii) $e^{\pm 2i\kappa x}V_i(x) = \frac{dW_i^{(\pm)}}{dx}$ *where* $|W_i^{\pm}(x)| \leq C_3|x|^{-1}$.

Let

$$V^{(R)} = \begin{cases} V_0(x) + V_1(x), & |x| < R, \\ V_0(x) + V_1(x) + V_2(x), & |x| > R, \end{cases}$$

with $V^{(\infty)}(x) = \lim_{R\to\infty} V^{(R)}(x)$. Then there exists a unique function $u^{(R)}(x)$ for $R \in [0,\infty]$ (including ∞) with ($u \equiv u^{(R)}$)

(a) $-u'' + V^{(R)}u = \kappa^2 u$,

(b) $|u(x) - \frac{\sin(\kappa x + \frac{1}{2}\varphi_0)}{1+|x|}| \leq C_4(1+x)^{-2}$ *and* $|u'(x) - \frac{\kappa\cos(\kappa x + \frac{1}{2}\varphi_0)}{1+|x|}| \leq C_5(1+x)^{-2}$.

In addition,

$$|||u^{(R)} - u^{(\infty)}||| \to 0 \tag{5}$$

as $R \to \infty$. Moreover, C_4, C_5, and the rate convergence in (5) *only depend on R_0, C_1, C_2, and C_3.*

Since this is a straightforward application of the methods of [5, 3], we omit the details.

The second input we'll need is the ability to undo small changes of Prüfer angles with small changes of potential. We'll need the following lemma:

Lemma 4. *Fix $k_1, \dots, k_n > 0$ distinct and $\theta_1^{(0)}, \dots, \theta_n^{(0)}$. Let*

$$f_j(x) = \sin^2(k_i x + \theta_i^{(0)}).$$

Fix $a < b$. Then $\{f_1, \dots, f_n\}$ are linearly independent on $[a,b]$.

Proof. Relabel so $0 < k_1 < k_2 < \cdots < k_n$. Suppose there is a dependency relation of the form $g(x) \equiv \sum_{i=1}^{n} \alpha_j f_j(x) \equiv 0$ on $[a,b]$. Without loss, we can suppose that $\alpha_n \neq 0$ (for otherwise, decrease n). Writing $\sin^2(y) = (e^{2iy} + e^{-2iy} - 2)/4$, we see that high order derivatives of $g(x)$ are dominated by the f_n term, so α_n must be zero after all. □

It will be convenient to use modified Prüfer angles, $\varphi(x)$, defined by

$$u'(x) = kR(x)\cos(\varphi); \quad u(x) = R(x)\sin(\varphi) \tag{6}$$

where u obeys $-u'' + V(x)u = k^2u(x)$. Then φ obeys

$$\frac{d\varphi}{dx} = k - k^{-1}V(x)\sin^2(\varphi(x)). \tag{7}$$

Explicitly, given $V(x)$ on $[0,b]$ and $\theta^{(0)}$, let $\varphi(x;\theta,V)$ solve the differential equation (7) on $[0,b]$ with initial condition $\varphi(0;\theta,V)=\theta^{(0)}$. Obviously,

$$\varphi(x;\theta,V\equiv 0)=kx+\theta. \tag{8}$$

Theorem 5. *Fix $[a,b]\subset(0,\infty)$, $k_1,\dots,k_n>0$ and distinct, and angles $\theta_1^{(0)},\dots,\theta_n^{(0)}$. Define $F:C[a,b]\to T^n$ (with T^n the n-torus) to be the generalized Prüfer angles $\varphi_i(b)$ solving (7) (with $k=k_i$ and $V(x)=0$ on $[0,a)$ and the argument of F on $[a,b]$) with $\varphi_i(0)=\theta_i^{(0)}$. Then for any ϵ, there is a δ such that for any $\theta_1^{(1)},\dots,\theta_n^{(1)}$ with*

$$|\theta_i^{(1)}-k_ib-\theta_i^{(0)}|<\delta,$$

there is a $V\in C[a,b]$ with $\|V\|_\infty<\epsilon$ and

$$F(V)=(\theta_1^{(1)},\dots,\theta_n^{(1)}).$$

Proof. $F(V=0)$ is $(\theta_1^{(0)}+k_1b,\dots,\theta_n^{(0)}+k_nb)$ by (8), so this theorem merely asserts that F takes a neighborhood of $V=0$ onto a neighborhood of $F(V=0)$. By the implicit function theorem, it suffices that the differential is surjective. But

$$\left.\frac{\delta F_i}{\delta V(x)}\right|_{V\equiv 0}=-\frac{1}{k_i}\,\sin^2(k_ix+\theta_i^{(0)})$$

by (7) and (8). By the lemma, this derivative is surjective. □

We now turn to the proof of Theorem 2. The overall strategy will be to use an inductive construction. We'll write

$$W(x)=\sum_{m=1}^{\infty}(\delta W_m)(x) \tag{9}$$

with δW_m supported on $[2^{-m},2^{-(m-1)}]$ so that the W_m of equation (3) is $W_m=\sum_{k=1}^{m}\delta W_k$. Then assuming we have V_{m-1}, we'll choose R_m, φ_m, δW_m in successive order, so

(1) R_m is so large that

$$|8\kappa_m\chi_m(x)|\le 2^{-m}g(x) \tag{10}$$

on all $(0,\infty)$, that is, $g(R_m)\ge 2^m(8\kappa_m)$.

(2) R_m is chosen so large that steps (3), (4) work.

(3) Let $u^{(0)}(x)$ solve $-u''+V_{m-1}u=\kappa_m^2u$ with $u'(0)/u(0)=\cot(\theta_m)$. We show that (so long as R_m is chosen large enough) we can pick φ_m so this u matches to the decaying solution guaranteed by Theorem 3.

(4) By choosing R_m large, we can be sure that $|||u_n^{(m-1)}-\tilde u_n^{(m)}|||\le 2^{-m-1}$ where $\tilde u_n^{(m)}$ obeys the equation for $V_m-\delta W_m$ and that the modified Prüfer angles for $\tilde u_n^{(m)}$ at $b_m=2^{-m+1}$ are within a range that can apply Theorem 5 with

$$[a,b]=[2^{-m},2^{-m+1}]$$

and $\epsilon<\frac{1}{2}$. By applying Theorem 5, we'll get δW_{m+1} to assure $u_n^{(m)}$ obeys the boundary conditions at zero.

Here are the formal details:

Proof of Theorem 2. Let

$$(\delta V_n)(x) = 4\kappa_n \chi_n(x) \frac{\sin(2\kappa_n x + \varphi_n)}{x} \tag{11}$$

where χ_n is the characteristic function of $[R_n, \infty)$ and φ_n, R_n are parameters we'll pick below. R_n will be picked to have many properties, among them

$$R_n \to \infty,\ R_n \geq 1, \qquad g(R_n) \geq 2^n(8\kappa_n). \tag{12}$$

δW_n will be a function supported on $[2^{-n}, 2^{-n+1})$ chosen later but obeying

$$\|\delta W_n\|_\infty \leq \frac{1}{2}. \tag{13}$$

We'll let

$$V_m(x) = \sum_{n=1}^{m} (\delta V_n + \delta W_n)(x)$$

and

$$V(x) = \lim_{m\to\infty} V_m(x)$$

where the limit exists since $V_m(x)$ is eventually constant for any x.

By (12), (13), we have

$$|V_m(x)| \leq g(x)/(|x|+1), \qquad m = 1, 2, \ldots, \infty. \tag{14}$$

For each m and each $n = 1, \ldots, m$, we have by Theorem 3 a unique function $u_n^{(m)}(x)$ obeying

$$-u'' + V_m u = \kappa_n^2 u, \tag{15}$$

$$|\!|\!| u - \sin((\kappa_n + \tfrac{1}{2}\varphi_n)\,\cdot\,)(1 + |\cdot|)^{-1} |\!|\!| < \infty. \tag{16}$$

We will choose $\delta V_n, \delta W_m$ so that

$$|\!|\!| u_n^{(m)} - u_n^{(m-1)} |\!|\!| \leq 2^{-m}, \qquad n = 1, 2, \ldots, m-1, \tag{17}$$

$$u_n^{(m)} \text{ obeys eqn. (1)}, \qquad n = 1, \ldots, m. \tag{18}$$

Let $u_n = |\!|\!|\cdot|\!|\!|\text{-}\lim_{m\to\infty} u_n^{(m)}$. Writing the differential equation as an integral equation, we see that u_n obeys $-u'' + V(u) = \kappa_n^2 u$. By (18), u_n obeys equation (1) and by $|\!|\!|\cdot|\!|\!|$ convergence, u_n obeys (16) and so lies in L^2. Thus as claimed, $-\frac{d^2}{dx^2} + V$ has $\{\kappa_n^2\}_{n=1}^\infty$ as eigenvalues.

Thus we are reduced to showing that $\delta V_m, \delta W_m$ can be chosen so that (17), (18) hold.

Let $\theta_i^{(0)}$ be defined by $\kappa_i \cot(\theta_i^{(0)}) = \cot(\theta_i)$ so $\theta_i^{(0)}$ are the generalized Prüfer angles associated to the originally specified Prüfer angles. Look at the solutions $u_i^{(n-1)}$, $i = 1, \ldots, m-1$. These match to the generalized Prüfer angles $\kappa_i 2^{-m+1} + \theta_i^{(0)}$ at $x = 2^{-m+1}$.

We'll choose δV_m so that the new solutions $\tilde{u}_i^{(m)}$ $(i = 1, \ldots, m-1)$ with δV_m added obey $|\!|\!| \tilde{u}_i^{(m)} - u_i^{(m-1)} |\!|\!| < 2^{-m-1}$. We can find ϵ_m such that if $\|\delta W_m\| < \epsilon_m$, then the new solutions $u_i^{(m)}$ obey $|\!|\!| u_i^{(m)} - \tilde{u}_i^{(m)} |\!|\!| < 2^{-m-1}$. So using Theorem 5, pick δ so small that the resulting V given is that theorem with $a = 2^{-m}, b = 2^{-m+1}$ has $\|\cdot\|$ bounded by $\min(\frac{1}{2}, \epsilon_n)$. In that theorem, use $\kappa_1, \ldots, \kappa_m$ and $\theta_i^{(0)}$, $i = 1, \ldots, m$.

According to Theorem 3, we can take R_m so large that uniformly in φ_m (in $[0, 2\pi/2\kappa_m]$), we have $\|u_i^{(m-1)} - \tilde{u}_i^{(m)}\| < 2^{-m-1}$ for $i = 1, \ldots, m-1$ and so large that again uniformly in φ_m, the generalized Prüfer angles $\theta_i^{(0)}$ for $\tilde{u}_i^{(m)}$ at $b_m \equiv 2^{-m+1}$ obey $|\theta_i^{(1)} - \theta_i^{(0)} - \kappa_i b_i| < \delta$ for $i = 1, \ldots, m-1$.

Thus, if we can pick the angle φ_m in (11) so that $\tilde{u}_m^{(m)}$ obeys the boundary condition at zero (and so $\theta_m^{(1)} - \theta_m^{(0)} - \kappa_m b_m = 0$), then the construction is done.

By condition (b) of Theorem 3, for $|x|$ large, as φ_m runs from 0 to $2\pi/2\kappa_m$, $(|x|u(x), |x|u'(x))$ runs through a complete half-circle. Thus, by taking R_m at least that large and choosing φ_m appropriately, we can match the angle of the solution of $u'' + V_{m-1}u = \kappa_m^2 u$ which obeys the boundary condition at $x = 0$. □

Note added in proof

A. Kiselev, Y. Last, and this author (paper in preparation) have shown that if $C = \overline{\lim}\, |x|\, |V(x)| < \infty$, then the positive eigenvalues $\{E_n\}_{n=1}^{\infty}$ obey $\sum E_n \leq \frac{C^2}{2}$. This partially answers the questions in Remark 4 after Theorem 2.

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