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The Golinskii-Ibragimov Method and a Theorem of Damanik and Killip

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1 Introduction

Let $\partial\mathbb{D}$ be the unit circle $\{z \mid |z| = 1\}$ in $\mathbb{C}$ and $\mathbb{D}$ the open disc $\{z \mid |z| < 1\}$. Let μ be a probability measure on $\partial\mathbb{D}$ which is not supported on a finite number of points. Then using the Gram-Schmidt procedure, we can define monic orthogonal polynomials on the unit circle (OPUC) $\Phi_n(z; d\mu)$ and normalized polynomials $\varphi_n(z; d\mu)$. These obey the Szegö recursion formulae [9, 6]:

$$\Phi_{n+1}(z) = z\Phi_n(z) - \bar{\alpha}_n \Phi_n^*(z), \tag{1.1}$$

where

$$\Phi_n^*(z) = z^n \overline{\Phi_n\left(\frac{1}{\bar{z}}\right)}. \tag{1.2}$$

The parameters α_n are called the Verblunsky coefficients of $d\mu$ (also called Schur, Szegö, Geronimus, or reflection parameters or coefficients). They lie in $\mathbb{D}$, and any $\alpha \in \times_{n=0}^{\infty}\mathbb{D}$ is the Verblunsky coefficient of a unique measure [2, 9]. In this paper, we are mainly interested in the spectral problem of going from information on the Verblunsky coefficients to information on the measure.

Our starting point is a method from a lovely 1971 paper of Golinskii and Ibragimov [4], who used this method to prove the following theorem.

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Theorem 1.1 (Golinskii and Ibragimov [4]). If $\sum_{n=0}^{\infty} n|\alpha_n|^2 < \infty$, then $d\mu_s = 0$. □

Here, we make a canonical decomposition:

$$d\mu = w(\theta)\frac{d\theta}{2\pi} + d\mu_s, \tag{1.3}$$

where $d\mu_s$ is singular with respect to $d\theta$. We are interested in α's that obey the following weaker condition:

$$\sum_{n=0}^{N} n\left|\alpha_n\right|^2 \leq A \log N + C \tag{1.4}$$

with A and C constants. Think of $\alpha_n = \sqrt{A}/n$ as a prototypical example. First, we prove the following theorem.

Theorem 1.2. If (1.4) holds with $A < 1/4$, then $d\mu_s = 0$. □

The Golinskii-Ibragimov (GI) method most directly only gets $A < 1/16$, but by replacing their L^1 method by an L^2 method, we bring things up to $A < 1/4$.

Theorem 1.2 is almost optimal in that there are examples with $A > 1/4$ but $|A - 1/4|$ arbitrarily small, where $d\mu$ has an eigenvalue. However, we can do better if we assume the α's are real.

Theorem 1.3. If all α_n are real (equivalently, $d\mu$ is invariant under complex conjugation) and (1.4) holds with $A < 1/2$, then $d\mu_s$ can only consist of possible pure points at $z = +1$ or $z = -1$. If $A = 1/4$, $d\mu_s = 0$. □

We use an idea motivated by Damanik and Killip [1] to prove Theorem 1.3. In this regard, we prove the following special looking result which, as we explain, is related to [1].

Theorem 1.4. Let all α_n be real and obey the following conditions:

(i) the bound

$$\sum_{j=0}^{N} j\left|\alpha_{2j}\right|^2 \leq A \log N + C, \tag{1.5}$$

(ii) $|\alpha_{2j-1}| \leq |\alpha_{2j+1}|$ for $j = 1, 2, \ldots$ with either all $\alpha_{2j-1} \geq 0$ or all $\alpha_{2j-1} \leq 0$,

(iii) $\sum_{j=1}^{\infty} |\alpha_{2j-1}|^2 < \infty$.

If $A < 1/2$, $d\mu_s$ consists only of possible pure points at $z = \pm 1$ or $z = \pm i$. Moreover, if $A = 1/4$, the only possible pure points are at $z = \pm 1$, and if $A = 1/4$ and $\alpha_{2j-1} \leq 0$, then $d\mu_s = 0$. □

This theorem is custom-made to provide part of a proof of the following recent striking result of Damanik and Killip [1].

Theorem 1.5 (Damanik and Killip [1]). Let H be a half-line Schrödinger operator on $\ell^2(\mathbb{Z}_+)$,

$$(Hu)_n = u_{n+1} + u_{n-1} + v_n u_n \tag{1.6}$$

(u_0 interpreted as 0). Suppose $\mathrm{spec}(H) = [-2, 2]$. Then H has purely a.c. spectrum, that is, $\sigma_{sc} = \sigma_{pp} = \emptyset$. □

Their proof has the following steps (they use γ_n for our α_n).

(i) Following Szegö [6], map H to an associated measure on $\partial\mathbb{D}$ by using $z = e^{i\theta} \mapsto E = 2\cos\theta$ to pull back the spectral measure $d\rho$ on $[-2, 2]$ for H to a measure μ on $\partial\mathbb{D}$. Note (following Geronimus) that the Verblunsky coefficients for $d\mu$ obey that

$$v_{n+1} = \big(1 - \alpha_{2n-1}\big)\alpha_{2n} - \big(1 + \alpha_{2n-1}\big)\alpha_{2n-2}, \tag{1.7}$$

$$1 = \big(1 - \alpha_{2n-1}\big)\big(1 - \alpha_{2n}^2\big)\big(1 + \alpha_{2n+1}\big) \tag{1.8}$$

(for general Jacobi matrices, the left-hand side of (1.8) is a_{n+1}^2; the initial conditions are $\alpha_{-1} = -1$).

(ii) Analyze (1.8) with α_n real and $|\alpha_n| < 1$ to conclude the following:

(a) $\alpha_{2n-1} \le \alpha_{2n+1} \le 0$, [1, Lemma 4.1],

(b) $|\alpha_{2n+1}| \le 1/(n+2)$ so $\sum_{n=1}^{\infty} |\alpha_{2n-1}|^2 < \infty$, [1, Lemma 4.2],

(c) $\sum_{j=0}^{N} (j+1)\alpha_j^2 \le (1/4)\log N + C$, [1, Proposition 4.5].

(iii) Translate information on the α_n's to information on the v_n's.

(iv) Prove that $-2, 2, 0$ are not eigenvalues of H.

(v) Prove that solutions of

$$u_{n+1} + u_{n-1} + v_n u_n = Eu \tag{1.9}$$

with $E \in (-2, 0) \cup (0, 2)$ have $|u_n| \le cn^{\eta}$ for any $\eta > (1/2\sqrt{2})$.

(vi) Prove that the set of E in $(-2, 2)$, for which (1.9) has unbounded solutions, has Hausdorff dimension 0.

(vii) Use (v), (vi), Hausdorff dimension, and the Jitomirskaya-Last inequalities [7] to show that $d\mu$ has no singular part on $(-2, 0) \cup (0, 2)$.

Given step (ii), one can use Theorem 1.4 to replace steps (iii), (iv), (v), (vi), and (vii). For Theorem 1.4 says that $d\mu$ is purely a.c. and the pullback then implies that $d\rho$ is.

Since Theorem 1.4 depends on ideas closely related to steps (iv) and (v), what we are really doing is using an appeal to the GI method to replace steps (vi) and (viii) and, in particular, the use of Hausdorff dimension and the Jitomirskaya-Last inequalities. These steps follow the ideas of Remling [8] so, in essence, where [1] extends [8], we extend [4]. It is pointless to argue which approach is "simpler" (since some of their techniques have appeared extensively in the Schrödinger operator literature), but we believe that it is useful to have the alternate approach.

In Section 2, we discuss the ideas of Golinskii-Ibragimov and, in particular, prove Theorem 1.2. In Section 3, we use Prüfer variables for OPUC to prove Theorem 1.3. In Section 4, we prove Theorem 1.4.

2 The GI method

Golinskii and Ibragimov made the assumption that

$$\sum_{n=0}^{\infty} n|\alpha_n|^2 < \infty. \tag{2.1}$$

On $\partial\mathbb{D}$, D^{-1} is defined a.e. $d\theta$ as boundary values of D, the Szegö function (see [3, 9]), and so a.e. $d\mu_{ac}$. We extend it to $L^2(\partial\mathbb{D}, d\mu) = L^2(\partial\mathbb{D}, d\mu_{ac}) \oplus L^2(\partial\mathbb{D}, d\mu_s)$ by setting to 0 on the singular subspace. Then Golinskii and Ibragimov [4] proved that

$$\|\varphi_n^* - D^{-1}\|_{L^2(\partial\mathbb{D},d\mu)} \le Cn^{-1/2} \tag{2.2}$$

and that

$$\left\|(\varphi_n^*)^{-1}\right\|_\infty \le C_1 \exp\left(C_2\sqrt{\log n}\right), \tag{2.3}$$

where $\|\cdot\|_\infty$ means sup over $\partial\mathbb{D}$ or $\bar{\mathbb{D}}$ (equal since $(\varphi_n^*)^{-1}$ is analytic in a neighborhood of $\bar{\mathbb{D}}$). In (2.2), the Szegö function exists since (2.1) implies $\sum_{n=0}^\infty |\alpha_n|^2 < \infty$. We actually prove results like this below.

They then wrote

$$\begin{aligned} &\left\||\varphi_n^*|^{-2} - |D|^2\right\|_{L^1(\partial\mathbb{D},d\theta/2\pi)} \\ &\quad\le \left\|(\varphi_n^*)^{-1}\right\|_\infty^2 \left\|\left(|\varphi_n|^2 - |D|^{-2}\right)D^2\right\|_{L^1(\partial\mathbb{D},d\theta/2\pi)} &(2.4)\\ &\quad\le \left\|(\varphi_n^*)^{-1}\right\|_\infty^2 \left\|\left(|\varphi_n|^2 - |D|^{-2}\right)\right\|_{L^1(\partial\mathbb{D};d\mu)} &(2.5)\\ &\quad\le \left\|(\varphi_n^*)^{-1}\right\|_\infty^2 \|\varphi_n - D^{-1}\|_{L^2(\partial\mathbb{D};d\mu)}\left\||\varphi_n| + |D|^{-1}\right\|_{L^2(\partial\mathbb{D};d\mu)}. &(2.6)\end{aligned}$$

In (2.5), we use $D^2(d\theta/2\pi) = d\mu_{ac} \le d\mu$ and in (2.6) we use $\| \varphi_n| - |D|^{-1}| \le |\varphi_n - D^{-1}|$. Thus, by (2.2) and (2.3), $|\varphi_n|^{-2}(d\theta/2\pi) \to |D|^2(d\theta/2\pi)$ in norm on measures. But it is known that $|\varphi_n|^{-2}(d\theta/2\pi) \to d\mu$ weakly (see, e.g., [2]). Thus $d\mu = |D|^2(d\theta/2\pi)$ so $d\mu_s = 0$.

What is especially interesting about this approach is that it uses the divergent estimate (2.3). Clearly, we can have much more rapid growth of $\|(\varphi_n^*)^{-1}\|_\infty$ than in (2.3) and still have convergence. Basically, it suffices to have $\|(\varphi_n^*)^{-1}\|_\infty \le Cn^\beta$, with $2\beta < 1/2$. If one looks at the proof of (2.3) in [4], that translates to a bound like (1.4) with $\sqrt{A} < 1/4$, that is, $A < 1/16$. Our first observation is that instead of estimating L^1-norms as GI do, it pays to estimate L^2-norms. Then $\|(\varphi_n^*)^{-1}\|_\infty$ will occur as a first power, not second. Here are the key facts.

Theorem 2.1. Let μ obey the Szegö condition. Then

(a) $d\mu_s = 0$ if and only if $\|(\varphi_n^*)^{-1} - D\|_{L^2(\partial\mathbb{D}, d\theta/2\pi)} \to 0$;

(b) if I is an open interval and $\|\chi_I[(\varphi_n^*)^{-1} - D]\|_{L^2(\partial\mathbb{D}, d\theta/2\pi)} \to 0$, then $\mu_s(I) = 0$.

□

Proof. (a) As is well known (see [2]), $\|(\varphi_n^*)^{-1}\|_{L^2(\partial\mathbb{D}, d\theta/2\pi)} = 1$. Moreover, $(\varphi_n^*)^{-1}(z) \to D(z)$ uniformly on compact subsets of $\mathbb{D}$, so $(\varphi_n^*)^{-1} \to D$ weakly in L^2. Thus $(\varphi_n^*)^{-1} \to D$ in norm if and only if $1 = \|(\varphi_n)^{-1}\|^2_{L^2(\partial\mathbb{D}, d\theta/2\pi)} = \|D\|^2_{L^2(\partial\mathbb{D}, d\theta/2\pi)}$. But since $\|D\|^2 = \int |D(\theta)|^2(d\theta/2\pi) = \mu_{ac}(\partial\mathbb{D})$, norm convergence is equivalent to $\mu_s(\partial\mathbb{D}) = 0$.

(b) Let f be continuous and have support in I. Then

$$f|\varphi_n^*|^{-2} = \left(f(\varphi_n^*)^{-1}\chi_I\right)\left(\overline{(\varphi_n^*)^{-1}}\chi_I\right) \longrightarrow f|D|^2 \tag{2.7}$$

in L^1 and thus, by the weak convergence of $|\varphi_n^*|^{-2}(d\theta/2\pi)$ to $d\mu$, we have

$$\int f\, d\mu = \int f|D|^2 \left(\frac{d\theta}{2\pi}\right), \tag{2.8}$$

that is, $d\mu_s(I) = 0$. ■

Theorem 2.2. If

$$\left\|\chi_I(\varphi_n^*)^{-1}\right\|_\infty \|(\varphi_n^* - D^{-1})\|_{L^2(\partial\mathbb{D}, d\mu)} \longrightarrow 0 \tag{2.9}$$

for some open interval I (including $I = \partial\mathbb{D}$), then $\mu_s(I) = 0$. □

Proof. We have that

$$\begin{aligned}\left\|\chi_I\left[(\varphi_n^*)^{-1} - D\right]\right\|_{L^2(\partial\mathbb{D}, d\theta/2\pi)} &= \left\|\chi_I(\varphi_n^*)^{-1}(\varphi_n^* - D^{-1})D\right\|_{L^2(\partial\mathbb{D}, d\theta/2\pi)} \\ &\leq \left\|\chi_I(\varphi_n^*)^{-1}\right\|_\infty \left\|(\varphi_n^* - D^{-1})\right\|_{L^2(\partial\mathbb{D}, d\mu_{ac})} \\ &\leq \text{LHS of (2.9)}.\end{aligned} \tag{2.10}$$

Thus the result follows from Theorem 2.1. ■

Define $\rho_n = (1 - |\alpha_n|^2)^{1/2}$. A straightforward calculation shows that for $n < m$,

$$\langle \varphi_n^*, \varphi_m^* \rangle = \prod_{\ell=n}^{m-1} \rho_\ell \tag{2.11}$$

and, of course, $\langle \varphi_n^*, \varphi_n^* \rangle = 1$. Since [3] $\varphi_n^* \to D^{-1}$ in $L^2(\partial\mathbb{D}, d\mu)$, we have that

$$\|\varphi_n^* - D^{-1}\|_{L^2(\partial\mathbb{D}, d\mu)}^2 = 2\left(1 - \prod_{\ell=n}^{\infty} \rho_\ell\right). \tag{2.12}$$

Proposition 2.3. (a) We have that $\|\varphi_n^* - D^{-1}\|_{L^2(\partial\mathbb{D}, d\mu)}^2 \leq 2\sum_{\ell=n}^\infty |\alpha_\ell|^2$.

(b) If (1.4) holds, then $\sum_{n=0}^\infty |\alpha_n|^2 < \infty$.

(c) If (1.4) holds for any A, then

$$\|\varphi_n^* - D^{-1}\|_{L^2(\partial\mathbb{D}, d\mu)} \leq \frac{C \log n}{n^{1/2}} \tag{2.13}$$

□

for all large n.

Proof. (a) Since $\rho_\ell^2 \leq \rho_\ell$, $1 - \prod_{\ell=n}^\infty \rho_\ell \leq 1 - \prod_{\ell=n}^\infty \rho_\ell^2$. Moreover, $1 - \prod_{j=1}^\ell x_j \leq \sum_{j=1}^\ell (1 - x_j)$ by a simple induction. Thus (2.3) implies (a), (b), (c)

$$\sum_{n=2^\ell}^{2^{\ell+1}-1} |\alpha_n|^2 \leq 2^{-\ell} \sum_{n=0}^{2^{\ell+1}} n|\alpha_n|^2 \leq 2^{-\ell}\left[A \log\left(2^{\ell+1}\right) + C\right]. \tag{2.14}$$

Since

$$\sum_{\ell=k}^\infty \ell 2^{-\ell} \leq Ck2^{-k}, \tag{2.15}$$

we see that (1.4) implies $\sum_{n=0}^\infty |\alpha_n|^2 < \infty$ and

$$\sum_{\ell=n}^\infty |\alpha_\ell|^2 \leq Cn^{-1} \log n. \tag{2.16}$$

Given (a), we get (c). ■

Theorem 2.4. If the α's obey (1.4) and for some open interval $I \subset \partial\mathbb{D}$, and $B < 1/2$, then

$$\sup_{z \in I} \left[|\varphi_n(z)|^{-1} \right] \leq C(n+1)^B \tag{2.17}$$

for a constant C, then $\mu_s(I) = 0$. □

Proof. For $z \in \partial\mathbb{D}$, $|\varphi_n^*(z)| = |\varphi_n(z)|$ so (2.17) states that $\|\chi_I(\varphi_n^*)^{-1}\|_\infty \leq C(n+1)^B$. This and (2.13) imply that (2.9) holds, so $\mu_s(I) = 0$. ■

The second kind polynomials, ψ_n and Ψ_n, are defined by reversing the signs of all the Verblunsky coefficients. It is known (see, e.g., [5]) that for $z \in \partial\mathbb{D}$, $\mathrm{Re}(\varphi_n(z)\bar{\psi}_n(z)) = 1$, so $|\varphi_n(z)|^{-1} \leq |\psi_n(z)|$.

Since $\psi_n = (\prod_{j=0}^{n-1} \rho_j^{-1})\Psi_n$ and $\sum \alpha_j^2 < \infty$ implies $\prod_{j=0}^{\infty} \rho_j^{-1} < \infty$, we see that $|\psi_n(z)| \leq C|\Psi_n|$. Thus, Theorem 2.4 can be rewritten as the following corollary.

Corollary 2.5. If the α's obey (1.4) and for some open interval $I \subset \partial\mathbb{D}$, we have

$$\sup_{z \in I} |\Psi_n(z)| \leq C(n+1)^B \tag{2.18}$$

for a constant C, and $B < 1/2$, then $\mu_s(I) = 0$. □

Proof of Theorem 1.2. Equation (1.2) implies $|\Psi_n^*| = |\Psi_n|$ on $\partial\mathbb{D}$ and so (1.1) implies $|\Psi_{n+1}| \leq (1 + |\alpha_n|)|\Psi_n|$. Thus

$$\sup_{z \in \partial\mathbb{D}} |\Psi_n(z)| \leq \prod_{j=0}^{n-1} 1 + |\alpha_j| \leq \exp\left(\sum_{j=0}^{n-1} |\alpha_j| \right). \tag{2.19}$$

But, by the Schwartz inequality, (C is a "variable constant" and ε is arbitrarily small)

$$\begin{aligned} \sum_{j=0}^{n} |\alpha_j| &\leq \left(\sum_{j=0}^{n} (j+1)|\alpha_j|^2 \right)^{1/2} \left(\sum_{j=0}^{n} (j+1)^{-1} \right)^{1/2} \\ &\leq \left[A\log(n) + C \right]^{1/2} \left[\log(n) + C \right]^{1/2} \\ &\leq \sqrt{A+\varepsilon}\log(n) + C. \end{aligned} \tag{2.20}$$

Thus, by (2.19),

$$\sup_{z \in \mathbb{D}} |\Psi_n(z)| \leq Cn^{\sqrt{A+\varepsilon}}. \tag{2.21}$$

If $\sqrt{A+\varepsilon} < 1/2$, that is, $A < 1/4$, Corollary 2.5 is applicable. ■

We emphasize that, in essence, the calculation in Proposition 2.3(a), (2.19), and the basic strategy are all from [4]; the only real advance in this section is the use of L^2-norms allowing $A < 1/4$, where the method of [4] gets $A < 1/16$.

Finally, we note that $A = 1/4$ is a critical value for the appearance of bound states, on account of the following theorem.

Theorem 2.6. If all α_n are real, then $\varphi_n(1) > 0$ and

$$\varphi_n(1) = \prod_{j=0}^{n-1} (1 - \alpha_j). \tag{2.22}$$

In addition, $(-1)^n \varphi_n(-1) > 0$ and

$$(-1)^n \varphi_n(-1) = \prod_{j=0}^{n-1} \left(1 - (-1)^{j+1} \alpha_j\right). \tag{2.23}$$

□

Remark 2.7. In particular, if $\alpha_j = B(j+1)^{-1}$, then $\varphi_n(1) \sim n^{-B}$ and $\sum |\varphi_n(1)|^2 < \infty$ if $B > 1/2$. For that α, (1.4) holds for $A = B^2$, that is, there are examples with $d\mu_s \neq 0$ and (1.4) holding for any $A > 1/4$. At $A = 1/4$, our proof of Theorem 1.2 shows that $|\alpha_n(z)| \geq Cn^{-1/2}$, so at least there are no eigenvalues.

Proof. Note that $\varphi_n(1)$ is real by induction and then (1.1) states that

$$\varphi_{n+1}(1) = (1 - \alpha_n) \varphi_n(1), \tag{2.24}$$

proving (2.22). Similarly, $\varphi_n(-1)$ is real, (1.1) for $z = -1$ states that

$$(-1)^{n+1} \varphi_{n+1}(-1) = (-1)^n \varphi_n(-1) - \alpha_n (-1)^{n+1} (-1)^n \varphi_n(-1) \tag{2.25}$$

since $\varphi_n^*(-1) = (-1)^n \varphi_n(-1)$, which yields (2.23). ■

3 Prüfer variables and the proof of Theorem 1.3

Write $\Psi_n(z) = R_n(z) e^{i[\theta_n(z) + n\eta(z)]}$ where $e^{i\eta(z)} = z$ and θ_n is defined initially only modulo 2π. If α_n is real, (1.1) with $\alpha_n \to -\alpha_n$ becomes $R_{n+1}^2 = R_n^2 |e^{i\eta} e^{i[\theta_n + n\eta]} + \alpha_n e^{-i\theta_n}|^2$.

Thus,

$$\frac{R_{n+1}^2}{R_n^2} = 1 + \alpha_n^2 + 2\alpha_n \cos\left((n+1)\eta + 2\theta_n\right). \tag{3.1}$$

Secondly,

$$e^{i(\theta_{n+1}(z)-\theta_n(z))} = \frac{1+\alpha_n e^{-i(2\theta_n+(n+1)\theta_n)}}{\left[1+\alpha_n^2+2\alpha_n \cos\big((n+1)\eta+2\theta_n\big)\right]^{1/2}}. \tag{3.2}$$

These are the Prüfer variable equations for Ψ, (3.2) implies that $\cos(\theta_{n+1}-\theta_n) > 0$, so we can always pick θ_{n+1} so $|\theta_{n+1}-\theta_n| < \pi/2$, settling the 2π ambiguity.

Proposition 3.1. Let α_n be real and $\sum \alpha_n^2 < \infty$. Define $Q = \sup_n |\alpha_n| < 1$. Then

(a) for a z independent constant $C \in (1,\infty)$,

$$C^{-1} \le R_n \exp\left(-\sum_{j=0}^{n-1} \alpha_j \cos\big((j+1)\eta + 2\theta_j(z)\big)\right) \le C, \tag{3.3}$$

(b) for all n,

$$\left|\theta_{n+1}-\theta_n\right| \le \frac{\pi}{2}\frac{1}{1-Q}\left|\alpha_n\right|. \tag{3.4}$$

□

Remark 3.2. The constant C depends only on Q and $\sum_{j=0}^{\infty}\alpha_j^2$.

Proof. (a) Define $b_n(z)$ so the right-hand side of (3.1) is $1+b_n$. Then

$$(1-Q)^2 \le \big(1-|\alpha_n|\big)^2 \le 1+b_n \le \big(1+\alpha_n\big)^2 \le (1+Q)^2. \tag{3.5}$$

It follows that for Q dependent constant K, we have $e^{-Kb_n^2} \le (1+b_n)e^{-b_n} \le e^{Kb_n^2}$. Since $b_n \le 3|\alpha_n|$ and b_n has an α_n^2 in it, we have

$$\begin{aligned} \exp\big(-(9K+1)\alpha_n^2\big) &\le \big(1+b_n\big)\exp\big(-2\alpha_n\cos\big((n+1)\eta+2\theta_n\big)\big) \\ &\le \exp\big((9K+1)\alpha_n^2\big). \end{aligned} \tag{3.6}$$

Thus, by (3.1) and $R_0 = 1$, we have (3.3) with

$$C = \exp\left((9K+1)\sum_{j=0}^{\infty}\alpha_j^2\right). \tag{3.7}$$

(b) Taking imaginary parts of both sides of (3.2) and using the lower bound $1-Q$ on the denominator, we get

$$\left|\sin\big(\theta_{n+1}-\theta_n\big)\right| \le \left|\alpha_n\right|(1-Q)^{-1}. \tag{3.8}$$

Equation (3.2) also implies that $\cos(\theta_{n+1}-\theta_n) > (1-Q)/(1+Q) > 0$, so $|\theta_{n+1}-\theta_n| < \pi/2$. Since $|x_n| < \pi/2$ implies $|x| \le (\pi/2)|\sin x|$, (3.4) follows from (3.8). ■

The point of (3.3) is to control $|R_n|$, we need to control $\sum_{j=0}^{n-1} \alpha_j \cos((j+1)\eta + 2\theta_j)$. In using the Schwartz inequality, we decouple the α_j's and the cosines so the key will be the following lemma (essentially in [1]).

Lemma 3.3. (i) If k is not a multiple of 2π,

$$\left| \sum_{j=1}^{n} \frac{\cos(kj + \theta_j)}{j} \right| \leq \frac{1}{\left| \sin\left(\frac{k}{2}\right) \right|} \left[1 + \sum_{j=1}^{n-1} \frac{|\theta_{j+1} - \theta_j|}{j+1} \right]. \tag{3.9}$$

(ii) If k is not a multiple of π,

$$\sum_{j=1}^{N} \frac{\cos^2(kj + \theta_j)}{j} \leq \frac{1}{2}(\log N + 1 + C), \tag{3.10}$$

where

$$C = \frac{1}{|\sin(k)|} \left[1 + 2 \sum_{j=1}^{\infty} \frac{|\theta_{j+1} - \theta_j|}{j+1} \right]. \tag{3.11}$$

□

Proof. (i) Since $\cos(kj + \theta_j) = \mathrm{Re}(\exp(ikj)\exp(i\theta_j))$, it suffices to prove (3.9) with cosines replaced by complex exponentials. Define $b_n = \sum_{j=1}^{n} e^{ikj}$ so, by summing the geometric series,

$$|b_n| \leq \frac{1}{\left| \sin\left(\frac{k}{2}\right) \right|}. \tag{3.12}$$

If $a_j = e^{i\theta_j}/j$, then the sum we want to control is $\sum_{j=1}^{n} (b_j - b_{j-1}) a_j$ with $b_0 = 0$. But

$$\sum_{j=1}^{n} (b_j - b_{j-1}) a_j = -\sum_{j=1}^{n} b_j (a_{j+1} - a_j), \tag{3.13}$$

where $a_{n+1} = 0$. Thus

$$\left| \sum_{j=1}^{n} \frac{\cos(kj + \theta_j)}{j} \right| \leq \frac{1}{\sin\left(\frac{k}{2}\right)} \left[\sum_{j=1}^{n-1} |a_{j+1} - a_j| + |a_n| \right]. \tag{3.14}$$

Clearly,

$$\begin{aligned} |a_{j+1} - a_j| &\leq \left| \left(e^{i\theta_{j+1}} - e^{i\theta_j}\right)(j+1)^{-1} \right| + \left| e^{i\theta_j} \left(j^{-1} - (j+1)^{-1}\right) \right| \\ &\leq |\theta_{j+1} - \theta_j| (j+1)^{-1} + j^{-1} - (j+1)^{-1}. \end{aligned} \tag{3.15}$$

Since $\sum_{j=1}^{n-1} j^{-1} - (j+1)^{-1} + n^{-1} = 1$, (3.14) and (3.15) yield (3.9).

(ii) Since $\cos^2(x) = (1/2)(1+\cos(2x))$ and

$$\sum_{j=1}^{n} \frac{1}{j} \leq 1 + \int_1^n \frac{dx}{x} = 1 + \log(n), \tag{3.16}$$

(3.9) implies (3.10). ∎

Proof of Theorem 1.3. Write

$$\begin{aligned} &\left| \sum_{j=0}^{n-1} \alpha_j \cos\big((j+1)\eta + 2\theta_j\big) \right| \\ &\quad \leq \left(\sum_{j=0}^{n-1} (j+1)|\alpha_j|^2 \right)^{1/2} \left(\sum_{j=0}^{n-1} \frac{\cos^2\big((j+1)\eta + 2\theta_j\big)}{(j+1)} \right)^{1/2}. \end{aligned} \tag{3.17}$$

By hypothesis, the first sum on the right-hand side of (3.17) is bounded by $A \log N + C$. By the lemma, if η is not a multiple of π (i.e., $z \neq \pm 1$), the second sum is bounded by

$$\frac{1}{2}\Big(\log n + \widetilde{C} |\sin(\eta)|^{-1} \Big), \tag{3.18}$$

where $\widetilde{C}$ can be chosen independently of z, since

$$\begin{aligned} \sum_{j=1}^{\infty} \frac{|\theta_{j+1} - \theta_j|}{j+1} &\leq C \sum_{j=0}^{\infty} \frac{|\alpha_j|}{j+2} \quad \text{(by (3.8))} \\ &\leq C \left(\sum_{j=0}^{\infty} |\alpha_j|^2 \right)^{1/2} \left(\sum_{j=0}^{\infty} \frac{1}{(j+2)^2} \right)^{1/2} \end{aligned} \tag{3.19}$$

is finite. Thus, for $\eta \in [\theta_0, \pi - \theta_0]$,

$$\begin{aligned} |R_n| &\leq C \exp \big| [A(\log n) + C] \big|^{1/2} \left[\frac{1}{2} \log n + \widetilde{C} |\sin\theta_0|^{-1} \right]^{1/2} \\ &\leq C_\varepsilon n^{\sqrt{(1/2)A + \varepsilon}}, \end{aligned} \tag{3.20}$$

where C_ε depends on ε and θ_0. So long as $\sqrt{(1/2)A} < 1/2$, we can apply Corollary 2.5, that is, $A < 1/2$. We conclude $d\mu_s(I) = 0$ for $I = \pm(\theta_0, \pi - \theta_0)$, that is, $d\mu_s$ is supported on $\{\pm 1\}$. As already noted, at ± 1, $|\varphi_n(\pm 1)| \geq n^{-\sqrt{A}}$, so if $A = 1/4$, $\varphi_n(\pm 1)$ are not in L^2 and $d\mu_s = 0$.
∎

4 Sequences of bounded variation and the proof of Theorem 1.4

To obtain Theorem 1.4, we need one more summation-by-parts argument that will supplement Lemma 3.3.

Lemma 4.1. If k is not a multiple of 2π,

$$\left|\sum_{j=1}^{n} c_j \cos\left(kj+\theta_j\right)\right| \leq \frac{1}{\sin\left(\frac{k}{2}\right)}\left[\sum_{j=1}^{\infty}\left|c_{j+1}-c_j\right| + \sup_n \left|c_n\right| + \sum_{j=1}^{\infty}\left|c_{j+1}\right|\left|\theta_{j+1}-\theta_j\right|\right]. \tag{4.1}$$

□

Proof. As in the proof of Lemma 3.3, let $b_n = \sum_{j=1}^{n} e^{ikj}$, so (3.12) holds and $a_k = c_j e^{i\theta_j}$. Then summing by parts as in the earlier lemma,

$$\left|\sum_{j=1}^{n} c_j \cos\left(kj+\theta_j\right)\right| \leq \frac{1}{\sin\left(\frac{k}{2}\right)}\left[\left|c_n\right| + \sum_{j=1}^{n-1}\left|a_{j+1}-a_j\right|\right]. \tag{4.2}$$

But

$$\left|a_{j+1}-a_j\right| \leq \left|a_{j+1}\right|\left|e^{i\theta_{j+1}}-e^{i\theta_j}\right| + \left|c_{j+1}-c_j\right|, \tag{4.3}$$

so (4.1) follows. ■

Proof of Theorem 1.4. By (3.3) and the hypothesis that $\sum_{j=0}^{\infty}|\alpha_j|^2<\infty$,

$$\left|\Psi_n(z)\right| \leq C \exp\left(\sum_{j=0}^{n-1}\alpha_j \cos\left((j+1)\eta+2\theta_j\right)\right). \tag{4.4}$$

Write

$$\sum_{j=0}^{n-1}\alpha_j \cos\left((j+1)\eta+2\theta_j\right) = O_n + E_n, \tag{4.5}$$

where O_n is the sum over odd values of j and E_n over even values.

By Lemma 4.1,

$$\left|O_n\right| \leq \frac{1}{\left|\sin\left(\frac{\eta}{2}\right)\right|}\left[\sup_n \left|\alpha_{2n-1}\right| + \sum_{n=1}^{\infty}\left|\alpha_{2n+1}-\alpha_{2n-1}\right| + \sum_{n=1}^{\infty}\left|\alpha_{2n+1}\right|\left|\theta_{2n+1}-\theta_{2n-1}\right|\right]. \tag{4.6}$$

Since α_{2n-1} is monotone, $\sum_{n=1}^{\infty} |\alpha_{2n+1} - \alpha_{2n-1}| = |\alpha_1|$. By (3.4), $|\theta_{2n+1} - \theta_{2n-1}| \leq C(|\alpha_{2n-1}| + |\alpha_{2n}|)$ so, since $\sum_{n=0}^{\infty} |\alpha_n|^2 < \infty$, (4.6) implies

$$|O_n| \leq C\left|\sin\left(\frac{\eta}{2}\right)\right|^{-1}. \tag{4.7}$$

For E_n, we use the same argument as in the proof of Theorem 1.3, taking into account that the change in frequency from $2n$ to $2n+2$ is $2\eta + (\theta_{2n+2} - \theta_{2n})$. Thus $|\sin(\eta)|^{-1}$ becomes $|\sin(2\eta)|^{-1}$, and we find that

$$|E_n| \leq (A \log N + C)^{1/2}\left(\frac{1}{2}\log N + C|\sin(2\eta)|^{-1}\right)^{1/2}. \tag{4.8}$$

Inequalities (4.7) and (4.8) imply that for any $\theta_0 > 0$ for all $z \in (\theta_0, \pi/2 - \theta_0) \cup (\pi/2 + \theta_0, \pi - \theta_0)$,

$$|\Psi_n| \leq C_{\theta_0,\varepsilon} n^{\varepsilon + \sqrt{A/2}} \tag{4.9}$$

which, by Corollary 2.5, implies μ_s is restricted to ± 1 and $\pm i$.

To obtain the result on eigenvalues when $A = 1/4$, note first that since $|E_n| \leq \sum_{m=1}^{[n/2]} |\alpha_{2m}|$, uniformly in z,

$$e^{|E_n|} \leq Cn^{1/2}, \tag{4.10}$$

if $A = 1/4$. At $z = \pm i$, (4.7) implies $e^{|O_n|}$ is bounded, so $|\varphi_n(\pm i)| \geq Cn^{-1/2}$ is not in L^2.

At ± 1, we use Theorem 2.6. Since $\alpha_{2j-1} \leq 0$, (2.22) implies that

$$|\varphi_n(\pm 1)| \geq \prod_{\substack{j=0 \\ j \text{ even}}}^{n} (1 - |\alpha_j|) \geq Cn^{-1/2} \tag{4.11}$$

since $1 - \alpha_{2j-1} \geq 1$. So ± 1 are not eigenvalues if $\alpha_{2j-1} \leq 0$. ■

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