

## A Feynman-Kac Formula for Unbounded Semigroups

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**ABSTRACT.** We prove a Feynman-Kac formula for Schrödinger operators with potentials  $V(x)$  that obey (for all  $\varepsilon > 0$ )

$$V(x) \geq -\varepsilon|x|^2 - C_\varepsilon.$$

Even though  $e^{-tH}$  is an unbounded operator, any  $\varphi, \psi \in L^2$  with compact support lie in  $D(e^{-tH})$  and  $\langle \varphi, e^{-tH}\psi \rangle$  is given by a Feynman-Kac formula.

### 1. Introduction

One of the most useful tools in the study of Schrödinger operators, both conceptually and analytically, is the Feynman-Kac formula. All the standard proofs, (see, e.g., [7]) assume the Schrödinger operator  $H$  is bounded below, so the Schrödinger semigroup  $e^{-tH}$  is bounded. This means, for example, that Stark Hamiltonians are not included.

But the restriction to semibounded  $H$  is psychological, not real. We deal with unbounded  $H$ 's all the time, so why not unbounded  $e^{-tH}$ ? Once one considers the possibility, the technical problems are mild, and it is the purpose of this note to show that.

The form of the Feynman-Kac formula we will discuss is in terms of the Brownian bridge (Theorem 6.6 of [7]). Once one has this, it is easy to extend to the various alternate forms of the Feynman-Kac formula.

The  $\nu$ -dimensional Brownian bridge consists of  $\nu$  jointly Gaussian processes,  $\{\alpha_i(t)\}_{i=1, 0 \leq t \leq 1}^\nu$  with covariance

$$\begin{aligned} E(\alpha_i(t)\alpha_j(s)) &= \delta_{ij} \min(t, s)[1 - \max(t, s)] \\ E(\alpha_i(t)) &= 0. \end{aligned}$$

If  $b$  is Brownian motion, then  $\alpha(s) = b(s) - sb(1)$  is an explicit realization of the Brownian bridge.

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For any real function  $V$  on  $\mathbb{R}^\nu$  and  $t > 0$ , define (the expectation may be infinite):

$$(1.1) \quad Q(x, y; V, t) = E \left( \exp \left( - \int_0^t V \left( \left(1 - \frac{s}{t}\right)x + \frac{s}{t}y + \sqrt{t}\alpha \left(\frac{s}{t}\right) \right) ds \right) \right).$$

Throughout this paper, let

$$H_0 = -\frac{1}{2}\Delta$$

on  $L^2(\mathbb{R}^\nu)$ , so

$$(1.2) \quad e^{-tH_0}(x, y) = (2\pi t)^{-\nu/2} \exp \left( -\frac{|x-y|^2}{2t} \right).$$

The Feynman-Kac formula I'll start with — one of many in [7] — is

**THEOREM 1.1.** *Suppose  $V$  is a continuous function on  $\mathbb{R}^\nu$  which is bounded from below. Let  $H = H_0 + V$ . Then for any  $t > 0$  and  $\varphi, \psi \in L^2(\mathbb{R}^\nu)$ :*

$$(1.3) \quad \langle \varphi, e^{-tH}\psi \rangle = \int \overline{\varphi(x)} \psi(y) e^{-tH_0}(x, y) Q(x, y; V, t).$$

In this paper, we will consider potentials  $V(x)$  for which for any  $\varepsilon > 0$ , there is  $C_\varepsilon$  so that

$$(1.4) \quad V(x) \geq -\varepsilon|x|^2 - C_\varepsilon.$$

It is known (see [5], Theorem X.38) that for such  $V$ ,  $H = H_0 + V$  is essentially self-adjoint on  $C_0^\infty(\mathbb{R}^\nu)$ , so we can use the functional calculus to define  $e^{-tH}$  which might be unbounded. Our main goal here is to prove:

**THEOREM 1.2.** *Suppose  $V$  is a continuous function which obeys (1.4). Then for all  $x, y \in \mathbb{R}^\nu$ ,  $t > 0$ , (1.1) is finite. Let  $\varphi, \psi \in L^2(\mathbb{R}^\nu)$  have compact support. Then for all  $t > 0$ ,  $\varphi, \psi \in D(e^{-tH})$  and (1.3) holds.*

**REMARKS.** 1. It isn't necessary to suppose that  $\varphi, \psi$  have compact support. Our proof shows that it suffices that  $e^{\varepsilon x^2}\psi, e^{\varepsilon x^2}\varphi \in L^2$  for some  $\varepsilon > 0$ . In particular,  $\varphi, \psi$  can be Gaussian.

2. Using standard techniques [1],[3],[7], one can extend the proof to handle  $V = V_1 + V_2$  where  $V_1$  obeys (1.4) but is otherwise in  $L^1_{loc}$  and  $V_2$  is in the Kato class,  $K_\nu$ .

3. If one only has  $V(x) \geq -C_1 - C_2x^2$  for a fixed  $C_2$ , our proof shows that the Feynman-Kac formula holds for  $t$  sufficiently small. It may not hold if  $t$  is large since it will happen if  $V(x) = -x^2$  that  $E(\exp(-\int_0^t V(\alpha(s)) ds))$  will diverge if  $t$  is large.

As for applications of Theorem 1.2, one should be able to obtain various regularity theorems as in [6]. Moreover, for  $H = -\Delta + F \cdot x$ , one can compute  $e^{-tH}(x, y)$  explicitly and so obtain another proof of the explicit formula of Avron and Herbst [2].

**Dedication.** Sergio Albeverio has been a master of using and extending the notion of path integrals. It is a pleasure to dedicate this to him on the occasion of his 60th birthday.

### 2. A Priori Bounds on Path Integrals

Our goal in this section is to prove

**THEOREM 2.1.** *Let  $V$  obey (1.4) and let  $Q$  be given by (1.1). Then, for each  $t > 0$  and  $\delta > 0$ , we have that*

$$Q(x, y; V, t) \leq D \exp(\delta x^2 + \delta y^2),$$

where  $D$  depends only on  $t, \delta$  and the constants  $\{C_\varepsilon\}$ .

**LEMMA 2.2.** *Let  $X$  be a Gaussian random variable. Suppose  $\varepsilon \text{Exp}(X^2) < \frac{1}{2}$ . Then  $E(\exp(\varepsilon X^2)) < \infty$  (and is bounded by a function of  $\varepsilon \text{Exp}(X^2)$  alone).*

**PROOF.** A direct calculation. Alternately, we can normalize  $X$  so  $\text{Exp}(X^2) = 1$ . Then  $E(\exp(\varepsilon X^2)) = (2\pi)^{-1/2} \int \exp((\varepsilon - \frac{1}{2})x^2) dx < \infty$ . □

**PROOF OF THEOREM 2.1.** Note that if  $0 < \theta < 1$ , and  $x, y, \alpha \in \mathbb{R}^\nu$ , then

$$\begin{aligned} |\theta x + (1 - \theta)y + \alpha|^2 &\leq 2|\theta x + (1 - \theta)y|^2 + 2|\alpha|^2 \\ &\leq 2(x^2 + y^2 + |\alpha|^2). \end{aligned}$$

Thus, by (1.4),

$$(2.1) \quad Q(x, y; V, t) \leq E \left( \exp \left( C_\varepsilon t + 2\varepsilon t(x^2 + y^2) + 2\varepsilon \int_0^1 t^2 \alpha(s)^2 ds \right) \right).$$

By Jensen's inequality,

$$(2.2) \quad E \left( \exp \left( 2 \int_0^1 \varepsilon t^2 \alpha(s)^2 ds \right) \right) \leq \int_0^1 E(\exp(2\varepsilon t^2 \alpha(s)^2) ds).$$

Since  $E(\alpha(s)^2)$  is maximized at  $s = \frac{1}{2}$  when it is  $\frac{1}{4}$ , we see that

$$\text{RHS of (2.2)} \leq E(\exp(2\varepsilon t^2 \alpha(\frac{1}{2})^2))$$

is finite if  $\varepsilon t^2 < 1$ , so we can pick  $\varepsilon = \delta_0/t^2$  with  $\delta_0 < 1$  and find (using the explicit value of  $E(\exp(X^2))$  in that case

$$Q(x, y; V, t) \leq \sqrt{2} (1 - \delta_0)^{-1/2} \exp(C_\varepsilon t + 2\delta_0(x^2 + y^2)/t),$$

which proves Theorem 2.1. □

### 3. A Convergence Lemma

In this section, we will prove:

**THEOREM 3.1.** *Let  $A_n, A$  be self-adjoint operators on a Hilbert space  $\mathcal{H}$  so that  $A_n \rightarrow A$  in strong resolvent sense. Let  $f$  be a continuous function on  $\mathbb{R}$  and  $\psi \in \mathcal{H}$  with  $\psi \in D(f(A_n))$  for all  $n$ . Then*

- (i) *If  $\sup_n \|f(A_n)\psi\| < \infty$ , then  $\psi \in D(f(A))$ .*
- (ii) *If  $\sup_n \|f(A_n)^2\psi\| < \infty$ , then  $f(A_n)\psi \rightarrow f(A)\psi$ .*

**REMARK.** Let  $\mathcal{H} = L^2(0, 1)$ ,  $\psi(x) \equiv 1$ ,  $A_n =$  multiplication by  $n^{1/2}$  times the characteristic function  $[0, 1/n]$ , and  $A \equiv 0$ . Then  $A_n \rightarrow A$  in strong resolvent sense and  $\sup_n \|A_n\psi\| < \infty$ , but  $A_n\psi$  does not converge to  $A\psi$  so one needs more than  $\sup_n \|f(A_n)\psi\| < \infty$  to conclude that  $f(A_n)\psi \rightarrow f(A)\psi$ . The square is overkill. We need only  $\sup_n \|F(f(A_n))\psi\| < \infty$  for some function  $F: \mathbb{R} \rightarrow \mathbb{R}$  with  $\lim_{|x| \rightarrow \infty} |F(x)|/x = \infty$ .

PROOF. Suppose that  $\sup_n \|f(A_n)\psi\| < \infty$ . Let

$$f_m(x) = \begin{cases} m & \text{if } f(x) \geq m \\ f(x) & \text{if } |f(x)| \leq m \\ -m & \text{if } f(x) \leq -m. \end{cases}$$

Then ([4], Theorem VIII.20) for each fixed  $m$ ,  $f_m(A_n) \rightarrow f_m(A)$  strongly. It follows that

$$\begin{aligned} \|f_m(A)\psi\| &= \lim_n \|f_m(A_n)\psi\| \\ &\leq \sup_n \|f_m(A_n)\psi\| \leq \sup_n \|f(A_n)\psi\|. \end{aligned}$$

Thus,  $\sup_m \|f_m(A)\psi\| < \infty$ , which implies that  $\psi \in D(f(A))$ .

Now suppose  $\sup_n \|f(A_n)^2\psi\| < \infty$ . Then

$$\|(f(A_n) - f_m(A_n))\psi\| \leq \frac{1}{m} \|f(A_n)^2\psi\|.$$

Thus  $f_m(A_n)\psi \rightarrow f(A_n)\psi$  uniformly in  $n$  which, given that  $f_m(A_n)\psi \rightarrow f_m(A)\psi$ , implies that  $f(A_n)\psi \rightarrow f(A)\psi$ .  $\square$

#### 4. Putting It Together

We are now ready to prove Theorem 1.2. Let  $V$  be continuous and obey (1.4). Let  $V_n(x) = \max(V(x), -n)$ . Then  $V_n$  is bounded from below, so Theorem 1.1 applies, and so (1.3) holds. Let  $\varphi \in L^2$  with compact support. By Theorem 2.1, we have

$$\sup_n \|\exp(-tH_n)\varphi\| < \infty$$

for each  $t$  positive.

By the essential self-adjointness of  $H$  on  $C_0^\infty(\mathbb{R}^\nu)$  and  $(V_n - V)\eta \rightarrow 0$  for any  $\eta \in C_0^\infty$ , we see that  $H_n$  converges to  $H$  in strong resolvent sense. Hence setting  $A_n = H_n$ ,  $A = H$ ,  $f(x) = e^{-tx}$ , and  $\psi = \varphi$ , we can use Theorem 3.1 to see that  $\varphi \in D(\exp(-tH))$  and  $\|[\exp(-tH_n) - \exp(-tH)]\varphi\| \rightarrow 0$ . Thus as  $n \rightarrow \infty$ , the left-hand side of the Feynman-Kac formula converges. By the a priori bound in Theorem 2.1 and the dominated convergence theorem, the right-hand side converges. So Theorem 1.2 is proven.

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