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Connectedness of the Isospectral Manifold for One-Dimensional Half-Line Schrödinger Operators[1]

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Let V_0 be a real-valued function on $[0, \infty)$ and $V \in L^1([0, R])$ for all $R > 0$ so that $H(V_0) = -\frac{d^2}{dx^2} + V_0$ in $L^2([0, \infty))$ with $u(0) = 0$ boundary conditions has discrete spectrum bounded from below. Let $\mathscr{M}(V_0)$ be the set of V so that $H(V)$ and $H(V_0)$ have the same spectrum. We prove that $\mathscr{M}(V_0)$ is connected.

KEY WORDS: Isospectral sets of potentials; half-line Schrödinger operators; inverse problems.

The limitation of our knowledge of inverse spectral theory for Schrödinger operators $H_{\mathbb{R}}(V) = -\frac{d^2}{dx^2} + V$ in $L^2(\mathbb{R})$ is shown by the following open question: Let $\mathscr{M}_{\mathbb{R}}(x^2)$ denote the set of all V's with $\sigma(H_{\mathbb{R}}(V)) = \{1, 3, 5, \ldots\}$, the spectrum of the harmonic oscillator $-\frac{d^2}{dx^2} + x^2$ in $L^2(\mathbb{R})$. Is $\mathscr{M}_{\mathbb{R}}(x^2)$ connected? One can ask the same question for restricted sets of V's, say requiring that V is $C^k(\mathbb{R})$ for some $k \in \mathbb{N}$ or $C^\infty(\mathbb{R})$. The question remains open for sets so large that they include the translates of V .

If one demands that V be close to x^2 in a strong sense, there are some results that go back to McKean–Trubowitz[(9)] (see also Levitan[(7)]), culminating in the recent beautiful paper of Chelkak, Kargaev, and Korotyaev[(1)] who require $V(x) = x^2 + q(x)$ with $\int [|q'(x)|^2 + x^2 \, |q^2(x)|^2] \, dx < \infty$ (by a Sobolev estimate such q's have $|q(x)| \to 0$ as $|x| \to \infty$). Their analysis implies the set of $V \in \mathscr{M}_{\mathbb{R}}(x^2)$ obeying this estimate is connected.

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[1] To Elliott Lieb on his 70th birthday, with our best wishes.

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The purpose of this note is to make what turns out to be an elementary observation: The corresponding problem for the half-line is easy! Suppose V is real-valued and in $L^1([0, R])$ for all $R > 0$. We consider $H(V) = -\frac{d^2}{dx^2} + V$ in $L^2([0, \infty))$ with $u(0) = 0$ (i.e., Dirichlet) boundary conditions. If V is merely in $L^1([0, R])$ for all $R > 0$, we will use that to define a topology on $\mathcal{M}(V)$. If V is in $C^k([0, \infty))$ for some $k \in \mathbb{N}$, we will use the norm on $C^k([0, R])$ for all $R > 0$ to define the topology on $\mathcal{M}(V)$. Here is our result:

Theorem 1. Let V_0, V_1 be real-valued and in $L^1([0, R])$ for all $R > 0$ so that $H(V_\ell)$ is bounded from below (which means that $-\frac{d^2}{dx^2} + V_\ell$ is in the limit point case at ∞, cf. ref. 5) and each has discrete spectrum $\mathrm{spec}(H(V_\ell)) = \{E_j(V_\ell)\}_{j \in \mathbb{N}}$, $\ell = 0, 1$.

Suppose

$$\mathrm{spec}(H(V_0)) = \mathrm{spec}(H(V_1)). \tag{1}$$

Then there exists $\{V_t\}_{0 \leqslant t \leqslant 1}$, with V_t real-valued and in $L^1([0, R])$ for all $R > 0$, interpolating V_0 and V_1 so that

$$\mathrm{spec}(H(V_t)) = \mathrm{spec}(H(V_0)), \qquad t \in [0, 1]$$

and $t \to V_t|_{[0, R]}$ is continuous in $L^1([0, R])$ for all $R > 0$. Moreover,

(i) $t \to V_t|_{[0, R]}$ is real analytic in $L^1([0, R])$ for all $R > 0$.

(ii) If V_0 and V_1 are $C^k([0, \infty))$ for some $k \in \mathbb{N}$, then V_t is $C^k([0, \infty))$ and $t \to V_t|_{[0, R]}$ is real analytic in $C^k([0, R]))$ for all $R > 0$.

We recall that all eigenvalues $\{E_j(V)\}_{j \in \mathbb{N}}$ of the Dirichlet operator $H(V)$ in $L^2([0, \infty))$ are simple.

The proof exploits the A-function studied by us in refs. 4 and 13. A can be defined in terms of the spectral measure defined in the standard way (see, e.g., ref. 8) so that the Weyl m-function satisfies

$$m(z) = c + \int_{\mathbb{R}} \left[\frac{1}{\lambda - z} - \frac{\lambda}{1 + \lambda^2} \right] d\rho(\lambda), \qquad z \in \mathbb{C} \setminus \mathrm{spec}(H(V)). \tag{2}$$

We let $d\rho_0$ denote the spectral measure for the case $V = 0$. It is well-known, since

$$m(z, V = 0) = i\sqrt{z}, \tag{3}$$

that

$$d\rho_0(E) = \frac{1}{\pi} \sqrt{E}\, \chi_{[0, \infty)}(E)\, dE. \tag{4}$$

A is then defined by

$$A(\alpha) = -2 \int_{-\infty}^{\infty} \lambda^{-\frac{1}{2}} \sin(2\alpha \sqrt{\lambda})\, [d\rho(\lambda) - d\rho_0(\lambda)], \tag{5}$$

where the integral is intended in the distributional sense on $(-\infty, \infty)$ (so, a priori, A is only a distribution, not a function). Of course, $A = 0$ for $V = 0$.

One can also define a distribution A by (5) with ρ_0 dropped (cf., ref. 4) but then this distribution is only L^1_{loc} away from 0 and has a δ' singularity at 0. We will not use this approach in this note.

The fact on which Theorem 1 depends is the following:

Theorem 2. $d\rho$ is the spectral measure of an $H(V)$ with $V \in L^1([0, R])$ for all $R > 0$, V real-valued, if and only if $A \in L^1_{\text{loc}}(\mathbb{R})$. $V \in C^k([0, \infty))$ for some $k \in \mathbb{N}$ if and only if $A \in C^k(\mathbb{R})$. If $d\rho_t$ is a family so that $A_t|_{[-R, R]}$ is real analytic in t in $L^1([-R, R])$ (resp. $C^k([-R, R])$) for all $R > 0$, then $t \to V_t$ is real analytic in t in $L^1([0, R])$ (resp. $C^k([0, R])$) for all $R > 0$.

This theorem combines results from Gesztesy–Simon[4] and Simon[13] (who show V is $C^k([0, \infty))$ if and only if A is, once one knows V exists) and work of Remling[10] or a suitable version of the Gel'fand–Levitan theory,[6] Chap. 2, ref. 8, Section 2.3, to get the existence part of Theorem 2.

Once one has Theorem 2, Theorem 1 is immediate.

Proof of Theorem 1. Let $\{E_j\}_{j \in \mathbb{N}}$ be the common spectrum of V_0 and V_1 so

$$d\rho_\ell(E) = \sum_{j \in \mathbb{N}} a_{j,\ell}\, \delta(E - E_j), \qquad \ell = 1, 2. \tag{6}$$

Define

$$d\rho_t(E) = \sum_{j \in \mathbb{N}} [t a_{j,1} + (1-t)\, a_{j,0}]\, \delta(E - E_j), \qquad t \in [0, 1].$$

The associated A-functions satisfy

$$A_t = t A_1 + (1-t)\, A_0, \qquad t \in [0, 1]. \tag{7}$$

Clearly, if A_0, A_1 are $L^1_{\text{loc}}(\mathbb{R})$, so is A_t, and if A_0, A_1 are $C^k(\mathbb{R})$, so are A_t. Thus, $d\rho_t$ is the spectral measure of a potential V_t that, by Theorem 2, has the claimed properties. ∎

Remarks. 1. The C^k result extends to C^∞ and one can also extend it to real analyticity.

2. Because V on $[0, x_0]$ only depends on A on $[0, x_0]$ (see refs. 4 and 13), if V_0 and V_1 are C^k on $[0, x_0]$, so is each V_t.

The key to our proof is the fact that, while not all measures $\sum_{j\in\mathbb{N}} a_j\, \delta(E-E_j)$ are spectral measures (the fact that the A-transform of $\rho-\rho_0$ has no singularity at $\alpha=0$, which means the ρ-term alone has a specific singularity that places restrictions on the a_j), those that are form a convex, hence, connected set.

The difficulty of extending this to potentials on $\mathbb{R}$ is that there is no known way to describe when a candidate spectral measure is, in fact, the spectral measure for a potential on $\mathbb{R}$. Typically, one reduces the whole-line problem on $\mathbb{R}$ to two half-line problems on $(-\infty, x_0)$ and (x_0, ∞) for some fixed $x_0 \in \mathbb{R}$ (cf. ref. 6, Chap. 7, and refs. 11 and 12) but, in general, loses control over the potential at the point x_0 (in the sense that generally the potential V will be discontinuous at $x=x_0$). To determine the potential, the spectral measure is a 2×2 matrix in this case,[2] Section 9.5, ref. 3, Section 10.3.5, and because of restrictions on this matrix, convex combinations of the matrix measures coming from a potential will *not* come from a potential, so our method cannot extend.

We, of course, believe that for the whole-line case, there is also a result on connectedness of the spectral manifold for a potential like $V(x)=x^2$. But we wonder about a case like

$$V(x) = \ln(|x|+1) + \exp(x) \tag{8}$$

with very different asymptotics if $x\to\infty$ and $x\to-\infty$. $V(x)$ and $V(-x)$ are obviously isospectral, but we wonder if there is a path between them in the isospectral manifold. It might be that the correct conjecture is that for any V, $\mathcal{M}_{\mathbb{R}}(V)$ consists of either one or two connected components.

We hope that this note will stimulate more work on this problem.

We close by thanking Elliott Lieb for his many years of research results, insights, and service to the mathematics and physics communities, and by expressing the hope that he has enjoyed this birthday bouquet.

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