

Summability Methods, the Strong Asymptotic Condition, and Unitarity in Quantum Field Theory*

Barry Simon†

Departments of Mathematics and Physics, Princeton University, Princeton, New Jersey 08540

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We discuss a summability mechanism which preserves nonlinear perturbative conditions such as unitarity of the Feynman series. This condition, which relates a function $f(z)$ and a series $\sum a_n z^n$ by the requirement

$$|f(z) - \sum_{n=0}^N a_n z^n| \leq A \sigma^{N+1} (N+1)! |z|^{N+1},$$

is applicable to certain divergent series.

It has been known for some time that the perturbation series in super-renormalizable self-coupled boson field theories diverge,¹ and there have been a variety of mechanisms proposed which suggest the divergence of Feynman series in theories with fermions.^{2,3} While it has certainly not been proven that fantastic cancelations do not occur in renormalizable but nonsuper-renormalizable series,⁴ it seems unlikely that perturbation theory converges in any Lagrangian field theory. On the other hand, much hard work has been done on proving properties of the Feynman perturbation series order by order. The most interesting properties are the nonlinear ones that do not hold exactly if the series is truncated to any fixed finite order, but only hold in the sense of formal power series. Among these are formal unitarity of the S matrix⁵ and Ward identities, both nonanomalous and anomalous.⁶ It is our goal in this note to describe a mechanism which could make perturbative equations exact in an actual theory. We will use unitarity as an illustrative example—the same mechanism could operate in other cases.⁷

To put this mechanism in perspective, let us first discuss some simple summability techniques. In recent years, several techniques for summing divergent series have been proposed as computational devices⁸ or have been proven to be valid⁹ in relatives of the simplest of all divergent perturbation series: the ground-state energy of an anharmonic oscillator.¹⁰ Let us see how these summability techniques respect three types of conditions:

(i) *Invariance conditions.*—We have in mind properties like crossing symmetry in amplitudes with trivial crossing matrix or Lorentz invariance. These are clearly preserved automatically by any convergent summability method.

(ii) *Linear covariances.*—We have in mind

crossing symmetry in amplitudes with nontrivial crossing matrices and Lorentz covariance for currents or other tensorial quantities. These are clearly preserved by *linear* summation procedures like Borel summability, but not *automatically* preserved by nonlinear summation processes like Padé summation.

(iii) *Nonlinear perturbative properties, as mentioned above.*—In general, these are not preserved by any summability methods although it is known that partial-wave unitarity in the elastic region is preserved by the diagonal Padé method.¹¹ We thus see that a summability statement alone is not, in general, enough to carry formal computations from the power-series level to the “summed-series” level.

The mechanism which we wish to discuss has already been discussed in “physical situations” by the author elsewhere^{3,12}; it is implicit in the classical work of Watson and Carleman¹³ for formal asymptotic series. We say a function f obeys a *strong asymptotic condition* (SAC) and has $\sum a_n z^n$ as a strong asymptotic series (SAS) if and only if (a) f is analytic in a sectorial region S of the form $\{z \mid |\arg z| < \theta, 0 < |z| < B\}$ for some $\theta > \frac{1}{2}\pi$ and some $B > 0$ (in field theories, one expects that θ can be chosen arbitrarily close to $\pi^{9,3}$); and (b) there exist constants A and σ so that for all N and $z \in S$

$$|f(z) - \sum_{n=0}^N a_n z^n| \leq A \sigma^{N+1} (N+1)! |z|^{N+1}. \quad (1)$$

There are two critical properties of functions obeying a SAC:

(I) *Any function f with SAS whose coefficients are all 0 is identically 0.*—Equivalently, there is at most one function with any given SAS. This is just Carleman’s theorem¹³ or follows alternately from the fact that f can be recovered from $\sum a_n z^n$ by Borel summation.

(II) *Products of functions obeying a SAC obey a SAC.*—So do sums and so do quotients f/g if the lowest-order term in g is nonzero. At first sight, one might expect that products have remainders growing like $[(N+1)!]^2$, not $(N+1)!$. However, a simple proof shows that if f has $\sum a_n z^n$ as a SAS and g has $\sum b_n z^n$ as a SAS, then fg has $\sum c_n z^n$ as a SAS with

$$c_n = \sum_{m=0}^n a_m b_{n-m}.$$

As a typical element of the proof let us show that $|c_n| < A\sigma^N N!$ for all N . Since (I) holds for f and g , $|a_n| \leq A_1 \sigma_1^N N!$ and $|b_n| \leq A_2 \sigma_2^N N!$, then

$$\begin{aligned} |c_n| &\leq \sum_{m=0}^N |a_m| |b_{N-m}| \\ &\leq A_1 A_2 \sum_{m=0}^N \sigma_1^m \sigma_2^{N-m} m! (N-m)! \\ &\leq N! (\sigma_1 + \sigma_2)^N (A_1 A_2), \end{aligned}$$

since $[m!(N-m)!]^2 < (N!)^2$.

As a typical “nonlinear summation theorem” that follows from (I) and (II), we note the following:

Proposition.—Suppose at some fixed energy E , there are $n < \infty$ open channels for $a+b$ scattering given by amplitudes $T_1(E; \{p_1\}, g), \dots, T_n(E; \{p_n\}, g)$, where g is the coupling constant in some underlying field theory and $\{p_i\}$ is a set of kinematic variables for channel i . Suppose in some region

$$S = \{g | 0 < |g| < G, |\arg g| < \theta; \theta > \frac{1}{2}\pi\};$$

the $T_i(E; \{p_i\}, g)$ have the Feynman series for the process $a+b \rightarrow i$ as a SAS with the constants σ, A in (I) bounded uniformly as the $\{p_i\}$ vary over their (compact) set of allowable phase space. Then forward unitarity holds at energy E with $0 < g < G$.

The proof of this proposition is trivial given (I) and (II); for unitarity says that

$$\begin{aligned} \text{Im} T_{ei}(0, g) \\ = \sum_{i=1}^n \int d\rho(p_i) T_i(p_i, g^*)^* T_i(p_i, g). \end{aligned} \quad (2)$$

Both sides of (2) are analytic functions obeying a SAC [using (II) and the hypothetical uniform estimates on σ, A]. By formal unitarity⁵ their SAS are equal, so by (I) they are equal.

Similar arguments hold for any other nonlinear relation. It is a difficult and open mathematical question¹⁴ as to when there is any function with a

given SAS, so we are far from using the ideas of this Letter in constructing quantum fields. Nevertheless, we have presented a simple mechanism that could validate formal perturbation theory arguments even when the series in question diverge.

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¹A. Jaffe, *Commun. Math. Phys.* **1**, 127 (1965); C. Hurst, *Proc. Cambridge Phil. Soc.* **48**, 625 (1952); W. Thirring, *Helv. Phys. Acta* **26**, 33 (1953); A. PETERMAN, *Helv. Phys. Acta* **26**, 291 (1953).

²For quantum electrodynamics, see F. Dyson, *Phys. Rev.* **85**, 631 (1952).

³For Y_2 , a mechanism whereby renormalization counterterms cause a divergence is proposed by B. Simon, in the Coral Gables Conference on Fundamental Interactions at High Energy, University of Miami, Coral Gables, Florida, January 1972 (Gordon and Breach, New York, to be published). See also, F. GUERRA and M. MARIANO, *Nuovo Cimento* **42A**, 285 (1966).

⁴In fact, the indications are that renormalization makes convergence worse; see B. Simon, *Nuovo Cimento* **59A**, 166 (1969).

⁵K. Hepp, in *Field Theory and Statistical Mechanics: 1970 Les Houches Lectures*, edited by R. Stora and C. DeWitt (Gordon and Breach, New York, 1972); H. Epstein and V. Glaser, *ibid.*

⁶Nonanomalous: see, e.g., J. Ward, *Phys. Rev.* **78**, 1824 (1950); Anomalous: see, e.g., S. Adler, *Phys. Rev.* **177**, 2426 (1969).

⁷More complex arguments such as the derivation of the Callan-Symanzik relations [K. Symanzik, *Commun. Math. Phys.* **18**, 227 (1970); G. Callan, *Phys. Rev. D* **2**, 1541 (1970)], which rely on Weinberg’s theorem and would require Eq. (1) to hold uniformly as certain variables approached infinity, are more dubious from our point of view.

⁸The Padé method was proposed in L. Copley and D. Masson, *Phys. Rev.* **164**, 2059 (1967); D. Bessis and L. Pusterla, *Nuovo Cimento* **60A**, 243 (1969).

⁹For Padé, see J. J. Loeffel *et al.*, *Phys. Lett.* **30B**, 656 (1969); for Borel summability, see S. Graffi *et al.*, *Phys. Lett.* **32B**, 631 (1970); B. Simon, *Phys. Rev. Lett.* **25**, 1583 (1970); for Euler summability, see J. Gunson and D. Ng, to be published.

¹⁰C. Bender and T. T. Wu, *Phys. Rev.* **184**, 1231 (1969); B. Simon, *Ann. Phys. (New York)* **58**, 79 (1970).

¹¹This is most thoroughly discussed in J. Basdevant *et al.*, *Nuovo Cimento* **60A**, 185 (1969).

¹²B. Simon, *Advan. Math.* **7**, 240 (1971).

¹³G. Watson, *Phil. Trans. Roy. Soc. London, Ser. A* **211**, 279 (1912); T. Carleman, *Les Fonctions Quasi-analytiques* (Gauthier-Villars, Paris, 1926).

¹⁴For example, $\sum (-1)^n n! z^n$ is a SAS for a some function, while $\sum n! z^n$ and $\sum n! [(-2^n + 1)]$ are not the SAS for any function.