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UNIFORM CROSSNORMS

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A crossnorm on a pair of Banach spaces (X, Y) is a norm, α, on the algebraic tensor product $X \odot Y$ obeying $\alpha(x \otimes y) = \|x\| \|y\|$ for all $x \in X$, $y \in Y$. When Schatten introduced crossnorms, he singled out two general classes of crossnorms: the dualizable crossnorms (called by him "crossnorms whose associates are crossnorms") and the uniform crossnorms. These are crossnorms which induce in a natural way other crossnorms: in the dualizable case, a crossnorm, α_d, on $X^* \odot Y^*$, and in the uniform case, a crossnorm, $\tilde{\alpha}$, on $\mathscr{L}(X) \odot \mathscr{L}(Y)$ where $\mathscr{L}(X)$ is the algebra of bounded operators on X. Our main new result is a proof that if α is a uniform crossnorm, then $\tilde{\alpha}$, the induced crossnorm on $\mathscr{L}(X) \odot \mathscr{L}(Y)$ is dualizable.

This result will be applied to the theory of tensor products of commutative Banach algebras.

§1. Basic definitions and facts.

We recall several definitions from Schatten [5]:

DEFINITION 1. A norm α on $X \odot Y$, the algebraic tensor product of two Banach spaces X and Y, is called a *crossnorm* if and only if $\alpha(x \otimes y) = \|x\| \|y\|$ for all $x \in X$, $y \in Y$.

DEFINITION 2. A crossnorm, α, on $X \odot Y$ is called *dualizable* if and only if for all $l \in X^*$, $\mu \in Y^*$, $z \in X \odot Y$:

$$|(l \otimes \mu)(z)| \leqq \|l\| \|\mu\| \alpha(z) .$$

REMARKS. 1. We have replaced Schatten's awkward "crossnorm whose associate is a crossnorm" whith the term "dualizable crossnorm".

2. It is a simple exercise [5] to show that if α is dualizable, and $\lambda \in X^* \odot Y^*$, then

$$\alpha_d(\lambda) \equiv \sup_{z \in X \odot Y} |\lambda(z)|/\alpha(z)$$

defines a crossnorm, α_d, on $X^* \odot Y^*$.

DEFINITION 3. α_d is called the *dual* crossnorm of α (Schatten uses the term associated crossnorm).

We will use $\mathscr{L}(X)$ to denote the Banach algebra of all bounded operators on X.

DEFINITION 4. A crossnorm, α, on $X \odot Y$ is called *uniform* if and only if for all $A \in \mathscr{L}(X)$, $B \in \mathscr{L}(Y)$ and $z \in X \odot Y$:

$$\alpha((A \otimes B)z) \leqq \|A\| \, \|B\| \alpha(z) .$$

Similar to dualizable crossnorms, for $C \in \mathscr{L}(X) \odot \mathscr{L}(Y)$, the quantity

$$\tilde{\alpha}(C) = \sup_{z \in X \odot Y} \alpha(Cz)/\alpha(z)$$

defines a crossnorm $\tilde{\alpha}$ on $\mathscr{L}(X) \odot \mathscr{L}(Y)$.

DEFINITION 5. $\tilde{\alpha}$ is called the induced *crossnorm* of α.

There is an elementary fact about crossnorms which does not seem to have been noted in the literature:

THEOREM 1. *Every uniform crossnorm is dualizable.*

Proof. Let $l \in X^*$, $\mu \in Y^*$. Pick $x \neq 0$ in X, $y \neq 0$ in Y. Let $A \in \mathscr{L}(X)$ be given by $Ax' = l(x')x$ and $B \in \mathscr{L}(Y)$ by $By' = \mu(y')y$. Then $\|A\| = \|l\| \, \|x\|$, $\|B\| = \|\mu\| \, \|y\|$ and

$$\alpha((A \otimes B)z) = |(l \otimes \mu)(z)| (\alpha(x \otimes y) = \|x\| \, \|y\| \, |(l \otimes \mu)(z)| .$$

It follows that if

$$\alpha((A \otimes B)z) \leqq \|A\| \, \|B\| \alpha(z) ,$$

then

$$|l \otimes \mu(z)| \leqq \|l\| \, \|\mu\| \alpha(z) .$$

Finally we recall the two "canonical" crossnorms of Schatten and some facts about them:

DEFINITION 6. γ is the function on $X \odot Y$ given by

$$\gamma(z) = \inf \{ \sum_{i=1}^{n} \|x_i\| \, \|y_i\| \mid z = \sum_{i=1}^{n} x_i \otimes y_i \} .$$

DEFINITION 7. Given $z \in X \odot Y$, define $\prod_z \in \mathscr{L}(X^*, Y)$, the bounded operators from X^* to Y by

$$\prod_{\left(\sum_{i=1}^{n} x_i \otimes y_i\right)} (l) = \sum_{i=1}^{n} l(x_i) y_i .$$

λ is the function on $X \odot Y$ given by

$$\lambda(z) = \|\prod_z\|_{\mathscr{L}(X^*,Y)}$$

where $\|\cdot\|_{\mathscr{L}(X^*,Y)}$ is the operator norm.

THEOREM 2. (Schatten [5])
(a) γ and λ *are uniform (dualizable) crossnorms.*
(b) *If* α *is any crossnorm* $\alpha \leqq \gamma$.
(c) *A norm is a dualizable crossnorm if and only if* $\lambda \leqq \alpha \leqq \gamma$.

REMARKS. 1. Schatten calls γ the *greatest crossnorm* and λ the *least crossnorm whose associate is a crossnorm.*

2. We will use the symbols $\gamma_{X\odot Y}$ and $\lambda_{X\odot Y}$ where there might be some confusion as to which algebraic tensor tensor product is intended.

3. The completion of $X \odot Y$ in the crossnorm, α, will be denoted $X \otimes_\alpha Y$.

2. The main result.

The main new result of this paper is:

THEOREM 3. *Let* α *be a uniform crossnorm on* $X \odot Y$. *Then the induced norm* $\tilde{\alpha}$ *on* $\mathscr{L}(X) \odot \mathscr{L}(Y)$ *is dualizable.*

This is a rather technical looking result but it is motivated by a fairly simple problem which we discuss in §3. The heart of the proof is the following density lemma.

LEMMA 1. *Let X be a Banach space. Endow* $\mathscr{L}(X)^*$ *with the weak-* topology. Given* $l \in \mathscr{L}(X)$ *and* $x \in X$, *let* $L_{l,x} \in \mathscr{L}(X)^*$ *be defined by* $L_{l,x}(A) = l(Ax)$. *Then the (weak *-) closed convex hull of* $\{L_{l,x} \mid \|l\| = \|x\| = 1\}$ *is the entire unit ball in* $\mathscr{L}(X)^*$.

Proof. Suppose $L_0 \in \mathscr{L}(X)^*$ and L_0 is not in the closed convex hull of $\{L_{l,x} \mid \|l\| = \|x\| = 1\}$. Then by the Hahn-Banach theorem, there exists a weak *-continuous linear functional, Λ, on $\mathscr{L}(X)^*$ with $\operatorname{Re} \Lambda(L_{l,x}) \leqq a$ for all l and x with $\|l\| = 1$, $\|x\| = 1$, and with $\operatorname{Re} \Lambda(L_0) > a$. Since $L_{l,cx} = cL_{l,x}$ for any scalar c, by rescaling Λ, we can suppose $|\Lambda(L_{l,x})| \leqq 1$; $\Lambda(L_0) > 1$. But every weak *-continuous functions Λ is of the form $\Lambda(L) = L(A)$ for some $A \in \mathscr{L}(X)$ (see [4], pp. 114-115). Thus $\sup_{\|l\|=\|x\|=1} |l(Ax)| \leqq 1$ and $L_0(A) > 1$. The first inequality implies $\|A\| \leqq 1$ so the second implies $\|L_0\| > 1$.

Proof of Theorem 3. By Theorem 2, we need only show that $\lambda_{\mathscr{L}(X)\odot\mathscr{L}(Y)} \leqq \tilde{\alpha}$. But, by definition, if $C = \sum_{i=1}^n A_i \otimes B_i \in \mathscr{L}(X) \odot \mathscr{L}(Y)$, then $\lambda(C) = \|\prod_c\|$ where $\prod_c \colon \mathscr{L}(X)^* \to \mathscr{L}(Y)$ by $\prod_c(L) = \sum_{i=1}^n L(A_i)B_i$.

Since $\prod_c$ has this form, it is weak *-continuous, i.e., if $L_\alpha \to L$ in the $\mathscr{L}(X)^*$ -weak* topology, then $\prod_c(L_\alpha) \to \prod_c(L)$ in $\mathscr{L}(Y)$-norm. Thus $\|\prod_c\| = \sup_{L \in S} \|\prod_c(L)\|_{\mathscr{L}(Y)}$ for any set S whose closed convex hull is the unit ball of $\mathscr{L}(X)^*$. Using the lemma and

$$\|B\| = \sup\{|\mu(By)| \mid \|\mu\| = \|y\| = 1\},$$

we conclude:

$$\begin{aligned}&\lambda_{\mathscr{L}(X)\odot\mathscr{L}(Y)}(C) = \\ &\sup\{|(l \otimes \mu)[C(x \otimes y)]| \mid l \in X^*, \mu \in Y^*, x \in X, y \in Y; \|l\| \\ &= \|\mu\| = \|x\| = \|y\| = 1\}.\end{aligned}$$

Let α be a uniform crossnorm, then since α is dualizable,

$$\begin{aligned}|(l \otimes \mu)[C(x \otimes y)]| &\leqq \alpha_d(l \otimes \mu)\alpha(C(x \otimes y)) \\ &\leqq \tilde{\alpha}(C)\alpha_d(l \otimes \mu)\alpha(x \otimes y) \\ &= \tilde{\alpha}(C)\,\|l\|\,\|\mu\|\,\|x\|\,\|y\|.\end{aligned}$$

We conclude $\lambda \leqq \tilde{\alpha}$ and with that, the theorem.

3. Tensor products of commutative Banach algebras. Now let $\mathfrak{A}_1, \mathfrak{A}_2$ be Banach algebras with identities.

DEFINITION 8. If $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are Banach algebras (with identity) a crossnorm, α, on $\mathfrak{A}_1 \odot \mathfrak{A}_2$ (which is an algebra) is called a *B-algebra crossnorm* if and only if $\alpha(xy) \leqq \alpha(x)\alpha(y)$ for all $x, y \in \mathfrak{A}_1 \odot \mathfrak{A}_2$.

Surprisingly, the following question is open.

Question 1. Let $\mathfrak{A}_1, \mathfrak{A}_2$ be commutative Banach algebras with identity. Let $\sigma(\cdot)$ denote the spectrum of the algebra $\cdot$. Let α be a B-algebra crossnorm on $\mathfrak{A}_1 \odot \mathfrak{A}_2$. Then $\mathfrak{A}_1 \otimes_\alpha \mathfrak{A}_2$ is a commutative B-algebra. Is $\sigma(\mathfrak{A}_1 \otimes_\alpha \mathfrak{A}_2) = \sigma(\mathfrak{A}_1) \times \sigma(\mathfrak{A}_2)$?

One can be more explicit. If l is a multiplicative linear functional on $\mathfrak{A}_1 \otimes_\alpha \mathfrak{A}_2$, then $l(\cdot \otimes 1)$ and $l(1 \otimes \cdot)$ define elements $l_1 \in \sigma(\mathfrak{A}_1)$ and $l_2 \in \sigma(\mathfrak{A}_2)$ with $l = l_1 \otimes l_2$. Thus, to conclude that $\sigma(\mathfrak{A}_1 \otimes_\alpha \mathfrak{A}_2) = \sigma(\mathfrak{A}_1) \times \sigma(\mathfrak{A}_2)$, it is sufficient to show that for any $l_1 \in \sigma(\mathfrak{A}_1)$ and $l_2 \in \sigma(\mathfrak{A}_2)$, $l_1 \otimes l_2$ defines an α-bounded linear functional on $\mathfrak{A}_1 \odot \mathfrak{A}_2$ which then extends to $\mathfrak{A}_1 \otimes_\alpha \mathfrak{A}_2$ We conclude:

LEMMA 2. [7] *If α is a Banach algebra crossnorm on commutative algebras wnich is dualizable, then $\sigma(\mathfrak{A}_1 \otimes_\alpha \mathfrak{A}_2) = \sigma(\mathfrak{A}_1) \times \sigma(\mathfrak{A}_2)$.*

Question 2. Is every Banach algebra crossnorm on commutative

Banach algebras dualizable?

An affirmative answer to Question 2 would, of course, imply an affirmative answer to Question 1. Our main remark is that Theorem 3 implies that question 1 has an affirmative answer in a situation which arises quite often in practice.

THEOREM 4. *Let X and Y be Banach spaces and let α be a uniform crossnorm. Let $\mathfrak{A}_1$ be a commutative subalgebra of $\mathscr{L}(X)$ and let $\mathfrak{A}_2$ be a commutative subalgebra of $\mathscr{L}(Y)$. Let $\mathfrak{A}$ be the subalgebra of $\mathscr{L}(X \otimes_\alpha Y)$ generated by*

$$\{A \otimes B \mid A \in \mathfrak{A}_1; B \in \mathfrak{A}_2\} .$$

Then

$$\mathfrak{A} = \mathfrak{A}_1 \otimes_{\tilde{\alpha}} \mathfrak{A}_2, \sigma(\mathfrak{A}) = \sigma(\mathfrak{A}_1) \times \sigma(\mathfrak{A}_2) .$$

Proof. That $\mathfrak{A} = \mathfrak{A}_1 \otimes_{\tilde{\alpha}} \mathfrak{A}_2$ is a trivial fact. That $\sigma(\mathfrak{A}) = \sigma(\mathfrak{A}_1) \times \sigma(\mathfrak{A}_2)$ follows from Theorem 3 and Lemma 2.

REMARKS. 1. This theorem is not new in the case X and Y are Hilbert spaces and α is the Hilbert space inner product. For $\tilde{\alpha}$ on $\mathscr{L}(X) \otimes \mathscr{L}(Y)$ is a C^*-norm, so by a result of Takesaki [6] $\tilde{\alpha} \geqq \lambda_{\mathscr{L}(X) \otimes \mathscr{L}(Y)}$. By the "local nature" of λ[5], one concludes $\tilde{\alpha} \geqq \lambda_{\mathfrak{A}_1 \otimes \mathfrak{A}_2}$.

2. The special case of this theorem where X and Y are commutative Banach algebras and $\mathfrak{A}_1 = \{L_x \mid x \in X\}$, $\mathfrak{A}_2 = \{L_y \mid y \in Y\}$ with $L_a b = ab$, is due to J. Gil de Lamadrid [2]. He proves in his special case that $\tilde{\alpha} \geqq \lambda$ without requiring a Hahn-Banach argument as in Lemma 1.

3. The special case of this theorem where $\mathfrak{A}_1$ and $\mathfrak{A}_2$ are generated by the resolvents of a single operator has been proven by Reed-Simon [4] using the fact that the only compact analytic subvarieties of C^2 are points. We note in passing that Theorem 3 does not allow a simplification of [4] since the machinery needed to prove the special use of Theorem 4 is needed to prove other results.

4. Under the hypotheses of the theorem it is also quite easy to prove $\partial_{\mathfrak{A}} = \partial_{\mathfrak{A}_1} \times \partial_{\mathfrak{A}_2}$ where ∂ is the Shilov boundary. The proof is the same as in the special case $\mathfrak{A}_1 \otimes_\gamma \mathfrak{A}_2$ [1].

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