**IMRN** International Mathematics Research Notices
2004, No. 22

# Necessary and Sufficient Conditions in the Spectral Theory of Jacobi Matrices and Schrödinger Operators

**David Damanik, Rowan Killip, and Barry Simon**

## 1 Introduction

In this paper, we want to describe some new results in the spectral and inverse spectral theory of half-line Schrödinger operators and Jacobi matrices. Given $V \in L^1_{\mathrm{loc}}(0,\infty)$ with a mild regularity condition at infinity (ensuring limit-point case there, cf. [22]), one can define a unique selfadjoint operator which is formally

$$H = -\frac{d^2}{dx^2} + V(x) \tag{1.1}$$

with the boundary condition $u(0) = 0$ (see, e.g., [22]). For any $z \notin \mathbb{R}$, there is a solution $u_+(x;z)$ of $-u'' + Vu = zu$ which is $L^2$ at infinity and unique up to a constant. The Weyl $m$-function is then defined by

$$m(z) = \frac{u_+'(0;z)}{u_+(0;z)}. \tag{1.2}$$

It obeys $\operatorname{Im} m(z) > 0$ when $\operatorname{Im} z > 0$, which implies that $\operatorname{Im} m(E+i\varepsilon)$ has a boundary value as $\varepsilon \downarrow 0$ in distributional sense:

$$d\rho(E) = \operatorname*{w-lim}_{\varepsilon\downarrow 0} \frac{1}{\pi} \operatorname{Im} m(E+i\varepsilon)\,dE. \tag{1.3}$$

We call $d\rho$ the spectral measure.

Received 11 September 2003. Revision received 3 December 2003.
Communicated by Percy Deift.

In this way, each $V$ gives rise to a spectral measure $d\rho$. In fact, the correspondence is one to one: Gel'fand and Levitan [11, 12] (see also Simon [29]) found an inverse procedure to go from $d\rho$ to $V$.

Similarly, given a Jacobi matrix, $a_n > 0$, $b_n \in \mathbb{R}$,

$$J = \begin{pmatrix} b_1 & a_1 & 0 & 0 & \cdots \\ a_1 & b_2 & a_2 & 0 & \cdots \\ 0 & a_2 & b_3 & a_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix} \tag{1.4}$$

on $\ell^2(\mathbb{Z}_+)$, we define $d\mu$ to be the measure associated to the vector $\delta_1$ by the spectral theorem; that is,

$$m(z) \equiv \left\langle \delta_1, (J-z)^{-1}\delta_1 \right\rangle = \int \frac{d\mu(E)}{E-z}. \tag{1.5}$$

In this setting, the inverse procedure dates back to Jacobi, Chebychev, Markov, and Stieltjes. It is easy to describe: by applying Gram-Schmidt to $\{1, E, E^2, \dots\}$ in $L^2(d\mu)$, we obtain the orthonormal polynomials $p_n(E)$. These obey the three-term recursion relation

$$Ep_n(E) = a_{n+1}p_{n+1}(E) + b_{n+1}p_n(E) + a_n p_{n-1}(E). \tag{1.6}$$

Alternatively, one can obtain $a_n$ and $b_n$ from a continued fraction expansion of $m$ (see [33, 40]).

The main subject of spectral theory is to find relations between general properties of the spectral measures $d\rho$ or $d\mu$ and of the differential/difference equation parameters $V$ or $a_n$ and $b_n$. Clearly, the gems of the subject are ones that provide necessary and sufficient conditions, that is, a one-to-one correspondence between some explicit family of measures and some explicit set of parameters. In this paper, we announce three such results (one involving asymptotics of orthogonal polynomials rather than the measures) whose details will appear elsewhere [4, 5, 21].

In the context of orthogonal polynomials on the unit circle [31], Verblunsky's form [39] of Szegő's theorem [35, 36, 37] is such a one-to-one correspondence between a measure and the recurrence coefficients for its orthogonal polynomials. Baxter's theorem [1, 2] and Ibragimov's theorem [15, 19] can be viewed as other examples.

Our work here is related to and motivated by the more recent result of Killip and Simon [20].

**Theorem 1.1** (see [20]). The matrix $J - J_0$ is Hilbert-Schmidt, that is,

$$\sum_{n=1}^{\infty} (a_n - 1)^2 + b_n^2 < \infty \tag{1.7}$$

if and only if the spectral measure $d\mu$ obeys the following:

(i) (Blumenthal-Weyl) $\operatorname{supp}(d\mu) = [-2, 2] \cup \{E_j^+\}_{j=1}^{N_+} \cup \{E_j^-\}_{j=1}^{N_-}$ with $E_1^+ > E_2^+ > \cdots > 2$ and $E_1^- < E_2^- < \cdots < -2$ with $\lim_{j\to\infty} E_j^{\pm} = \pm 2$ if $N_{\pm} = \infty$;

(ii) (normalization) $\mu$ is a probability measure;

(iii) (Lieb-Thirring bound)

$$\sum_{\pm, j} (|E_j^{\pm}| - 2)^{3/2} < \infty; \tag{1.8}$$

(iv) (quasi-Szegő condition) let $d\mu_{ac}(E) = f(E)dE$. Then

$$\int_{-2}^{2} \log[f(E)] \sqrt{4 - E^2} dE > -\infty. \tag{1.9}$$

□

Our first result is the analog of this theorem for Schrödinger operators. This is discussed in Section 2.

Our second result concerns Szegő asymptotics for orthogonal polynomials. In 1922, Szegő [38] proved that if $d\mu = f(E)dE$, where $f$ is supported on $[-2, 2]$ and

$$\int \log[f(E)] \frac{dE}{\sqrt{4 - E^2}} > -\infty, \tag{1.10}$$

then

$$\lim_{n\to\infty} z^n p_n(z + z^{-1}) \tag{1.11}$$

exists and is nonzero (and finite) for all $z \in \mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$. There are works by Gončar [16], Nevai [24], and Nikishin [26] that allow point masses outside $[-2, 2]$. The following summarizes more recent results on this subject by Peherstorfer and Yuditskii [27], Killip and Simon [20], and Simon and Zlatoš [32].

**Theorem 1.2.** Suppose $d\mu = f(E)dE + d\mu_s$ with $\operatorname{supp}(d\mu_{sc}) \cup \operatorname{supp}(f) \subset [-2, 2]$ and

$$\sum_{j,\pm} (|E_j^{\pm}| - 2)^{1/2} < \infty. \tag{1.12}$$

Then the following are equivalent:

(i) $\inf(a_1 \cdots a_n) > 0$;
(ii) (a) $\sum_{n=1}^{\infty} |a_n - 1|^2 + |b_n|^2 < \infty$;
(b) $\lim_{n\to\infty} a_n \cdots a_1$ exists and is finite and nonzero;
(c) $\lim_{n\to\infty} \sum_{j=1}^{n} b_j$ exists;
(iii) $d\mu$ obeys the Szegő condition

$$\int_{-2}^{2} \log\left[f(E)\right] \frac{dE}{\sqrt{4-E^2}} > -\infty. \tag{1.13}$$

Moreover, if these hold, then the limit (1.11) exists and is finite for all $z \in \mathbb{D}$ and is nonzero if $z + z^{-1} \notin \{E_j^{\pm}\}$. □

Because (1.12) is required a priori here, this result is not a necessary and sufficient condition with only parameter information on one side and only spectral information on the other. In Section 3, we will discuss a necessary and sufficient condition for the asymptotics (1.11) to hold, thereby closing a chapter that began in 1922.

Finally, in Section 4, we discuss necessary and sufficient conditions on the measure for the $a$'s and $b$'s to obey

$$\limsup \left(\left|a_n - 1\right| + \left|b_n\right|\right)^{1/2n} \leq R^{-1} \tag{1.14}$$

for some $R > 1$. Namely, $d\mu$ must give specified weight to those eigenvalues $E_j$ with $|E_j| < R + R^{-1}$, and the Jost function must admit an analytic continuation to the disk $\{z : |z| < R\}$.

The Jost function is naturally defined in terms of scattering; however, there is a simple procedure for determining it from the measure and vice versa; see (4.2).

## 2 Schrödinger operators with $L^2$ potential

The proofs of the results in this section will appear in [21]. Given a measure $d\rho$ on $\mathbb{R}$, define $\tilde{\sigma}$ on $[0, \infty)$ by

$$\int_0^{\infty} g\left(\sqrt{E}\right) d\rho(E) = \int_0^{\infty} g(k)\, d\tilde{\sigma}(k), \tag{2.1}$$

that is, formally $d\tilde{\sigma}(k) = \chi_{(0,\infty)}(k^2)\, d\rho(k^2)$. For the Schrödinger operator with $V = 0$,

$$\begin{aligned} d\rho_0(E) &= \pi^{-1} \chi_{[0,\infty)}(E) \sqrt{E}\, dE, \\ d\tilde{\sigma}_0(p) &= 2\pi^{-1} \chi_{[0,\infty)}(p) p^2\, dp. \end{aligned} \tag{2.2}$$

Given $\rho$, define $\tilde{F}$ by

$$\tilde{F}(q) = \pi^{-1/2} \int_{p \geq 1} p^{-1} e^{-(q-p)^2} \left[ d\tilde{\sigma}(p) - d\tilde{\sigma}_0(p) \right]. \tag{2.3}$$

Since $d\rho$ obeys

$$\int \frac{d\rho(E)}{1+E^2} < \infty, \tag{2.4}$$

the integral in (2.3) is convergent.

If $d\rho$ is the spectral measure corresponding to the potential $V$, $d\rho$ and $m$ are related by the Herglotz representation:

$$m(z) = c + \int \left[ \frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2} \right] d\rho(\lambda). \tag{2.5}$$

The constant $c$ is determined by the known asymptotics (see [14]), $m(-k^2) = -k + o(1)$.

Define $M(k)$ by $M(k) = m(k^2)$. Here is our main result on $L^2$ potentials.

**Theorem 2.1.** Let $d\rho$ be the spectral measure associated to a potential $V$. Then $V \in L^2([0, \infty))$ if and only if

(i) (Weyl) $\operatorname{supp}(d\rho) = [0,\infty) \cup \{E_j\}_{j=1}^N$ with $E_1 < E_2 < \cdots < 0$ and $\lim_j E_j = 0$ if $N = \infty$;

(ii) (local solubility)

$$\int_0^\infty \left| \tilde{F}(q) \right|^2 dq < \infty; \tag{2.6}$$

(iii) (Lieb-Thirring)

$$\sum_j \left| E_j \right|^{3/2} < \infty; \tag{2.7}$$

(iv) (quasi-Szegő)

$$\int \log \left[ \frac{\left| M(k) + ik \right|^2}{4k \operatorname{Im} M(k)} \right] k^2 \, dk < \infty. \tag{2.8}$$

□

Remark 2.2. (1) While there is a parallelism with Theorem 1.1, there are two significant differences. First, the innocuous normalization condition is replaced by (2.6) and, second, (2.8) involves $M$ and not just $\mu$.

(2) Inequality (2.6) (assuming (2.8) holds) is an expression of the fact that $d\rho$ is the spectral measure of an $L^2_{loc}$ potential essentially because it implies (by [14]) that the $A$-function of [29] is in $L^2_{loc}$.

(3) That $M$ has a.e. boundary values is standard; compare [8, Chapter 1] or [28, Chapter 17].

(4) The integrand in (2.8) is $-\log(1-|R(k)|^2)$, where $R$ is a reflection coefficient. Weak lower semicontinuity of the negative of the entropy used in [20] is replaced by lower semicontinuity of the $L^{2n}$-norm.

(5) The key to the proof of Theorem 2.1 is a strong version of the Zaharov-Faddeev [41] sum rules. Essentially following [20, 30, 32], we provide a step-by-step sum rule for $V \in L^2_{loc}$ and take suitable limits. What is interesting is that we use a whole-line, not half-line, sum rule.

We note that prior to our work, $V \in L^2 \Rightarrow$ (2.7) was proved by Gardner, Greene, Kruskal, and Miura in [10]. Bounds of this type are often called Lieb-Thirring inequalities after their work on moments of eigenvalues for $V \in L^p(\mathbb{R}^d)$; see [23]. Deift and Killip [6] proved that $V \in L^2$ implies $f(E) > 0$ for a.e. $E > 0$. There are related works when $-d^2/dx^2 + V \geq 0$ in Sylvester and Winebrenner [34] and Denisov [7].

## 3 Szegő asymptotics

The proofs of the results in this section will appear in [4]. For the study of Szegő asymptotics, it is useful to map $\mathbb{D} = \{z : |z| < 1\}$ to $\mathbb{C} \setminus [-2, 2]$ by $z \to E = z + z^{-1}$. Our main result on this issue uses the following conditions:

$$\sum_{n=1}^{\infty} |a_n - 1|^2 + |b_n|^2 < \infty, \tag{3.1}$$

$$\lim_{N\to\infty} \sum_{n=1}^{N} \log(a_n) \quad \text{exists (and is finite)}, \tag{3.2}$$

$$\lim_{N\to\infty} \sum_{n=1}^{N} b_n \quad \text{exists (and is finite)}. \tag{3.3}$$

**Theorem 3.1.** If for some $\varepsilon > 0$, $z^n p_n(z + z^{-1})$ converges uniformly on compact subsets of $\{z : 0 < |z| < \varepsilon\}$ to a nonzero value, then (3.1), (3.2), and (3.3) hold.

Conversely, if (3.1), (3.2), and (3.3) hold, then $z^n p_n(z + z^{-1})$ converges uniformly on compact subsets of $\mathbb{D}$ and has a nonzero limit for those $z \neq 0$, where $z + z^{-1}$ is not an eigenvalue of $J$. $\square$

Remark 3.2. (1) By Theorem 1.1, (3.1) implies only the quasi-Szegő condition (1.9) whereas all prior discussions of Szegő asymptotics have assumed the stronger Szegő condition (1.13). We have examples in [4] where (3.1), (3.2), and (3.3) hold and $\sum(|E_n^{\pm}|-2)^{1/2} = \infty$ which, by [32], implies that (1.13) fails; so we have examples where Szegő asymptotics hold, although the Szegő condition fails.

(2) The first step in the proof is to show that for fixed $z \in \mathbb{D}$, Szegő asymptotics hold if and only if there is a solution with Jost asymptotics, that is, for which $\lim z^{-n} u_n(z)$ exists and is nonzero.

(3) We have two constructions of the Jost solution when (3.1), (3.2), and (3.3) hold; one using the nonlocal step-by-step sum rule of [30], and the other using perturbation determinants [20]. In either case, one makes a renormalization: in the first approach, one renormalizes Blaschke products and Poisson-Fatou representations, and, in the second case, one uses renormalized determinants for Hilbert-Schmidt operators.

While these are the first results we know for Szegő/Jost asymptotics for Jacobi matrices with only $L^2$ conditions, Hartman [17] and Hartman and Wintner [18] (see also Eastham [9, Chapter 1]) have found Jost asymptotics for Schrödinger operators with $V \in L^2$ with conditionally convergent integral.

## 4 Jacobi parameters with exponential decay

The proofs of the results in this section will appear in [5].

If $m$ is given by (1.5), we define $M(z)$ by

$$M(z) = -m\big(z + z^{-1}\big). \tag{4.1}$$

Suppose that $M(z)$ is the $M$-function of a Jacobi matrix and that it has an analytic continuation to a neighborhood of $\bar{\mathbb{D}}$ with the only poles in $\bar{\mathbb{D}}$ lying in $\bar{\mathbb{D}} \cap \mathbb{R}$ and all such poles are simple. Then we can define

$$u(z) = \prod_{k=1}^{N} b\big(z, z_k\big) \exp\left(\int \left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right) \log\left(\frac{\sin\theta}{\operatorname{Im} M\big(e^{i\theta}\big)}\right) \frac{d\theta}{4\pi}\right), \tag{4.2}$$

where $\{z_k\}_{k=1}^{\infty}$ are the poles of $M$ in $\mathbb{D}$. This agrees with the Jost function from scattering theory (see [20]), so we call it by this name.

Given $M$ and the Jost function $u$, suppose $u$ is analytic in $\{z : |z| < R\}$ and $z_0$ is a zero of $u$ (pole of $M$) with $R^{-1} < |z_0| < 1$. We say that $M$ has a canonical weight at $z_0$ if

$$\lim_{\substack{z\to z_0\\ z\neq z_0}} (z - z_0)M(z) = \left(z_0 - z_0^{-1}\right)\left[u'(z_0)u(z_0^{-1})\right]^{-1}. \tag{4.3}$$

**Theorem 4.1.** Let $M$ be the $M$-function of a Jacobi matrix $J$. Then $J - J_0$ is finite rank if and only if

(i) $M$ is rational and has only simple poles in $\bar{\mathbb{D}}$ with all such poles in $\mathbb{R}$;

(ii) $u$ is a polynomial;

(iii) $M$ has canonical weight at each $z \in \mathbb{D}$ which is a pole of $M$. □

**Theorem 4.2.** Let $M$ be the $M$ function of a Jacobi matrix $J$. Then the parameters of $J$ obey

$$\left(\left|a_n - 1\right| + \left|b_n\right|\right) \leq C_\varepsilon\left(R^{-1} + \varepsilon\right)^{2n} \tag{4.4}$$

for some $R > 1$ and all $\varepsilon > 0$ if and only if

(i) $M$ is meromorphic on $\{z : |z| < R\}$ with only simple poles inside $\bar{\mathbb{D}}$ with all such poles in $\mathbb{R}$;

(ii) $u$ is analytic in $\{z : |z| < R\}$;

(iii) $M$ has canonical weight at each pole of $M$, $z_0 \in \mathbb{D}$, with $R^{-1} < |z_0| < 1$. □

Given $u$ and not $m$, there is a normalization issue, so it is easier to discuss the perturbation determinant [20] which obeys

$$L(z) = \frac{u(z)}{u(0)} = \left(\prod_{n=1}^{\infty} a_n\right)u(z). \tag{4.5}$$

**Theorem 4.3.** Let $L$ be a polynomial with $L(0) = 1$. Then $L$ is a perturbation determinant of a Jacobi matrix $J$, with $J - J_0$ finite rank if and only if

(1) $L(z)$ is real if $z \in \mathbb{R}$;

(2) the only zeros of $L$ in $\bar{\mathbb{D}}$ lie on $\bar{\mathbb{D}} \cap \mathbb{R}$ and are simple.

In this case, there is a unique $J$ with $J - J_0$ finite rank, so $L$ is its perturbation determinant.

□

Remark 4.4. (1) While there is a unique $J$ with $J - J_0$ finite rank, if $L$ has $k$ zeros in $\mathbb{D}$, there is a $k$-parameter family of other $J$'s with $L$ as their perturbation determinant (each such $J$ has $|a_n - 1| + |b_n|$ decaying exponentially, but only one has $J - J_0$ finite rank).

(2) There is an analog of Theorem 4.3 if $L$ is only analytic in $\{z : |z| < R\}$.

(3) The proofs of these results depend on analyzing the map $(u, M) \to (u^{(1)}, M^{(1)})$, where $u^{(1)}, M^{(1)}$ are the Jost function and $M$-function for $J^{(1)}$, the Jacobi matrix with the top row and leftmost column removed. We control $|||u^{(1)}|||$ in terms of $|||u|||$, where

$$|||u|||^2 = \int \left|u\left(R_1 e^{i\theta}\right) - u(0)\right|^2 \frac{d\theta}{2\pi} \tag{4.6}$$

with $R_1 < R$, and are thereby able to show that $|||u^{(n)}|||$ goes to zero exponentially.

While we are aware of no prior work on the direct subject of this section, we note that Nevai and Totik [25] solved the analogous problem for orthogonal polynomials on the unit circle, that Geronimo [13] has related results for Jacobi matrices (but only under an a priori hypothesis on $M$), and that there are related results in the Schrödinger operator literature (see, e.g., Chadan and Sabatier [3]).

## Acknowledgments

We would like to thank Roman Romanov for drawing our attention to [9]. David Damanik was supported in part by National Science Foundation (NSF) Grant DMS-0227289, and Barry Simon was supported in part by NSF Grant DMS-0140592.

## References

[1] G. Baxter, *A convergence equivalence related to polynomials orthogonal on the unit circle*, Trans. Amer. Math. Soc. **99** (1961), 471–487.

[2] ———, *A norm inequality for a "finite-section" Wiener-Hopf equation*, Illinois J. Math. **7** (1963), 97–103.

[3] K. Chadan and P. C. Sabatier, *Inverse Problems in Quantum Scattering Theory*, 2nd ed., Texts and Monographs in Physics, Springer-Verlag, New York, 1989.

[4] D. Damanik and B. Simon, *Jost functions and Jost solutions for Jacobi matrices*, *I. A necessary and sufficient condition for Szegő asymptotics*, in preparation.

[5] ———, *Jost functions and Jost solutions for Jacobi matrices*, *II. Decay and analyticity*, in preparation.

[6] P. Deift and R. Killip, *On the absolutely continuous spectrum of one-dimensional Schrödinger operators with square summable potentials*, Comm. Math. Phys. **203** (1999), no. 2, 341–347.

[7] S. A. Denisov, *On the coexistence of absolutely continuous and singular continuous components of the spectral measure for some Sturm-Liouville operators with square summable potential*, J. Differential Equations **191** (2003), no. 1, 90–104.

[8] P. L. Duren, *Theory of* $H^p$ *Spaces*, Dover Publications, New York, 2000.

[9] M. S. P. Eastham, *The Asymptotic Solution of Linear Differential Systems. Applications of the Levinson Theorem*, London Mathematical Society Monographs. New Series, vol. 4, The Clarendon Press, Oxford University Press, New York, 1989.

[10] C. S. Gardner, J. M. Greene, M. D. Kruskal, and R. M. Miura, *Korteweg-deVries equation and generalization. VI. Methods for exact solution*, Comm. Pure Appl. Math. **27** (1974), 97–133.

[11] I. M. Gel′fand and B. M. Levitan, *On the determination of a differential equation from its spectral function*, Izv. Akad. Nauk SSSR Ser. Mat. **15** (1951), 309–360 (Russian).

[12] ———, *On the determination of a differential equation from its spectral function*, Amer. Math. Soc. Transl. Ser. 2 **1** (1955), 253–304.

[13] J. S. Geronimo, *Scattering theory, orthogonal polynomials, and* q*-series*, SIAM J. Math. Anal. **25** (1994), no. 2, 392–419.

[14] F. Gesztesy and B. Simon, *A new approach to inverse spectral theory. II. General real potentials and the connection to the spectral measure*, Ann. of Math. (2) **152** (2000), no. 2, 593–643.

[15] B. L. Golinskii and I. A. Ibragimov, *On Szegő's limit theorem*, Math. USSR-Izv. **5** (1971), 421–444.

[16] A. A. Gončar, *On convergence of Padé approximants for some classes of meromorphic functions*, Math. USSR-Sb. **26** (1975), 555–575.

[17] P. Hartman, *Unrestricted solution fields of almost-separable differential equations*, Trans. Amer. Math. Soc. **63** (1948), 560–580.

[18] P. Hartman and A. Wintner, *Asymptotic integrations of linear differential equations*, Amer. J. Math. **77** (1955), 45–86; erratum: Amer. J. Math. **77** (1955), 404.

[19] I. A. Ibragimov, *A theorem of Gabor Szegő*, Mat. Zametki **3** (1968), 693–702 (Russian).

[20] R. Killip and B. Simon, *Sum rules for Jacobi matrices and their applications to spectral theory*, Ann. of Math. (2) **158** (2003), no. 1, 253–321.

[21] ———, *Sum rules and spectral measures of Schrödinger operators with* $L^2$ *potentials*, in preparation.

[22] B. M. Levitan and I. S. Sargsjan, *Introduction to Spectral Theory: Selfadjoint Ordinary Differential Operators*, Translations of Mathematical Monographs, vol. 39, American Mathematical Society, Rhode Island, 1975.

[23] E. H. Lieb and W. Thirring, *Inequalities for the moments of the eigenvalues of the Schrödinger Hamiltonian and their relation to Sobolev inequalities*, Studies in Mathematical Physics. Essays in Honor of Valentine Bargmann, Princeton University Press, New Jersey, 1976, pp. 269–303.

[24] P. Nevai, *Orthogonal polynomials*, Mem. Amer. Math. Soc. **18** (1979), no. 213, v+185.

[25] P. Nevai and V. Totik, *Orthogonal polynomials and their zeros*, Acta Sci. Math. (Szeged) **53** (1989), no. 1-2, 99–104.

[26] E. M. Nikishin, *Discrete Sturm-Liouville operators and some problems of function theory*, J. Soviet Math. **35** (1986), 2679–2744.

[27] F. Peherstorfer and P. Yuditskii, *Asymptotics of orthonormal polynomials in the presence of a denumerable set of mass points*, Proc. Amer. Math. Soc. **129** (2001), no. 11, 3213–3220.

[28] W. Rudin, *Real and Complex Analysis*, 3rd ed., McGraw-Hill, New York, 1987.

[29] B. Simon, *A new approach to inverse spectral theory. I. Fundamental formalism*, Ann. of Math. (2) **150** (1999), no. 3, 1029–1057.

[30] ———, *A canonical factorization for meromorphic Herglotz functions on the unit disk and sum rules for Jacobi matrices*, to appear in J. Funct. Anal.

[31] ———, *Orthogonal Polynomials on the Unit Circle*, AMS Colloquium Publications Series, American Mathematical Society, Rhode Island, in press.

[32] B. Simon and A. Zlatoš, *Sum rules and the Szegő condition for orthogonal polynomials on the real line*, Comm. Math. Phys. **242** (2003) 393–423.

[33] T. Stieltjes, *Recherches sur les fractions continues*, Ann. Fac. Sci. Univ. Toulouse, **8** (1894–1895), 1–122; **9**, A5–A47.

[34] J. Sylvester and D. P. Winebrenner, *Linear and nonlinear inverse scattering*, SIAM J. Appl. Math. **59** (1999), no. 2, 669–699.

[35] G. Szegő, *Ein Grenzwertsatz über die Toeplitzschen Determinanten einer reellen positiven Funktion*, Math. Ann. **76** (1915), 490–503 (German).

[36] ———, *Beiträge zur Theorie der Toeplitzschen Formen*, *I*, Math. Z. **6** (1920), 167–202 (German).

[37] ———, *Beiträge zur Theorie der Toeplitzschen Formen*, *II*, Math. Z. **9** (1921), 167–190 (German).

[38] ———, *Über den asymptotischen Ausdruck von Polynomen*, *die durch eine Orthogonalitätseigenschaft definiert sind*, Math. Ann. **85** (1922), 114–139 (German).

[39] S. Verblunsky, *On positive harmonic functions (second part)*, Proc. London Math. Soc. **40** (1936), 290–320.

[40] H. S. Wall, *Analytic Theory of Continued Fractions*, AMS Chelsea, Rhode Island, 2000.

[41] V. E. Zaharov and L. D. Faddeev, *The Korteweg-de Vries equation is a fully integrable Hamiltonian system*, Funkcional. Anal. i Priložen. **5** (1971), no. 4, 18–27 (Russian).



David Damanik: Department of Mathematics, California Institute of Technology, Pasadena, CA 91125, USA
E-mail address: damanik@its.caltech.edu

Rowan Killip: Department of Mathematics, University of California, Los Angeles, CA 90095, USA
E-mail address: killip@math.ucla.edu

Barry Simon: Department of Mathematics, California Institute of Technology, Pasadena, CA 91125, USA
E-mail address: bsimon@caltech.edu