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The Lyapunov Exponents for Schrödinger Operators with Slowly Oscillating Potentials*

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By studying the integrated density of states, we prove the existence of Lyapunov exponents and the Thouless formula for the Schrödinger operator $-d^2/dx^2 + \cos x^\nu$ with $0 < \nu < 1$ on $L^2[0, \infty)$. This yields an explicit formula for these Lyapunov exponents. By applying rank one perturbation theory, we also obtain some spectral consequences. © 1996 Academic Press, Inc.

1. INTRODUCTION

Our goal in this paper is to prove Lyapunov behavior and compute a Lyapunov exponent for the one-dimensional half-line Schrödinger operator

$$H_\nu = -\frac{d^2}{dx^2} + \cos x^\nu \qquad x \in [0, \infty) \tag{1.1}$$

with $0 < \nu < 1$.

It is clear that H_ν is regular at 0 and is limit point at infinity. (For the definition of limit point, see [8, 15].) Therefore, for each $\theta \in [0, \pi)$, H_ν has a unique self-adjoint realization on $L^2[0, \infty)$ with boundary condition at 0 given by

$$u(0) \cos \theta + u'(0) \sin \theta = 0$$

which will be denoted by H_ν^θ.

In the spectral theory of Schrödinger operators, most work has concentrated on the potential $V(x)$, either $V(x) \to 0$ as $|x| \to \infty$ or $V(x)$ is periodic or almost periodic. Such models have been investigated particularly well. Comparatively new are the models with oscillating but not periodic or almost periodic potentials. Due to recent discoveries of Behncke [2],

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Kirsch *et al.* [6], and Stolz [12, 13], it is clear that some such models may yield very interesting spectra. As one of his particular examples, Stolz has studied the spectral properties for (1.1) in [13]. Let $\sigma(H)$, $\sigma_{\mathrm{ac}}(H)$, $\sigma_{\mathrm{sing}}(H)$, $\sigma_{\mathrm{sc}}(H)$, and $\sigma_{\mathrm{pp}}(H)$ denote the spectrum, absolutely continuous spectrum, singular spectrum, singular continuous spectrum, and pure point spectrum resp. for H. Then from Stolz's paper, we know that $\sigma(H_\nu) = [-1, \infty)$, $\sigma_{\mathrm{ac}}(H_\nu) = [1, \infty)$, and $\sigma_{\mathrm{sing}}(H_\nu) = [-1, 1]$. In fact, from an unpublished result of Kirsch and Stolz (see [6]), we also know that H_ν^θ has pure point spectrum in $[-1, 1]$ for almost all boundary conditions θ.

We already see that this model has some subtle and fascinating spectral properties, especially for $E \in (-1, 1)$. We will continue working on this model. In particular, we will prove Lyapunov behavior and compute a Lyapunov exponent formula.

We know that the Lyapunov exponent is an important tool in the spectral theory for one-dimensional Schrödinger operators with almost periodic or random potentials. In [10, 11], the rank one perturbation theory shows that Lyapunov behavior can also be used to study Schrödinger operators with deterministic potentials. For almost periodic or random potentials, we have the subadditive ergodic theorem to guarantee the existence of the Lyapunov exponent; but for deterministic potentials, it's often difficult to prove Lyapunov behavior. In this paper, we first study the integrated density of states in detail, then we directly study the existence of the Lyapunov exponent and prove the Thouless formula for a.e. E (with Lebesgue measure).

Now, our formula for $\gamma(E)$, $E \in (-1, 1)$, which we prove as an explicitly given set of measure 0, is strictly positive. It is known (see [3]) that since $(-1, 1) \subset \sigma(H_\nu)$, the complement of $\{E \mid \gamma(E)$ exists and is $>0\}$ is a dense G_δ in $[-1, 1]$. By our construction, this dense G_δ has measure zero; indeed, it has Hausdorff dimension zero.

We are unaware of any other explicit (nonrandom) Schrödinger operators with a computable positive Lyapunov exponent. The explicit formula (3.22) is quasiclassical.

2. THE INTEGRATED DENSITY OF STATES

To prove the Thouless formula, we need to study the integrated density of states, $k(E)$, and the existence of the Lyapunov exponent. Also, we need information on how rapidly $k^{(l)}(E)$ converges to $k(E)$ to establish the existence of the Lyapunov exponent. So, we first study the main technical object, the integrated density of states for equation (1.1). We will prove a formula for the integrated density states, and more importantly, we will estimate how fast $k^{(l)}(E)$ converges to $k(E)$.

The basic idea to compute the integrated density of states uses the standard Dirichlet–Neumann bracketing technique. Since the potentials in our problem are slowly oscillating, Dirichlet–Neumann bracketing works perfectly.

First, let us introduce some notation and definitions. In the following, when we write H_ν, we always mean the Schrödinger operator given by (1.1). Define

$$L = S_\nu(l) = (2\pi l)^{1/\nu}, \qquad \Omega_l = [S_\nu(l-1), S_\nu(l)], \qquad \text{for} \quad l = 1, 2, \ldots.$$

Ω_l is the lth potential well for the potential $V(x) = \cos x^\nu$ $(0 < \nu < 1)$. Let $H_D(\Omega)$ (resp. $H_N(\Omega)$) denote the self-adjoint operator $H_0 + V(x)$ on $L^2(\Omega)$ with Dirichlet (resp. Neumann) boundary conditions, where $H_0 = -\Delta$. When $\Omega = (0, L)$, we use $H_D(L)$ (resp. $H_N(L)$) to denote $H_D(\Omega)$, (resp. $H_N(\Omega)$). In this case, we use $H_{DN}(L)$ (resp. $H_{ND}(L)$) to denote the self-adjoint operator $H_0 + V(x)$ on $L^2(0, L)$ with Dirichlet (resp. Neumann) boundary condition at 0 and Neumann (resp. Dirichlet) boundary condition at L.

Definition. For any self-adjoint operator A, define

$$N(E, A) = \dim P_{(-\infty, E)}(A) = \sum_{E_k < E} 1,$$

where $P_\Omega(A)$ is the spectral projection for the operator A, and $\{E_k\}$ are the eigenvalues of A with $E_1 \leqslant E_2 \leqslant E_3 \leqslant \cdots$.

Now, let $H_{\mathrm{bc}}(S_\nu(l))$ be any self-adjoint realization of H_ν on $L^2(0, S(l))$ with some given boundary conditions at 0 and $S_\nu(l)$. Let $N_{\mathrm{bc}}(E, l) = N(E, H_{\mathrm{bc}}(S_\nu(l)))$.

Definition. Let $N_{\mathrm{bc}}(E, l)$ be as above, then we define

$$k^{(l)}(E) = \frac{1}{S_\nu(l)} N_{\mathrm{bc}}(E, l) \qquad \text{and} \qquad k(E) = \lim_{l \to \infty} k^{(l)}(E).$$

$k(E)$ is called the integrated density of states for (1.1).

We will show that in the above definition, the limit $k(E)$ exists and is independent of the choice of boundary conditions.

By standard Dirichlet–Neumann bracketing (see [9]),

$$\sum_{j=1}^{l} N(E, H_D(\Omega_j)) \leqslant N(E, H_D(L)) \leqslant N(E, H_N(L)) \leqslant \sum_{j=1}^{l} N(E, H_N(\Omega_j)). \tag{2.1}$$

By explicit construction and counting in boxes, we have

LEMMA 2.1. *If we let $N_D(E; a, b)$ (resp. $N_N(E; a, b)$) denote the dimension of the spectral projection $P_{(-\infty, E]}$ for $-\Delta_D$ (resp. $-\Delta_N$) on $L^2(a, b)$. Then for $E<0$, we have*

$$N_D(E; a, b) = N_N(E; a, b) = 0 \tag{2.2}$$

and for $E \geqslant 0$, we have

$$\left| N_D(E; a, b) - \frac{\sqrt{E}}{\pi}(b-a) \right| \leqslant 1 \tag{2.3}$$

$$\left| N_N(E; a, b) - \frac{\sqrt{E}}{\pi}(b-a) \right| \leqslant 1. \tag{2.4}$$

First, let us estimate $N(E, H_D(\Omega_j))$ and $N(E, H_N(\Omega_j))$. Let $a_k \in \Omega_j$ and $b_k = a_{k+1}$ such that $\bigcup_k [a_k, b_k] = \Omega_j$ and $b_k - a_k = j^\alpha$, where $\alpha > 0$ (depending on v) will be determined later. Let $I_k^{(j)} = (a_k, b_k)$ and

$$V_k^D = \sup\{ V(x) \mid x \in [a_k, b_k]\}, \qquad V_k^N = \inf\{ V(x) \mid x \in [a_k, b_k]\}.$$

Define $B_D(I_k^{(j)}) = -\Delta_D(I_k^{(j)}) + V_k^D$ and $B_N(I_k^{(j)}) = -\Delta_N(I_k^{(j)}) + V_k^N$, then

$$0 \leqslant H_D(I_k^{(j)}) \equiv -\Delta_D(I_k^{(j)}) + V(x) \leqslant B_D(I_k^{(j)})$$

and

$$0 \leqslant B_N(I_k^{(j)}) \leqslant -\Delta_N(I_k^{(j)}) + V(x) \equiv H_N(I_k^{(j)}).$$

Obviously,

$$N(E, B_D(I_k^{(j)})) \leqslant N(E, H_D(I_k^{(j)})), \qquad N(E, H_N(I_k^{(j)})) \leqslant N(E, B_N(I_k^{(j)}))$$

and by Dirichlet–Neumann bracketing,

$$\begin{aligned} N(E, H_D(\Omega_j)) &\geqslant N\left(E, H_D\left(\bigcup I_k^{(j)}\right)\right) = \sum_k N(E, H_D(I_k^{(j)})) \\ &\geqslant \sum_k N(E, B_D(I_k^{(j)})) \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} N(E, H_N(\Omega_j)) &\leqslant N\left(E, H_N\left(\bigcup I_k^{(j)}\right)\right) = \sum_k N(E, H_N(I_k^{(j)})) \\ &\leqslant \sum_k N(E, B_N(I_k^{(j)})). \end{aligned} \tag{2.6}$$

So, we only need to estimate $N(E, B_N(I_k^{(j)}))$ and $N(E, B_D(I_k^{(j)}))$. But by (2.2) and (2.4),

$$N(E, B_N(I_k^{(j)})) = N_N(E - V_k^N; a_k, b_k)$$
$$= \begin{cases} \dfrac{\sqrt{E - V_k^N}}{\pi}(b_k - a_k) + C_0(k), & \text{if } \; E \geqslant V_k^N, \\ 0, & \text{if } \; E < V_k^N \end{cases}$$

where $|C_0(k)| \leqslant 1$.

Thus, if we use the notation that $[f(x)]_+ = \max\{0, f(x)\}$, then we have

$$\left| N(E, B_N(I_k^{(j)})) - \frac{[E - V_k^N]_+^{1/2}}{\pi}(b_k - a_k) \right| \leqslant 1. \tag{2.7}$$

But

$$\frac{1}{\pi}[E - V_k^N]_+^{1/2}(b_k - a_k) - \frac{1}{\pi}\int_{a_k}^{b_k}[E - V(x)]_+^{1/2}\,dx$$
$$= \frac{1}{\pi}\int_{a_k}^{b_k}\{[E - V_k^N]_+^{1/2} - [E - V(x)]_+^{1/2}\}\,dx \overset{\text{def}}{=} J. \tag{2.8}$$

Since

$$\{[E - V_k^N]_+^{1/2} - [E - V(x)]_+^{1/2}\}^2$$
$$\leqslant |[E - V_k^N]_+^{1/2} - [E - V(x)]_+^{1/2}|\,\{[E - V_k^N]_+^{1/2} - [E - V(x)]_+^{1/2}\}$$
$$\leqslant \frac{v}{a_k^{1-v}}(b_k - a_k) \qquad \text{for} \quad x \in I_k^{(j)}$$

by Schwartz inequality, we have,

$$|J| \leqslant \frac{1}{\pi}(b_k - a_k)^{1/2}\left[\int_{a_k}^{b_k}\{[E - V(x)]_+^{1/2} - [E - V_k^N]_+^{1/2}\}^2\,dx\right]^{1/2}$$
$$\leqslant \frac{\sqrt{v}}{\pi}\,a_k^{-1/2(1-v)}(b_k - a_k)^{3/2}$$
$$\leqslant j^{(3/2)\,\alpha - (1/2)((1-v)/v)}. \tag{2.9}$$

Therefore, by (2.7)–(2.9), we have

$$\left| N(E, B_N(I_k^{(j)})) - \frac{1}{\pi}\int_{a_k}^{b_k}[E - V(x)]_+^{1/2}\,dx \right| \leqslant j^{(3/2)\,\alpha - (1/2)((1-v)/v)} + 1.$$

Thus, by summing over k and using (2.6), we have

$$N(E, H_N(\Omega_j)) \leqslant \frac{1}{\pi} \int_{S_\nu(l-1)}^{S_\nu(l)} [E - V(x)]_+^{1/2}\, dx + C_1 j^{(1/2)(\alpha + ((1-\nu)/\nu))} + C_2 j^{((1-\nu)/\nu) - \alpha} \tag{2.10}$$

where C_1 and C_2 are independent of j.

Similarly, if we use (2.3) and (2.5) instead of (2.4) and (2.6), then we have

$$N(E, H_D(\Omega_j)) \geqslant \frac{1}{\pi} \int_{S_\nu(l-1)}^{S_\nu(l)} [E - V(x)]_+^{1/2}\, dx - C_1 j^{(1/2)(\alpha + ((1-\nu)/\nu))} - C_2 j^{((1-\nu)/\nu) - \alpha} \tag{2.11}$$

Now, by summing over j in (2.10), (2.11) and using (2.1), we have

$$\begin{aligned} &\frac{1}{\pi} \int_0^{S_\nu(l)} [E - V(x)]_+^{1/2}\, dx - C_1 j^{(1/2)(\alpha + ((1-\nu)/\nu)) + 1} - C_2 j^{((1-\nu)/\nu) - \alpha + 1} \\ &\quad \leqslant N(E, H_D(L)) \leqslant N(E, H_N(L)) \\ &\quad \leqslant \frac{1}{\pi} \int_0^{S_\nu(l)} [E - V(x)]_+^{1/2}\, dx + C_1 j^{(1/2)(\alpha + ((1-\nu)/\nu)) + 1} + C_2 j^{((1-\nu)/\nu) - \alpha + 1}. \end{aligned}$$

So, if we take $\alpha = \frac{1}{3}(1-\nu)/\nu$, then we have

$$\begin{aligned} &\frac{1}{\pi} \int_0^{S(l)} [E - V(x)]_+^{1/2}\, dx - Cl^{(2/3)((1-\nu)/\nu) + 1} \\ &\quad \leqslant N(E, H_D(L)) \leqslant N(E, H_N(L)) \\ &\quad \leqslant \frac{1}{\pi} \int_0^{S(l)} [E - V(x)]_+^{1/2}\, dx + Cl^{(2/3)((1-\nu)/\nu) + 1} \end{aligned} \tag{2.12}$$

where $C = C_1 + C_2$.

Also, we have the following estimate:

$$\begin{aligned} &\frac{1}{S_\nu(l)} \int_0^{S_\nu(l)} [E - V(x)]_+^{1/2}\, dx \\ &\quad = \frac{1}{\nu(2\pi l)^{1/\nu}} \int_0^{2\pi l} y^{(1-\nu)/\nu} [E - \cos y]_+^{1/2}\, dy \qquad (x^\nu = y) \\ &\quad = \frac{1}{\nu(2\pi l)^2} \sum_{k=1}^{l} \int_{-2\pi}^{0} (z + 2k\pi)^{(1-\nu)/\nu} [E - \cos z]_+^{1/2}\, dz \qquad (y = z + 2k\pi) \\ &\quad = \frac{1}{2\pi} \int_{-\pi}^{\pi} [E - \cos x]_+^{1/2}\, dx + O(l^{-1}). \end{aligned}$$

Thus, if we denote

$$k_N^{(l)}(E) \equiv \frac{1}{S(l)} N(E, H_N(L)),$$

then by the above estimate and (2.12), we have

$$\left| k_N^{(l)}(E) - \frac{1}{2\pi^2} \int_{-\pi}^{\pi} [E - \cos x]_+^{1/2} \, dx \right| = O(l^{-(1/3)((1-\nu)/\nu)}) + O(l^{-1}). \quad (2.13)$$

Since variations of boundary condition are rank one perturbation (see [10]),

$$|N(E, H_N(L)) - N(E, H_{\text{bc}}(L))| \leqslant 2, \quad (2.14)$$

where $H_{\text{bc}}(L)$ is defined by any other self-adjoint boundary condition.

Thus by (2.13) and (2.14), we have proved the following:

THEOREM 2.2. *The integrated density of states for the Schrödinger operator* (2.1) *exists, which is independent of the boundary conditions, and is given by*

$$k(E) = \frac{1}{2\pi^2} \int_{-\pi}^{\pi} [E - \cos x]_+^{1/2} \, dx.$$

Moreover, we have the following estimate:

$$|k^{(l)}(E) - k(E)| = O(l^{-\kappa(\nu)}) \quad (2.15)$$

where

$$\kappa(\nu) = \min \left\{ \frac{1}{3} \frac{(1-\nu)}{\nu}, 1 \right\}. \quad (2.16)$$

3. THE THOULESS FORMULA AND LYAPUNOV EXPONENT

Now, we begin to study the Lyapunov exponent by first proving the Thouless formula which relates the Lyapunov exponent to the integrated density of states. In [1], the Thouless formula is proved for almost periodic potentials and random potentials. To prove the Thouless formula in our case, we can closely follow the proof given in [1] for Schrödinger operators. However, we will prove the existence of the Lyapunov exponent by using the information on how fast $k^l(E)$ converges to $k(E)$ which is given in Theorem 2.1.

First, we define the transfer matrix for the Schrödinger operator (1.1) as follows. Let $u(x, a, E)$, $v(x, a, E)$ $(x \geqslant 0,\ a \geqslant 0)$ solve the equation $-\phi'' + (V(x) - E)\,\phi = 0$ with the boundary conditions given by $u(a) = 0$, $u'(a) = 1$; $v(a) = 1$, $v'(a) = 0$. Then the transfer matrix is defined by

$$T_{a,x}(E) = \begin{pmatrix} v(x, a, E) & u(x, a, E) \\ \dfrac{\partial v(x, a, E)}{\partial x} & \dfrac{\partial u(x, a, E)}{\partial x} \end{pmatrix}. \tag{3.1}$$

In particular, when $a = 0$, we use $T_x(E)$ to denote $T_{0,x}(E)$.

Definition. For a given E, if $\gamma(E) = \lim_{x \to \infty} x^{-1} \ln \|T_x(E)\|$ exists, then we say that for the energy E, H has Lyapunov behavior, and $\gamma(E)$ is called the Lyapunov exponent.

To give the Thouless formula, we need to define the resonance set first. In Section 2, we defined the operators $H_D(L)$, $H_N(L)$, $H_{DN}(L)$, and $H_{ND}(L)$. Now, let $\{E_k(l, D)\}$, $\{E_k(l, N)\}$, $\{E_k(l, DN)\}$, and $\{E_k(l, ND)\}$ be the corresponding eigenvalues.

Definition. For each given $\nu \in (0, 1)$, let ε_ν be a fixed small number such that $\varepsilon_\nu < \kappa(\nu)$, where $\kappa(\nu)$ is defined by (2.16). Then the resonance set, R_ν, for the operator H_ν is defined by

$$R_\nu = R_D \cup R_N \cup R_{DN} \cup R_{ND}, \tag{3.2}$$

where

$$R_D = \bigcup_{d=1}^{\infty} \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \bigcup_{k} \{E \in [-d, d] \mid |E - E_k(n, D)| < \exp(-n^{\kappa(\nu) - \varepsilon_\nu})\}. \tag{3.3}$$

R_N, R_{DN}, and R_{ND} are defined by replacing $\{E_k(l, D)\}$ in (3.3) by $\{E_k(l, N)\}$, $\{E_k(l, DN)\}$, and $\{E_k(l, ND)\}$ resp.

Remark. We conjecture that instead of (3.2) and (3.3), the resonance set in $[-1, 1]$ can be defined by

$$R_\nu = \bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} \bigcup_{k} \{E \in [-1, 1] \mid |E - E_k^{(n)}| < \exp(-n^{\min\{(1-\nu/2\nu),\, 1/2\}})\},$$

where $\{E_k^{(n)}\}$ are the eigenvalues of $H_\nu = H_0 + V(x)$ on the nth potential well, $[(2n\pi - 2\pi)^{1/\nu}, (2n\pi)^{1/\nu}]$, with Dirichlet boundary conditions. We believe that this is the reasonable definition of the resonance set. However, in our proof of the Thouless formula, we need to use the resonance set defined by (3.2) and (3.3).

From the definition, it is easy to show that

THEOREM 3.1. *Let R_v be the resonance set for H_v which is defined by* (3.2) *and* (3.3) *and let* $\dim_H$ *denote the Hausdorff dimension. Then*

$$|R_v| = \dim_H R_v = 0.$$

Now, we are ready to prove one of our main results.

THEOREM 3.2 (Thouless formula). *Let H_v be the Schrödinger operator given by* (1.1). *Let $\gamma_0(E) = [\max(0, -E)]^{1/2}$ and $k_0(E) = \pi^{-1}[\max(0, E)]^{1/2}$. Then for any $E \notin R_v$, where R_v is defined by* (3.2) *and* (3.3), *we have*

$$\gamma(E) = \gamma_0(E) + \int_{-\infty}^{\infty} \ln |E - E'| \, d(k - k_0)(E'), \tag{3.4}$$

where $\gamma(E)$ is the Lyapunov exponent for H_v, and $k(E)$ is the integrated density of states for H_v.

We prove this theorem by proving the following series of lemmas. The first three lemmas are already given in [1], so we will not give a proof for these results here.

LEMMA 3.3 [1]. *For a.e. E,*

$$\lim_{l \to \infty} l^{-1} \ln |u_0(l, E)| = \gamma_0(E), \tag{3.5}$$

the limit being through the integers.

LEMMA 3.4 [1]. *Let $E_k(l)$ be the eigenvalue of H_v on $L^2[0, S_v(l)]$ with vanishing boundary conditions, and let $E_k^{(0)}(l) = (\pi k / S_v(l))^2$ be the corresponding eigenvalue of H_0. Then*

$$|E_k(l) - E_k^{(0)}| \leqslant \|V\|_\infty = 1. \tag{3.6}$$

LEMMA 3.5 [1]. *For fixed l, we have that*

$$\frac{u(S_v(l), E)}{u_0(S_v(l), E)} = \prod_{k=1}^{\infty} \left[\frac{E - E_k(l)}{E - E_k^{(0)}(l)} \right]. \tag{3.7}$$

From [1], we also know that

$$\lim_{M\to\infty}\left[\int_{k(E')\leqslant M}\ln|E-E'|\,dk(E')-\int_{k_0(E')\leqslant M}\ln|E-E'|\,dk_0(E')\right]$$
$$=\int_{-\infty}^{\infty}\ln|E-E'|\,d(k-k_0)(E'). \tag{3.8}$$

LEMMA 3.6. *For $E\notin R_D$, we have*

$$\lim_{l\to\infty}\frac{1}{S_\nu(l)}\ln\prod_{k=1}^{\infty}\left|\frac{E-E_k(l)}{E-E_k^{(0)}(l)}\right|=\int_{-\infty}^{\infty}\ln|E-E'|\,d(k-k_0)(E'). \tag{3.9}$$

Proof. For a given $E\notin R_D$, without loss of generality, we can also suppose that $E\notin R_D^{(0)}$, where $R_D^{(0)}$ is the corresponding resonance set for H_0 with Dirichlet boundary condition. From now on, we always suppose that E is fixed and $E\notin R_D\cup R_D^{(0)}$.

For each fixed E, we can choose $M(l)$ such that $M(l)\to\infty$ as $l\to\infty$ and $a_i(l)>E+1$ $(i=0,1)$, where

$$a_0(l)=\sup\{E'\mid k_0^{(l)}(E')\leqslant M(l)\},\qquad a_1(l)=\sup\{E'\mid k^{(l)}(E')\leqslant M(l)\}.$$

For convenience, we define

$$f_l(E)=\frac{1}{S_\nu(l)}\ln\prod_{k=1}^{\infty}\left|\frac{E-E_k(l)}{E-E_k^{(0)}(l)}\right|,\qquad f(E)=\int_{-\infty}^{\infty}\ln|E-E'|\,d(k-k_0)(E').$$

Then we have

$$\begin{aligned}|f_l(E)-f(E)|=&\left|\frac{1}{S_\nu(l)}\ln\prod_{k\leqslant M(l)\,S_\nu(l)}|[E-E_k(l)]/[E-E_k^{(0)}(l)]|\right.\\&\left.+\frac{1}{S_\nu(l)}\ln\prod_{k>M(l)\,S_\nu(l)}|[E-E_k(l)]/[E-E_k^{(0)}(l)]|-f(E)\right|\\\leqslant&\left|\int_{-\infty}^{a_1(l)}\ln|E-E'|\,d(k^{(l)}-k)(E')\right.\\&\left.-\int_{-\infty}^{a_0(l)}\ln|E-E'|\,d(k_0^{(l)}-k_0)(E')\right|\\&+\left|\int_{a_1(l)}^{\infty}\ln|E-E'|\,dk(E')-\int_{a_0(l)}^{\infty}\ln|E-E'|\,dk_0(E')\right|\\&+\left|\frac{1}{S_\nu(l)}\ln\prod_{k>M(l)\,S_\nu(l)}|[E-E_k(l)]/[E-E_k^{(0)}(l)]|\right|.\end{aligned} \tag{3.10}$$

By (3.8), we have

$$\lim_{l \to \infty} \left| \int_{a_1(l)}^{\infty} \ln |E - E'| \, dk(E') - \int_{a_0(l)}^{\infty} \ln |E - E'| \, dk_0(E') \right| = 0. \quad (3.11)$$

Since $E_k^{(0)}(l) = (\pi k / S_\nu(l))^2$, by using Lemma 3.4, we have

$$\begin{aligned} \ln \prod_{k > M(l)\, S_\nu(l)} \left| \frac{E - E_k(l)}{E - E_k^{(0)}(l)} \right| &= \sum_{k > M(l)\, S_\nu(l)} \ln \left| 1 + \frac{E_k(l) - E_k^{(0)}}{E_k^{(0)}(l) - E} \right| \\ &\leqslant \sum_{k > M(l)\, S_\nu(l)} S_\nu^2(l) / [\pi^2 k^2 - S_\nu^2(l)\, E] \\ &\leqslant S_\nu(l) \int_{M(l)}^{\infty} \frac{dx}{\pi^2 x^2 - E}. \end{aligned}$$

Therefore,

$$\left| \frac{1}{S_\nu(l)} \ln \prod_{k > M(l)\, S_\nu(l)} |[E - E_k(l)] / [E - E_k^{(0)}(l)]| \right| = O\left(\frac{1}{M(l)} \right). \quad (3.12)$$

So, it remains to estimate

$$J_l \equiv \left| \int_{-\infty}^{a_1(l)} \ln |E - E'| \, d(k^{(l)} - k)(E') - \int_{-\infty}^{a_0(l)} \ln |E - E'| \, d(k_0^{(l)} - k_0)(E') \right|.$$

We define

$$I_l(E) = [E - \delta_l, E + \delta_l], \qquad \delta_l = \tfrac{1}{3} \exp(-l^{\kappa(\nu) - \varepsilon_\nu}), \quad (3.13)$$

where $\kappa(\nu)$ is defined by (2.16) and ε_ν is given in the definition of the resonance set.

Since $E \notin R_D \cup R_D^{(0)}$, there are no eigenvalues of $H_D(L)$ and $H_{0D}(L)$ on the interval $I_l(E)$ which is defined by (3.13). Thus, $k^{(l)}(E)$, $k_0^{(l)}(E)$ are constant on the interval $I_l(E)$. Also, we notice that

$$\left| \int_{I_l(E)} \ln |E - E'| \, dk(E') \right| \leqslant C_E [|I_l(E)|]^{1/2} \quad (3.14)$$

where C_E is a constant for a given E. So, we have

$$\begin{aligned} J_l = \Bigg| & \int_{(-\infty,\, a_1(l)] \setminus I_l(E)} \ln |E - E'| \, d(k^{(l)} - k)(E') \\ & + \int_{I_l(E)} \ln |E - E'| \, d(k^{(l)} - k)(E') \end{aligned}$$

$$-\left|\int_{(-\infty,\,a_0(l)]\setminus I_l(E)} \ln|E-E'|\, d(k_0^{(l)}-k_0)(E')\right.$$

$$\left.-\int_{I_l(E)} \ln|E-E'|\, d(k_0^{(l)}-k_0)(E')\right|$$

$$\leqslant \left|\int_{(-\infty,\,a_1(l)]\setminus I_l(E)} \ln|E-E'|\, d(k^{(l)}-k)(E')\right|$$

$$+\left|\int_{I_l(E)} \ln|E-E'|\, dk(E')\right|$$

$$+\left|\int_{(-\infty,\,a_0(l)]\setminus I_l(E)} \ln|E-E'|\, d(k_0^{(l)}-k_0)(E')\right|$$

$$+\left|\int_{I_l(E)} \ln|E-E'|\, dk_0(E')\right|. \tag{3.15}$$

By (3.14), we know that

$$\lim_{l\to\infty} \int_{I_l(E)} \ln|E-E'|\, dk(E')=0. \tag{3.16}$$

Similarly,

$$\lim_{l\to\infty} \int_{I_l(E)} \ln|E-E'|\, dk_0(E')=0. \tag{3.17}$$

Using integration by parts, we have

$$\left|\int_{(-\infty,\,a_1(l)]\setminus I_l(E)} \ln|E-E'|\, d(k^{(l)}-k)(E')\right|$$

$$\leqslant (k^{(l)}-k)(a_1(l)) \ln|E-a_1(l)|$$

$$+\{(k^{(l)}-k)(E+\delta_l)-(k^{(l)}-k)(E-\delta_l)\} \ln \delta_l$$

$$+\left|\int_{(-\infty,\,a_1(l)]\setminus I_l(E)} \frac{(k^{(l)}-k)(E')}{E'-E}\, dE'\right|.$$

By Theorem 2.1 and (3.13), we know that

$$\lim_{l\to\infty} (k^{(l)}-k)(a_1(l)) \ln|E-a_1(l)|=0$$

$$\lim_{l\to\infty} \{(k^{(l)}-k)(E+\delta_l)-(k^{(l)}-k)(E-\delta_l)\} \ln \delta_l=0$$

and

$$\left|\int_{(-\infty,\, a_1(l)]\setminus I_l(E)} \frac{(k^{(l)}-k)(E')}{E'-E}\, dE'\right| \leqslant C_1 l^{-\kappa(v)} \left|\int_{(-\infty,\, a_1(l)]\setminus I_l(E)} \frac{1}{E'-E}\, dE'\right|$$
$$\leqslant l^{-\kappa(v)}\{C_2 \ln \delta_l + C_3 \ln |a_1(l)-E|\}$$
$$\to 0 \qquad \text{as} \quad l \to \infty.$$

Thus,

$$\lim_{l\to\infty} \left|\int_{(-\infty,\, a_1(l)]\setminus I_l(E)} \ln |E-E'|\; d(k^{(l)}-k)(E')\right| = 0. \tag{3.18}$$

Similarly,

$$\lim_{l\to\infty} \left|\int_{(-\infty,\, a_0(l)]\setminus I_l(E)} \ln |E-E'|\; d(k_0^{(l)}-k_0)(E')\right|. \tag{3.19}$$

So by (3.15)–(3.19),

$$\lim_{l\to\infty} J_l = 0. \tag{3.20}$$

Now, by (3.10)–(3.12) and (3.20), we have proved that $\lim_{l\to\infty} |f_l(E) - f(E)| = 0$. Therefore, Lemma 3.6 is proved. ▮

Now, by combining the results of Lemma 3.5 and Lemma 3.6, we have proved the following result:

Then for $E \notin R_D$, we have that

$$\lim_{l\to\infty} \frac{1}{S_v(l)} \ln \left|\frac{u(S_v(l), E)}{u_0(S_v(l), E)}\right| = \int_{-\infty}^{\infty} \ln |E-E'|\; d(k-k_0)(E').$$

By using Lemma 3.3, we obtain the following control on the limit

$$\lim_{l\to\infty} \frac{1}{S_v(l)} \ln |u(S_v(l), E)| = \gamma_0(E) + \int_{-\infty}^{\infty} \ln |E-E'|\; d(k-k_0)(E').$$

By using different boundary conditions, we can obtain similar control of $(1/S_v(l)) \ln |v(S_v(l), E)|$, $(1/S_v(l)) \ln |\partial u(S_v(l), E)/\partial x|$, and $(1/S_v(l)) \ln |\partial v(S_v(l), E)/\partial x|$. Therefore, we obtain control of $(1/S(l)) \ln \|T_{S(l)}(E)\|$, namely, LEMMA 3.7. *For* $E \notin R_\nu$, *where* R_ν *is the resonance set defined by* (3.2) *and* (3.3), *then*

$$\lim_{l \to \infty} \frac{1}{S(l)} \ln \|T_{S(l)}(E)\| = \gamma_0(E) + \int_{-\infty}^{\infty} \ln |E - E'| \, d(k - k_0)(E'), \tag{3.21}$$

where $\|\cdot\|$ *denotes the matrix norm, and* $T_x(E)$ *is defined by* (3.1).

Now, Theorem 3.2 follows from Lemma 3.7 and the definition of the Lyapunov exponent.

Next, we want to compute an explicit formula for the Lyapunov exponent by using the Thouless formula and the formula for the integrated density of states. First, (3.4) asserts that $\pi k + i\gamma$ is the boundary value of an analytic function in the upper half plane. Let $F(z) = \pi k(z) + i\gamma(z)$ for $\operatorname{Im} z \geqslant 0$, and define $\tilde{F}(z) = (1/2\pi) \int_{-\pi}^{\pi} \sqrt{z - \cos x} \, dx$ with branch cut from -1 to ∞ along the real axis. Then $\tilde{F}(z)$ is analytic for $\operatorname{Im} z > 0$, and by Theorem 2.1, $\operatorname{Re} \tilde{F}(z) \to \pi k(E)$ as $z \to E$ ($\operatorname{Im} z > 0$, $E \in \mathbb{R}$). Therefore,

$$\gamma(E) = \lim_{\operatorname{Im} z > 0,\, z \to E} \operatorname{Im} \tilde{F}(z) + C,$$

where C is a real constant. That is,

$$\gamma(E) = \frac{1}{2\pi} \int_{-\pi}^{\pi} [\cos x - E]_+^{1/2} \, dx + C.$$

Notice that for $E > 1$, $\gamma(E) = 0$ and the integral in the right-hand side is also zero, so $C = 0$. Therefore, we have

THEOREM 3.8. *For all* $E \notin R_\nu$, *where* R_ν *is defined by* (3.2) *and* (3.3), *the operator* H_ν *in* (1.1) *has Lyapunov behavior with the Lyapunov exponent given by*

$$\gamma(E) = \frac{1}{2\pi} \int_{-\pi}^{\pi} [\cos x - E]_+^{1/2} \, dx \tag{3.22}$$

where $[f(x)]_+ = \max\{0, f(x)\}$.

Remarks. 1. In fact, there is no mystery for this beautiful Lyapunov exponent formula if we use the WKB (see [4, 5]) heuristic argument. However, it's not easy to justify the WKB solutions.

2. Note that while R_ν is ν-dependent, the right-hand side of (3.22) is ν-independent!

4. SOME SPECTRAL CONSEQUENCES

We have already proved that for a.e. $E \in [-1, 1]$, H_v has positive Lyapunov exponent. By simply applying the Kotani argument (see [7]) or rank one spectral theory (see [3, 10, 11]), we can get dense pure point spectrum on $(-1, 1)$ for almost all boundary conditions. Also, we can show that the eigenfunctions are exponentially decaying. The result on pure point spectrum is an unpublished result by Kirsch and Stolz which is stated in [6] by Kirsch, Molchanov, and Pastur, and the result on exponentially decaying eigenfunctions is proved by Stolz in [14]. Now, we can give an explicit decaying rate of eigenfunctions.

THEOREM 4.1. *Let H_v^θ be the operator H_v given by* (1.1) *with the θ boundary condition at* 0, $u(0)\cos\theta + u'(0)\sin\theta = 0$. *Then for a.e. $\theta \in [0, \pi)$* (*with respect to Lebesgue measure*), *H_v^θ has dense pure point spectrum on* $(-1, 1)$, *and the eigenfunctions of H_v^θ to all eigenvalues $E \in (-1, 1)$ decay like $e^{-\gamma(E)x}$ at ∞ for almost every θ, where $\gamma(E)$ is given by* (3.22).

Next, as we have shown that the resonance set has Hausdorff dimension zero, by applying rank one perturbation theory, we get a new result on singular continuous spectrum.

THEOREM 4.2. *Let H_v^θ be the operator H_v given by* (1.1) *with the boundary condition at* 0 *given by* $u(0)\cos\theta + u'(0)\sin\theta = 0$ *for* $\theta \in [0, \pi)$. *Then for $\theta \neq \pi/2$, the singular continuous part, $(d\mu_\theta)_{sc}$, of the spectral measure $d\mu_\theta$ for H_v^θ is supported on a Hausdorff dimension zero set.*

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