Available online at www.sciencedirect.com
SCIENCE DIRECT

Journal of Functional Analysis 223 (2005) 109–115

JOURNAL OF Functional Analysis

www.elsevier.com/locate/jfa

# On a theorem of Kac and Gilbert

Barry Simon[1]

*Department of Mathematics 253-37, California Institute of Technology, Pasadena, CA 91125, USA*

Received 22 April 2004; received in revised form 27 August 2004; accepted 27 August 2004
Communicated by M. Christ
Available online 8 February 2005

**Abstract**

We prove a general operator theoretic result that asserts that many multiplicity two selfadjoint operators have simple singular spectrum.


*Keywords:* Spectral theory; Singular spectrum; Kac–Gilbert

## 1. Introduction

In 1962, Kac [6,7] proved that whole-line Schrödinger operators, $-\frac{d}{dx^2} + V(x)$, for fairly general $V$'s have simple singular spectrum. It is well known (e.g., $V = 0$) that the absolutely continuous spectrum can have multiplicity two and that under limit point hypotheses, eigenvalues are simple. But the simplicity of the singular continuous spectrum is surprisingly subtle. Some insight into the result was obtained by Gilbert [3,4], who found a proof using the subordinacy theory of Gilbert–Pearson [5]. The proof is elegant but depends on the substantial machinery of subordinacy. Our purpose here is to note an abstract result that relates these things to the celebrated result of Aronszajn–Donoghue [1]; see also Simon [8]:

*E-mail address:* bsimon@caltech.edu
[1] Supported in part by NSF Grant DMS-0140592.


doi:10.1016/j.jfa.2004.08.015

**Theorem 1.** *Let $A$ be a bounded selfadjoint operator on $\mathcal{H}$ and $\varphi \in \mathcal{H}$ a cyclic vector for $A$. Suppose $\lambda \in \mathbb{R}\backslash\{0\}$ and*

$$B = A + \lambda\langle\varphi, \cdot\rangle\varphi. \tag{1.1}$$

*Then the singular spectral measures for $A$ and $B$ are disjoint.*

We state this and the next theorem in the bounded case for simplicity; we discuss the general case later. Here's the main result of this note:

**Theorem 2.** *Let $\mathcal{H} = \mathcal{K}_1 \oplus \mathcal{K}_2$ and $P : \mathcal{H} \to \mathcal{K}_1$, the canonical projection. Let $A_j \in \mathcal{L}(\mathcal{K}_j)$ for $j = 1, 2$, and $\varphi \in \mathcal{H}$ so that $\varphi_1 = P\varphi$ and $\varphi_2 = (1 - P)\varphi$ are cyclic for $A_1$ and $A_2$. Let $\lambda \in \mathbb{R}\backslash\{0\}$. Then*

$$C = A_1 \oplus A_2 + \lambda\langle\varphi, \cdot\rangle\varphi \tag{1.2}$$

*has simple singular spectrum.*

**Remark.** If $A_1$ is unitarily equivalent to $A_2$ and has a.c. spectrum, then $C$ has multiplicity two a.c. spectrum. So it is interesting that the singular spectrum is simple.

In Section 2, we prove Theorem 2. In Section 3, we apply it to whole-line Jacobi matrices. In Section 4, we discuss extensions of Theorem 2 to the case of unbounded selfadjoint operators and to unitary operators. In Section 5, we apply the results of Section 4 to Schrödinger operators and to extended CMV matrices.

## 2. Proof of Theorem 2

Let $d\mu_j$ be the spectral measure for $\varphi_j$ and $A_j$ and $B = A_1 \oplus A_2$. Thus $\mathcal{K}_j \cong L^2(\mathbb{R}, d\mu_j)$ in such a way that $A_j$ is multiplication by $x$ and $\varphi_j \cong 1$.

Pick disjoint sets $X, Y, Z \subset \mathbb{R}$ whose union is $\mathbb{R}$ so $d\mu_1 \restriction X$ is equivalent to $d\mu_2 \restriction X$ and $\mu_1(Y) = \mu_2(Z) = 0$. For example, if $d\mu_1 = f(d\mu_1 + d\mu_2)$, then one can take $X = \{x \mid 0 < f(x) < 1\}$, $Y = \{x \mid f(x) = 0\}$, $Z = \{x \mid f(x) = 1\}$.

Let $\mathcal{L}_1$ be the cyclic subspace generated by $\varphi$ and $B$ and $\mathcal{L}_2 = \mathcal{L}_1^{\perp}$. Then $\psi = ((1 - f)\chi_X, -f\chi_X)$ is a cyclic vector for $B \restriction \mathcal{L}_2$ and its spectral measure is

$$d\mu_\psi^B = \chi_X(x)[(1 - f)^2\, d\mu_1 + f^2\, d\mu_2]. \tag{2.1}$$

In particular, since $0 \leqslant f \leqslant 1$,

$$(d\mu_\psi^B)_{\mathrm{s}} \leqslant (d\mu_1 + d\mu_2)_{\mathrm{s}}. \tag{2.2}$$

To see that $\psi$ is cyclic for $B \restriction \mathcal{L}_2$, we note that in $L^2(\mathbb{R}, d\mu_1) \oplus L^2(\mathbb{R}, d\mu_2)$, we have $(h_1, h_2) \in \mathcal{L}_2$ if and only if $h_j$ are supported on $X$, and for any bounded Borel function $g$,

$$\int g h_1 \, d\mu_1 + \int g h_2 \, d\mu_2 = 0.$$

Writing $g = (1 - f)\tilde{g}$ and using $(1 - f)\, d\mu_1 = f \, d\mu_2$, we see this is true if and only if

$$\int [h_1 f + h_2(1 - f)]\tilde{g} \, d\mu_2 = 0.$$

This is true if and only if a.e. on $X$,

$$h_1 f + h_2(1 - f) = 0$$

and, since $0 < f < 1$ on $X$, this happens if and only if there a $q$ with

$$h_1 = q(1 - f), \quad h_2 = -qf,$$

showing that $\psi$ is cyclic for $B \restriction \mathcal{L}_2$.

By definition of $\mathcal{L}_1$, $\varphi$ is cyclic for $B \restriction \mathcal{L}_1$ and

$$C \restriction \mathcal{L}_1 = B + \lambda \langle \varphi, \cdot \rangle \varphi$$

so, by Theorem 1,

$$(d\mu_\varphi^C)_{\mathrm{s}} \perp (d\mu_\varphi^B)_{\mathrm{s}} = (d\mu_1 + d\mu_2)_{\mathrm{s}}. \tag{2.3}$$

Thus, the singular parts of $d\mu_\varphi^C$ and $d\mu_\psi^C = d\mu_\psi^B$ are disjoint, which implies that the singular spectrum of $C$ is simple.

The proof shows that the singular parts of $B$ and $C$ overlap in $\chi_X (d\mu_1 + d\mu_2)_{\mathrm{s}}$ and, in particular,

**Corollary 2.1.** *$B$ and $C$ have mutually singular singular parts if and only if $A_1$ and $A_2$ have mutually singular singular parts.*

## 3. Application to Jacobi matrices

A two-sided Jacobi matrix is defined by two two-sided sequences, $\{b_n\}_{n=-\infty}^{\infty}$ and $\{a_n\}_{n=-\infty}^{\infty}$ with $b_n \in \mathbb{R}$ and $a_n \in (0, \infty)$ and $\sup_n(|a_n| + |b_n|) < \infty$. It defines a bounded operator $J$ on $\ell^2(\mathbb{Z})$ by

$$(Ju)_n = a_{n-1}u_{n-1} + b_n u_n + a_n u_{n+1} \tag{3.1}$$

so

$$J = \begin{pmatrix} \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & b_{-1} & a_{-1} & 0 & 0 & \dots \\ \dots & a_{-1} & b_0 & a_0 & 0 & \dots \\ \dots & 0 & a_0 & b_1 & a_1 & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \end{pmatrix}. \tag{3.2}$$

**Theorem 3.1.** *The singular spectrum of $J$ is simple.*

**Proof.** Let $\mathcal{K}_1 = \ell^2((-\infty, -1])$, $\mathcal{K}_2 = \ell^2([0, \infty))$, and $\varphi$ the vector with components

$$\varphi_j = \begin{cases} 1 & j = -1, 0, \\ 0 & j \neq -1, 0, \end{cases} \tag{3.3}$$

so $P\varphi = \delta_{-1}$; $(1 - P)\varphi = \delta_0$.

Then

$$J - a_{-1}\langle \varphi, \cdot \rangle \varphi = A_1 \oplus A_2,$$

where $A_2$ is the one-sided Jacobi matrix with

$$A_2 = \begin{pmatrix} b_0 - a_{-1} & a_0 & 0 & \dots \\ a_0 & b_1 & a_2 & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}$$

and $A_1$ in $\delta_{-1}, \delta_{-2}, \dots$ basis is

$$A_1 = \begin{pmatrix} b_{-1} - a_{-1} & a_{-2} & 0 & \dots \\ a_{-2} & b_{-2} & a_{-3} & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix}.$$

Thus $P\varphi$ is cyclic for $A_1$ and $(1-P)\varphi$ is cyclic for $A_2$. Theorem 2 applies and implies the desired result. $\square$

## 4. Unitary and unbounded selfadjoint operators

Let $U_1$, $U_2$, and $W$ be unitary operators on $\mathcal{K}_1$, $\mathcal{K}_2$, and $\mathcal{H} = \mathcal{K}_1 \oplus \mathcal{K}_2$ so $W - U_1 \oplus U_2$ is rank one, and so if $\varphi \in \ker(W - U_1 \oplus U_2)^\perp$, then $P\varphi$ is cyclic for $U_1$ and $(1-P)\varphi$ is cyclic for $U_2$, where $P$ is the canonical projection of $\mathcal{H}$ to $\mathcal{K}_1$. Then

**Theorem 4.1.** *The singular spectrum of W is simple.*

**Proof.** We begin by proving the unitary analog of the Aronszajn–Donoghue theorem. If $W - V$ is rank one and nonzero, then for $\varphi$ a unit vector in $\ker(W-V)^\perp$, we have $W\varphi = \lambda V\varphi$ for some $\lambda \in \partial\mathbb{D} = \{z \in \mathbb{C} \mid |z| = 1\}$ with $\lambda \neq 1$. By a direct calculation (see, e.g., [9, Sections 1.4 and 4.5]),

$$\left(\varphi, \frac{W+z}{W-z}\,\varphi\right) = \frac{1+zg(z)}{1-zg(z)}, \tag{4.1}$$

$$\left(\varphi, \frac{V+z}{V-z}\,\varphi\right) = \frac{1+zf(z)}{1-zf(z)} \tag{4.2}$$

and

$$g(z) = \lambda^{-1} f(z). \tag{4.3}$$

If $\varphi$ cyclic for $W$, by the theory of Schur functions (see, e.g., [9, Section 1.3]), the singular spectrum in $W$ is supported on those $z \in \partial\mathbb{D}$ with

$$\lim_{r\uparrow 1} rz\, g(rz) = 1$$

and similarly, the singular spectrum of $V$ on the set of $z \in \partial\mathbb{D}$ with

$$\lim_{r\uparrow 1} rz\, f(rz) = 1.$$

By (4.3), these sets are disjoint.

This proves the Aronszajn–Donoghue theorem in the unitary case, and that implies this theorem by mimicking the proof of Theorem 2. □

Next, let $A_1$, $A_2$, and $C$ be potentially unbounded selfadjoint operators on $\mathcal{K}_1$, $\mathcal{K}_2$, and $\mathcal{H} = \mathcal{K}_1 \oplus \mathcal{K}_2$. Suppose $D \equiv (A_1 \oplus A_2 - i)^{-1} - (C - i)^{-1}$ is rank one with $\varphi \in (\ker D)^\perp$ so $P\varphi$ is cyclic for $A_1$ and $(1-P)\varphi$ for $A_2$. Then with

$$U_j = (A_j + i)(A_j - i)^{-1}, \quad W = (C+i)(C-i)^{-1}.$$

Theorem 4.1 applies, so

**Theorem 4.2.** *$C$ has simple singular spectrum.*

## 5. Extended CMV matrices and Schrödinger operators

Extended CMV matrices enter in the theory of the orthogonal polynomials on the unit circle (see [9,10, Chapter 4, Section 10.5]). They are defined by a family of Verblunsky coefficients $\{\alpha_j\}_{j=-\infty}^{\infty}$ with $\alpha_j \in \mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$ as follows. Let $\Theta(\alpha)$ be the $2 \times 2$ matrix

$$\Theta(\alpha) = \begin{pmatrix} \bar{\alpha} & \rho \\ \rho & -\alpha \end{pmatrix}, \tag{5.1}$$

where $\rho = (1 - |\alpha|^2)^{1/2}$.

Think of $\ell^2(\mathbb{Z})$, first as a direct sum $\oplus_{n=-\infty}^{n} \mathbb{C}^2$ with the $n$th factor spanned by $(\delta_{2n}, \delta_{2n+1})$ and let $\mathcal{M} = \oplus\Theta(\alpha_{2n})$, then as a direct sum with the $n$th factor spanned by $(\delta_{2n+1}, \delta_{2n+2})$ and $\mathcal{L} = \oplus\Theta(\alpha_{2n+1})$. Then $\mathcal{E} = \mathcal{L}\mathcal{M}$ is the extended CMV matrix.

We claim

**Theorem 5.1.** *$\mathcal{E}$ always has simple singular spectrum.*

**Proof.** Let

$$x = \frac{1 + \bar{\alpha}_{-1}}{1 + \alpha_{-1}}.$$

Then

$$\det\left(\Theta(\alpha_{-1}) - \begin{pmatrix} x & 0 \\ 0 & 1 \end{pmatrix}\right) = 0$$

by a simple calculation. It is thus rank one, so if $\tilde{\mathcal{E}}$ is defined by replacing $\Theta(\alpha_{-1})$ by $\left(\begin{smallmatrix} x & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $\tilde{\mathcal{E}} - \mathcal{E}$ is rank one. On the other hand, $\tilde{\mathcal{E}}$ is a direct sum of two half-line CMV matrices and it is easy to see $P\varphi$ and $(1-P)\varphi$ are cyclic. Thus Theorem 4.1 applies. □

Finally, we turn to the Schrödinger operator case that motivated us in the first place. Suppose $H = -\frac{d^2}{dx^2} + V$ where $V \in L^1_{\mathrm{loc}}(-\infty, \infty)$ is limit point at both $+\infty$ and $-\infty$. Let $H_1$ (resp. $H_2$) be $H$ on $L^2(0, \infty)$ (resp. $L^2(-\infty, 0)$) with $u(0) = 0$ boundary conditions. Then

$$(H - i) - (H_1 \oplus H_2 - i)^{-1}$$

is rank one by the explicit Green's function formulae [2], and the orthogonal complement of its kernel is spanned by a function $\varphi$ with

$$\left(-\frac{d^2}{dx^2}+V\right)\varphi = i\varphi, \quad x \neq 0, \tag{5.2}$$

with $\varphi$ $L^2$ at $+\infty$ and $-\infty$ and $\varphi(0_+) = \varphi(0_-)$. (5.2) implies $\varphi\restriction[0,\infty)$ (resp. $\varphi\restriction(-\infty,0]$) is cyclic for $H_1$ (resp. $H_2$). Theorem 4.2 applies and yields the Kac–Gilbert theorem:

**Theorem 5.2.** $-\frac{d^2}{dx^2}+V$ *has simple singular spectrum.*

## References

[1] N. Aronszajn, W.F. Donoghue, On exponential representations of analytic functions in the upper half-plane with positive imaginary part, J. Anal. Math. 5 (1957) 321–388.
[2] F. Gesztesy, B. Simon, The xi function, Acta Math. 176 (1) (1996) 49–71.
[3] D.J. Gilbert, On subordinacy and analysis of the spectrum of Schrödinger operators with two singular endpoints, Proc. Roy. Soc. Edinburgh Sect. A 112 (1989) 213–229.
[4] D.J. Gilbert, On subordinacy and spectral multiplicity for a class of singular differential operators, Proc. Roy. Soc. Edinburgh A 128 (1998) 549–584.
[5] D.J. Gilbert, D.B. Pearson, On subordinacy and analysis of the spectrum of one-dimensional Schrödinger operators, J. Math. Anal. Appl. 128 (1987) 30–56.
[6] I.S. Kac, On the multiplicity of the spectrum of a second-order differential operator, Soviet Math. Dokl. 3 (1962) 1035–1039.
[7] I.S. Kac, Spectral multiplicity of a second-order differential operator and expansion in eigenfunction, Izv. Akad. Nauk SSSR Ser. Mat. 27 (1963) 1081–1112 [Russian]. Erratum, Izv. Akad. Nauk SSSR 28 (1964) 951–952.
[8] B. Simon, Spectral analysis of rank one perturbations and applications, in: Mathematical Quantum Theory, II: Schrödinger Operators (Vancouver, BC, 1993), CRM Proceedings Lecture Notes, vol. 8, American Mathematical Society, Providence, RI, 1995, pp. 109–149.
[9] B. Simon, Orthogonal Polynomials on the Unit Circle, Part 1: Classical Theory, AMS Colloquium Series, American Mathematical Society, Providence, RI, 2005.
[10] B. Simon, Orthogonal Polynomials on the Unit Circle, Part 2: Spectral Theory, AMS Colloquium Series, American Mathematical Society, Providence, RI, 2005.