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Higher-order Szegő theorems with two singular points

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Abstract

We consider probability measures, $d\mu = w(\theta)\frac{d\theta}{2\pi} + d\mu_s$, on the unit circle, $\partial\mathbb{D}$, with Verblunsky coefficients, $\{\alpha_j\}_{j=0}^{\infty}$. We prove for $\theta_1 \neq \theta_2$ in $[0, 2\pi)$ that

$$\int [1 - \cos(\theta - \theta_1)][1 - \cos(\theta - \theta_2)] \log w(\theta)\, \frac{d\theta}{2\pi} > -\infty$$

if and only if

$$\sum_{j=0}^{\infty} \left| \left\{ (\delta - e^{-i\theta_2})(\delta - e^{-i\theta_1})\alpha \right\}_j \right|^2 + |\alpha_j|^4 < \infty,$$

where δ is the left shift operator $(\delta\beta)_j = \beta_{j+1}$. We also prove that

$$\int (1 - \cos\theta)^2 \log w(\theta)\, \frac{d\theta}{2\pi} > -\infty$$

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if and only if

$$\sum_{j=0}^{\infty} |\alpha_{j+2} - 2\alpha_{j+1} + \alpha_j|^2 + |\alpha_j|^6 < \infty.$$

Keywords: Orthogonal polynomials on the unit circle; Szegő theorem

1. Introduction

This paper is a contribution to the theory of orthogonal polynomials on the unit circle (OPUC); see [6,15,16,18] for background. Throughout, $d\mu$ will be a non-trivial probability measure on the unit circle, $\partial\mathbb{D}$, in $\mathbb{C}$, which we suppose has the form

$$d\mu = w(\theta)\, \frac{d\theta}{2\pi} + d\mu_{\mathrm{s}}, \tag{1.1}$$

where $d\mu_{\mathrm{s}}$ is singular with respect to Lebesgue measure $d\theta$ on $\partial\mathbb{D}$.

The Carathéodory and Schur functions, F and f, associated to $d\mu$ are given for $z \in \mathbb{D}$ by

$$F(z) = \int \frac{e^{i\theta} + z}{e^{i\theta} - z}\, d\mu(\theta) \tag{1.2}$$

$$= \frac{1 + zf(z)}{1 - zf(z)}. \tag{1.3}$$

The Verblunsky coefficients $\{\alpha_j\}_{j=0}^{\infty}$ can be defined inductively by the Schur algorithm

$$f(z) = \frac{\alpha_0 + zf_1(z)}{1 + z\bar{\alpha}_0 f_1(z)}, \tag{1.4}$$

which defines $\alpha_0 \in \mathbb{D}$ and f_1. Iterating gives $\alpha_1, \alpha_2, \ldots$ and $f_2, f_3, \ldots$. That $\alpha_j \in \mathbb{D}$ (rather than just $\bar{\mathbb{D}}$) follows from the assumption that $d\mu$ is non-trivial, that is, has infinite support so f is not a finite Blaschke product. Actually, (1.4) defines what are usually called Schur parameters; the Verblunsky coefficients are defined by a recursion relation on the orthogonal polynomials. The equality of these recursion coefficients and the Schur parameters of (1.4) is a theorem of Geronimus [5]; see [15]. We will use the definition in (1.4).

The most famous result in OPUC is Szegő's theorem which, in Verblunsky's format [19], says

$$\log\left(\prod_{j=0}^{\infty} (1 - |\alpha_j|^2)\right) = \int \log(w(\theta))\, \frac{d\theta}{2\pi}. \tag{1.5}$$

In this expression, both sides are non-positive (since $|\alpha_j| < 1$, and Jensen's inequality implies $\int \log(w(\theta))\, \frac{d\theta}{2\pi} \leqslant \log(\int w(\theta) \frac{d\theta}{2\pi}) \leqslant \log(\mu(\partial\mathbb{D}))$). Moreover, (1.5) includes the statement that both sides are finite (resp., $-\infty$) simultaneously. Thus (1.5) implies a spectral theory result.

Theorem 1.1.

$$\int \log(w(\theta))\, \frac{d\theta}{2\pi} > -\infty \Leftrightarrow \sum_{j=0}^{\infty} |\alpha_j|^2 < \infty. \tag{1.6}$$

This form of the theorem has caused considerable recent interest due to work of Deift–Killip [1] and Killip–Simon [7] which motivated a raft of papers [2,8–11,14,17,20].

In [15, Section 2.8], Simon found a higher-order analog to (1.6) that allows $\log(w(\theta))$ to be singular at a single point:

Theorem 1.2.

$$\int (1-\cos\theta) \log(w(\theta))\, \frac{d\theta}{2\pi} > -\infty \Leftrightarrow \sum_{j=0}^{\infty} |\alpha_{j+1}-\alpha_j|^2 + |\alpha_j|^4 < \infty. \tag{1.7}$$

Remark. This result allows a single singular point of order 1 in $\log(w(\theta))$ at $\theta = 0$. By a simple rotation argument [15], if $\cos(\theta)$ is replaced by $\cos(\theta-\theta_1)$, $|\alpha_{j+1}-\alpha_j|^2$ is replaced by $|\alpha_{j+1} - e^{-i\theta_1}\alpha_j|^2$.

Our goal in this paper is to analyze two singularities or a single double singularity. We will prove that

Theorem 1.3. *For* $\theta_1 \neq \theta_2$,

$$\int (1-\cos(\theta-\theta_1))(1-\cos(\theta-\theta_2)) \log(w(\theta))\, \frac{d\theta}{2\pi} > -\infty$$
$$\Leftrightarrow \sum_{j=0}^{\infty} \left| \left\{ (\delta - e^{-i\theta_2})(\delta - e^{-i\theta_1})\alpha \right\}_j \right|^2 + |\alpha_j|^4 < \infty. \tag{1.8}$$

In this theorem, δ is the operator on sequences

$$(\delta\alpha)_j = \alpha_{j+1}. \tag{1.9}$$

We will also prove a result for $\theta_1 = \theta_2$.

Theorem 1.4.

$$\int (1-\cos\theta)^2 \log(w(\theta))\, \frac{d\theta}{2\pi} > -\infty$$
$$\Leftrightarrow \sum_{j=0}^{\infty} |\alpha_{j+2} - 2\alpha_{j+1} + \alpha_j|^2 + |\alpha_j|^6 < \infty. \tag{1.10}$$

Again, one can replace $\cos(\theta)$ by $\cos(\theta-\theta_1)$ if $\alpha_{j+2} - 2\alpha_{j+1} + \alpha_j$ is replaced by $\{(\delta - e^{-i\theta_1})^2\alpha\}_j$.

Given the form of these theorems, it is natural to conjecture the situation for arbitrarily many singularities:

Conjecture 1.5. *For* $\{\theta_k\}_{k=1}^{\ell}$ *distinct in* $[0, 2\pi)$,

$$\int \prod_{k=1}^{\ell} (1-\cos(\theta-\theta_k))^{m_k} \log(w(\theta)) \frac{d\theta}{2\pi} > -\infty$$
$$\Leftrightarrow \sum_{k=0}^{\infty} \left| \left\{ \prod_{k=1}^{\ell} [\delta - e^{-i\theta_k}]^{m_k} \alpha \right\}_j \right|^2 + |\alpha_j|^{2\max(m_k)+2} < \infty. \tag{1.11}$$

Independently of our work, Denisov–Kupin [3] have found conditions on the α's equivalent to the left-hand side of (1.11) being finite. However, their conditions are complicated and even for the case $\sum_{k=1}^{\ell} m_k = 2$, it is not clear they are equivalent to the ones we have in Theorems 1.3 and 1.4 (although they must be!).

In Section 2, we review the features we need of the relative Szegő function which will play a critical role in our proofs, and we compute its first two Taylor coefficients. In Section 3, we prove Theorem 1.3 in the special case $\theta_1 = 0$, $\theta_2 = \pi$, and in Section 4, we prove Theorem 1.4. With these two warmups done, we turn to the general result, Theorem 1.3, in Section 5. The details of this are sufficiently messy that we do not think this direct approach is likely to yield our conjecture.

2. The relative Szegő function

In Section 2.9 of Simon [15], introduced the relative Szegő function, defined by

$$(\delta_0 D)(z) = \frac{1-\bar{\alpha}_0 f(z)}{\rho_0} \frac{1-zf_1(z)}{1-zf(z)}, \tag{2.1}$$

where

$$\rho_k = (1-|\alpha_k|^2)^{1/2} \tag{2.2}$$

and f, f_1 are given by (1.3) and (1.4).

The key property of $\delta_0 D$ we will need and the reason it was introduced is

Theorem 2.1 (*Simon [15, Theorem 2.9.3]*). *Let* $d\mu_1$ *be the measure whose Verblunsky coefficients are* $(\alpha_1, \alpha_2, \ldots)$. *Let* w *be given by* (1.1) *and* w_1 *by*

$$d\mu_1 = w_1(\theta) \frac{d\theta}{2\pi} + d\mu_{1,\mathrm{s}}. \tag{2.3}$$

Suppose $w(\theta) \neq 0$ *for a.e.* $e^{i\theta}$ *in* $\partial\mathbb{D}$. *Then the same is true for* w_1 *and*

$$(\delta_0 D)(z) = \exp\left(\frac{1}{4\pi} \int \frac{e^{i\theta}+z}{e^{i\theta}-z} \log\left(\frac{w(\theta)}{w_1(\theta)}\right) d\theta \right). \tag{2.4}$$

As in [7,14,17], this is the basis for step-by-step sum rules, as we will see.

To prove Theorems 1.3 and 1.4, we will need to start with computing the first three Taylor coefficients of $\log((\delta_0 D)(z))$.

Theorem 2.2. *We have that*

$$\log(\delta_0 D(z)) = A_0 + A_1 z + A_2 z^2 + O(z^3), \tag{2.5}$$

where

$$A_0 = \log \rho_0 \tag{2.6}$$

$$A_1 = \alpha_0 - \alpha_1 - \bar{\alpha}_0 \alpha_1, \tag{2.7}$$

$$A_2 = \tfrac{1}{2}\,\alpha_0^2 - \tfrac{1}{2}\,\alpha_1^2 + \alpha_1 - \alpha_2 - \alpha_1 |\alpha_0|^2 + \alpha_2 |\alpha_1|^2 - \bar{\alpha}_0 \alpha_2 \rho_1^2 + \tfrac{1}{2}\,\bar{\alpha}_0^2 \alpha_1^2. \tag{2.8}$$

Proof. $f_2(0) = \alpha_2$, so

$$f_1 = \frac{z f_2 + \alpha_1}{1 + z \bar{\alpha}_1 f_2} = \alpha_1 + z \alpha_2 \rho_1^2 + O(z^2).$$

Thus

$$f = \frac{z f_1 + \alpha_0}{1 + z \bar{\alpha}_0 f_1} = \alpha_0 + z \alpha_1 \rho_0^2 + z^2 \rho_0^2 (\alpha_2 \rho_1^2 - \bar{\alpha}_0 \alpha_1^2) + O(z^3).$$

Plugging these into (2.1) yields the required Taylor coefficients. □

Remarks. 1. Denisov–Kupin [3] do what is essentially the same calculation using the CMV matrix.

2. (3.2) and (3.3) below show that (2.4) implies

$$\int \log\left(\frac{w(\theta)}{w_1(\theta)}\right) \frac{d\theta}{2\pi} = 2A_0, \tag{2.9}$$

$$\int \log\left(\frac{w(\theta)}{w_1(\theta)}\right) e^{-im\theta}\, \frac{d\theta}{2\pi} = \begin{cases} A_m, & m = 1, 2, \\ \bar{A}_{-m}, & m = -1, -2. \end{cases} \tag{2.10}$$

3. Singularities at antipodal points

As a warmup, in this section we prove the following, which is Theorem 1.3 for $\theta_1 = 0$, $\theta_2 = \pi$. By the remark after Theorem 1.2 this also gives the result for any antipodal θ_1 and θ_2.

Theorem 3.1.

$$\int (1 - \cos^2(\theta)) \log w(\theta)\, \frac{d\theta}{2\pi} > -\infty \Leftrightarrow \sum_{j=0}^{\infty} |\alpha_{j+2} - \alpha_j|^2 + |\alpha_j|^4 < \infty. \tag{3.1}$$

Remark. Let α_j be given and let β_j be the sequence $(\alpha_0, 0, \alpha_1, 0, \alpha_2, 0, \ldots)$. Then (see [15, Example 1.6.14]), $w^{(\beta)}(\theta) = \frac{1}{2} w^{(\alpha)}(\frac{1}{2}\theta)$ and the RHS of (3.1) for $\beta =$ the RHS of (1.7) for α. Thus (3.1) for β is (1.7) for α. This shows, in particular, that if a result like (3.1) holds, it must involve $|\alpha_j|^4$, rather than, say, $|\alpha_j|^6$.

We begin by noting that if $Q(\theta)$ is real and

$$Q(\theta) = \sum_{n=-\infty}^{\infty} b_n e^{in\theta} \tag{3.2}$$

then

$$\int \frac{e^{i\theta} + z}{e^{i\theta} - z}\, Q(\theta) \frac{d\theta}{2\pi} = b_0 + 2 \sum_{n=1}^{\infty} b_n z^n \tag{3.3}$$

since $(e^{i\theta} + z)/(e^{i\theta} - z) = 1 + 2\sum_{n=1}^{\infty} z^n e^{-in\theta}$. Thus, by (2.9), (2.10), and

$$1 - \cos^2(\theta) = \tfrac{1}{4}\,(2 - e^{2i\theta} - e^{-2i\theta}) \tag{3.4}$$

we have

$$\int (1 - \cos^2(\theta)) \log\left(\frac{w(\theta)}{w_1(\theta)}\right) \frac{d\theta}{2\pi} = A_0 - \tfrac{1}{2}\,\mathrm{Re}(A_2) \tag{3.5}$$

with A_0 given by (2.6) and A_2 by (2.8).

Lemma 3.2. *We have that*

$$A_0 - \tfrac{1}{2}\,\mathrm{Re}(A_2) = B_0 + C_0 + D_0 + F_0 - F_1 + G_0 - G_2, \tag{3.6}$$

where

$$B_j = \tfrac{1}{2}\left[\log(1 - |\alpha_j|^2) + |\alpha_j|^2 + \tfrac{1}{2}|\alpha_j|^4\right], \tag{3.7}$$

$$C_j = -\tfrac{1}{4}\,(1 - |\alpha_{j+1}|^2)|\alpha_j - \alpha_{j+2}|^2, \tag{3.8}$$

$$D_j = -\tfrac{1}{8}\,(|\alpha_{j+1}^2 + \alpha_j^2|^2 + 4|\alpha_j\alpha_{j+1}|^2), \tag{3.9}$$

$$F_j = -\tfrac{1}{2}\,\mathrm{Re}(\tfrac{1}{2}\,\alpha_j^2 + \alpha_{j+1} - \alpha_{j+1}|\alpha_j|^2) + \tfrac{1}{4}\,|\alpha_{j+1}|^2|\alpha_j|^2 - \tfrac{1}{8}\,|\alpha_j|^4, \tag{3.10}$$

$$G_j = -\tfrac{1}{4}\,|\alpha_j|^2.$$

Remark. (3.5)/(3.6) is thus the step-by-step sum rule in the spirit of [7,14,17].

Proof. This is a straightforward but tedious calculation. The first term in B_0 is just A_0 (since $\log\rho_j = \frac{1}{2}\log(1 - |\alpha_j|^2)$). A_2 is responsible for the $\mathrm{Re}(\cdot)$ terms in $F_0 - F_1$ and the cross-terms in $|\alpha_j - \alpha_{j+2}|^2$ and $|\alpha_{j+1}^2 + \alpha_j^2|^2$. The $|\alpha_j|^2 + |\alpha_{j+2}|^2$ term in C_0 is turned into $2|\alpha_j|^2$ by $G_0 - G_2$, and then cancelled by the $|\alpha_j|^2$ term in B_0. Similarly, the $|\alpha_j|^4 + |\alpha_{j+1}|^4$

in D_0 (after adding the $|\alpha_j|^4$ terms in $F_0 - F_1$) cancels the $|\alpha_j|^4$ term in B_0. Finally, the $|\alpha_{j+1}|^2(|\alpha_j|^2+|\alpha_{j+2}|^2)$ term in C_0 (after being turned into $2|\alpha_{j+1}|^2|\alpha_j|^2$ by the $|\alpha_{j+1}|^2|\alpha_j|^2$ term in $F_0 - F_1$) cancels the $4|\alpha_j\alpha_{j+1}|^2$ term in D_0. □

By iterating (3.5)/(3.6) and noting the cancellations from the telescoping $F_j - F_{j+1}$ and $G_j - G_{j+2}$ yields

$$\begin{aligned}&\int (1-\cos^2(\theta))\log\left(\frac{w(\theta)}{w_{2m}(\theta)}\right)\frac{d\theta}{2\pi}\\ &\quad = F_0 - F_{2m} + G_0 + G_1 - G_{2m} - G_{2m+1} + \sum_{j=0}^{2m-1}(B_j + C_j + D_j). \qquad (3.11)\end{aligned}$$

As a final preliminary, we need,

Lemma 3.3. (i) $|F_j| \leqslant \frac{13}{8}$; $|G_j| \leqslant \frac{1}{4}$,
(ii) $|\alpha_j| < \frac{1}{2} \Rightarrow c_1|\alpha_j|^6 \leqslant -B_j \leqslant c_2|\alpha_j|^6$ *for some* $c_2 > c_1 > 0$,
(iii) $|\alpha_{j+1}|^4 + |\alpha_j|^4 \leqslant -8D_j \leqslant 4(|\alpha_{j+1}|^4 + |\alpha_j|^4)$.

Proof. (i) follows from $|\alpha_j| \leqslant 1$, (ii) from $-\log(1-x) = \sum_{j=1}^\infty x^j/j$, and (iii) by noting that $2\,\mathrm{Re}(\alpha_j^2\alpha_{j+1}^2) + 2|\alpha_j^2\alpha_{j+1}^2| \geqslant 0$ and repeated use of $|xy| \leqslant \frac{1}{2}(|x|^2 + |y|^2)$. □

Proof of Theorem 3.1. We follow the strategy of Killip and Simon [7] as modified by Simon and Zlatoš [17]. Suppose first that the RHS of (3.1) holds. Let $w^{(n)}$ be the weight for the nth Bernstein–Szegő approximation with Verblunsky coefficients $(\alpha_0, \alpha_1, \ldots, \alpha_{n-1}, 0, \ldots, 0, \ldots)$, and let w_n be the one for the measure μ_n with coefficients $(\alpha_n, \alpha_{n+1}, \ldots)$. By (3.11) and $(w^{(n)})_{2m} \equiv 1$ for large m,

$$\begin{aligned}&\int (1-\cos^2(\theta))\log(w^{(n)}(\theta))\,\frac{d\theta}{2\pi}\\ &\quad = F_0^{(n)} + G_0^{(n)} + G_1^{(n)} + \sum_{j=0}^{n-1}(B_j^{(n)} + C_j^{(n)} + D_j^{(n)})\end{aligned}$$

so, by Lemma 3.3, $|\alpha_j|^6 \leqslant |\alpha_j|^4 \to 0$, and RHS of (3.1),

$$\inf_n \left[\int (1-\cos^2(\theta))\log(w^{(n)}(\theta))\frac{d\theta}{2\pi}\right] > -\infty. \qquad (3.12)$$

Up to a constant, $\int (1-\cos^2(\theta))\log w(\theta)\frac{d\theta}{2\pi}$ is an entropy and so upper semicontinuous [7]. Thus (3.12) implies

$$\int (1-\cos^2(\theta))\log w(\theta)\,\frac{d\theta}{2\pi} > -\infty. \qquad (3.13)$$

Conversely, suppose (3.13) holds. Since $\int (1-\cos^2(\theta))\log(w_{2m}(\theta))\frac{d\theta}{2\pi}$ is an entropy up to a constant, it is bounded above [7], and so the left-hand side of (3.11) is bounded below as m varies.

Since F and G are bounded and B, C, D are negative, we conclude

$$\sum_{j=0}^{\infty} -(B_j + C_j + D_j) < \infty.$$

Since $\sum(-D_j) < \infty$, Lemma 3.3 implies $\sum |\alpha_j|^4 < \infty$. This implies $\alpha_j \to 0$, so $\sum(-C_j) < \infty$ implies $\sum |\alpha_j - \alpha_{j+2}|^2 < \infty$. $\square$

Notice that the redistribution of the terms in (3.6) insures that all the essential terms on the RHS of (3.11) (i.e., B_j, C_j, D_j) are sign definite. This ultimately allows us to recover (3.1) by passing to the limit $m \to \infty$ in (3.11). The same strategy will be applied in the proofs of Theorems 1.3 and 1.4.

4. Singularity of order 2

Our goal here is to prove Theorem 1.4. Since

$$\begin{aligned}(1-\cos\theta)^2 &= \tfrac{1}{4}(2 - e^{i\theta} - e^{-i\theta})^2 \\ &= \tfrac{3}{2} - e^{i\theta} - e^{-i\theta} + \tfrac{1}{4}e^{2i\theta} + \tfrac{1}{4}e^{-2i\theta}\end{aligned}$$

we see, by (2.9)/(2.10) that

$$\int \log\left(\frac{w(\theta)}{w_1(\theta)}\right)(1-\cos\theta)^2 \frac{d\theta}{2\pi} = 3A_0 - 2\,\mathrm{Re}(A_1) + \tfrac{1}{2}\,\mathrm{Re}(A_2) \tag{4.1}$$

with A_0, A_1, A_2 given by (2.6)–(2.8).

Lemma 4.1. *The RHS of* (4.1) $= H_0 + I_0 + J_0 + K_0 - K_1 + L_0 - L_2$ *where*

$$H_j = \tfrac{3}{2}[\log(1-|\alpha_j|^2) + |\alpha_j|^2],$$

$$I_j = -\tfrac{1}{4}|\alpha_{j+2} - 2\alpha_{j+1} + \alpha_j|^2,$$

$$J_j = \tfrac{1}{4}(\alpha_j\bar\alpha_{j+2} + \bar\alpha_j\alpha_{j+2})|\alpha_{j+1}|^2 + \tfrac{1}{8}(\alpha_j^2\bar\alpha_{j+1}^2 + \bar\alpha_j^2\alpha_{j+1}^2),$$

$$\begin{aligned}K_j &= -2\,\mathrm{Re}(\alpha_j) + \tfrac{1}{4}\,\mathrm{Re}(\alpha_j^2) \\ &\quad + \tfrac{1}{2}\,\mathrm{Re}(\alpha_{j+1}) - \tfrac{1}{2}\,\mathrm{Re}(\alpha_{j+1}|\alpha_j|^2) + \mathrm{Re}[\bar\alpha_{j+1}\alpha_j] - |\alpha_j|^2,\end{aligned}$$

$$L_j = -\tfrac{1}{4}|\alpha_j|^2.$$

Proof. The non-cross-terms in I_0 are

$$-\tfrac{1}{4}(|\alpha_2|^2 + 4|\alpha_1|^2 + |\alpha_0|^2) = -\tfrac{3}{2}|\alpha_0|^2 + (|\alpha_0|^2 - |\alpha_1|^2) + \tfrac{1}{4}(|\alpha_0|^2 - |\alpha_2|^2)$$

which cancel the $|\alpha_0|^2$ term in H_0, the final $|\alpha_j|^2$ term in $K_0 - K_1$, and the $L_0 - L_2$ term.

The cross-terms in I_0 are

$$-\tfrac{1}{2}\,\mathrm{Re}(\bar\alpha_2\alpha_0) + \mathrm{Re}(\bar\alpha_2\alpha_1 + \bar\alpha_1\alpha_0)$$
$$= -\tfrac{1}{2}\,\mathrm{Re}(\bar\alpha_2\alpha_0) + 2\,\mathrm{Re}(\bar\alpha_0\alpha_1) - \mathrm{Re}(\bar\alpha_0\alpha_1) + \mathrm{Re}(\bar\alpha_1\alpha_2).$$

The first term comes from a piece of $\frac{1}{2}\,\mathrm{Re}(A_2)$ (since $\bar\alpha_0\alpha_2\rho_1^2 = \bar\alpha_0\alpha_2(1-|\alpha_1|^2)$), the second from the last term in $-2\,\mathrm{Re}(A_1)$, and the last two are cancelled by the $\mathrm{Re}(\bar\alpha_{j+1}\alpha_j)$ term in $K_0 - K_1$.

The $\alpha_0 - \alpha_1$ term in A_1 leads to the first term in $K_0 - K_1$. The first term in J_0 comes from the second half of $\bar\alpha_0\alpha_2\rho_1^2 = \bar\alpha_0\alpha_2 - \bar\alpha_0\alpha_2|\alpha_1|^2$ (the first half in this expression gave a cross-term in I_j). The second term in J_0 is the $\frac{1}{2}\bar\alpha_0^2\alpha_1^2$ term in A_2.

The remaining terms in A_2, that is, the first six terms on the RHS of (2.8), give precisely the remaining terms in $K_0 - K_1$. $\square$

Lemma 4.2. *The RHS of* (4.1) $= \tilde H_0 + \tilde I_0 + \tilde J_0 + \tilde K_0 - \tilde K_1 + \tilde L_0 - \tilde L_2$, *where*

$$\tilde H_j = \tfrac{3}{2}\left[\log(1-|\alpha_j|^2) + |\alpha_j|^2 + \tfrac{1}{2}|\alpha_j|^4\right],$$
$$\tilde I_j = I_j,$$
$$\tilde J_j = -\tfrac{1}{4}\,|\alpha_{j+1}|^2|\alpha_j - \alpha_{j+2}|^2 - \tfrac{1}{8}\,|\alpha_{j+1}^2 - \alpha_j^2|^2 - \tfrac{1}{4}\,(|\alpha_{j+1}|^2 - |\alpha_j|^2)^2,$$
$$\tilde K_j = K_j - \tfrac{3}{8}\,|\alpha_j|^4 - \tfrac{1}{4}\,|\alpha_{j+1}|^2|\alpha_j|^2,$$
$$\tilde L_j = L_j.$$

Proof. The non-cross-terms in the last two terms in $\tilde J_0$ give

$$-\tfrac{3}{8}\,(|\alpha_0|^4 + |\alpha_1|^4) = -\tfrac{3}{4}\,|\alpha_0|^4 + \tfrac{3}{8}(|\alpha_0|^4 - |\alpha_1|^4).$$

The first term cancels the $\tilde H_0 - H_0$ term, and the second, the first term in $(\tilde K_0 - K_0) - (\tilde K_1 - K_1)$.

The cross-term in $-\frac{1}{4}(|\alpha_{j+1}|^2 - |\alpha_j|^2)^2$ and the non-cross-terms in $-\frac{1}{4}|\alpha_{j+1}|^2|\alpha_j - \alpha_{j+2}|^2$ combine to $-\frac{1}{4}|\alpha_{j+2}|^2|\alpha_{j+1}|^2 + \frac{1}{4}|\alpha_{j+1}|^2|\alpha_j|^2$ and are cancelled by the second term in $(\tilde K_0 - K_0) - (\tilde K_1 - K_1)$. The cross-term in $-\frac{1}{8}|\alpha_{j+1}^2 - \alpha_j^2|^2$ is the second term in J_0 and finally, the cross-term in $-\frac{1}{4}|\alpha_{j+1}|^2|\alpha_j - \alpha_{j+2}|^2$ is the first term in J_0. $\square$

Lemma 4.3. (i) $|\tilde K_j| \leqslant \frac{47}{8}$; $|\tilde L_j| \leqslant \frac{1}{4}$,

(ii) $|\alpha_j| < \frac{1}{2} \Rightarrow d_1|\alpha_j|^6 \leqslant -\tilde H_j \leqslant d_2|\alpha_j|^6$ *for some* $d_2 > d_1 > 0$,

(iii) $\tilde J_j \leqslant 0$,

(iv) $\sum_{j=0}^{\infty} (-\tilde I_j) + |\alpha_j|^6 < \infty \Rightarrow \sum_{j=0}^{\infty} |\alpha_{j+1} - \alpha_j|^3 < \infty$,

(v) $\sum_{j=0}^{\infty} (-\tilde I_j) + |\alpha_j|^6 < \infty \Rightarrow \sum_{j=0}^{\infty}(-\tilde J_j) < \infty$.

Remark. (iv) is essentially a discrete version of the inequality of Gagliardo [4] and Nirenberg [12].

Proof. (i) follows from $|\alpha_j| < 1$, (ii) is just (ii) of Lemma 3.3 (since $\tilde{H}_j = 3B_j$), and (iii) is trivial.

To prove (iv), we let δ be given by (1.9) and let

$$\partial = \delta - 1 \tag{4.2}$$

so since $\delta^* = \delta^{-1}$ (δ is unitary on ℓ^2), we have

$$\partial^* = \delta^* - 1 = -\delta^{-1}\partial = -\delta^*\partial. \tag{4.3}$$

As a result, if α is a finite sequence, then

$$\begin{aligned}\sum_n |(\partial\alpha)_n|^3 &= \sum_n (\partial\alpha)_n(\partial\bar{\alpha})_n|\partial\alpha|_n \\ &= -\sum_n (\delta\alpha)_n[\partial\{(\partial\bar{\alpha})|\partial\alpha|\}]_n.\end{aligned} \tag{4.4}$$

Moreover, we have a discrete Leibnitz rule,

$$\begin{aligned}\partial(fg) &= (\delta f)(\delta g) - fg \\ &= (\delta f)\partial g + (\partial f)g\end{aligned} \tag{4.5}$$

and since $|a - b| \geqslant |a| - |b|$ by the triangle inequality,

$$|\partial|f|| \leqslant |\partial f|, \tag{4.6}$$

which is a discrete Kato inequality.

By (4.5),

$$\partial\{(\partial\bar{\alpha})|\partial\alpha|\} = [\delta(\partial\bar{\alpha})]\partial|\partial\alpha| + (\partial^2\bar{\alpha})|\partial\alpha|$$

so, by (4.6),

$$|\partial\{(\partial\bar{\alpha})|\partial\alpha|\}| \leqslant |\partial^2\alpha|\,|\delta(\partial\bar{\alpha})| + |\partial^2\alpha|\,|\partial\alpha|.$$

Using Hölder's inequality with $\frac{1}{6} + \frac{1}{2} + \frac{1}{3} = 1$ and (4.4), we get

$$\|\partial\alpha\|_3^3 \leqslant 2\|\alpha\|_6\|\partial^2\alpha\|_2\|\partial\alpha\|_3$$

(because $\|\delta\alpha\|_p = \|\alpha\|_p$), so

$$\sum_n |(\partial\alpha)_n|^3 \leqslant 2^{3/2}\left(\sum_n |\alpha_n|^6\right)^{1/4}\left(\sum_n |(\partial^2\alpha)_n|^2\right)^{3/4}. \tag{4.7}$$

Having proven (4.7) for α's of finite support, we get it for any α with the right-hand side finite since $\sum_n |\alpha_n|^6 < \infty$ implies $\alpha_n \to 0$, which allows one to cut off α at N and take $N \to \infty$ in (4.7). But (4.7) implies (iv).

To prove (v), we control the individual terms in $\sum(-\tilde{J}_j)$. First,

$$\||\alpha|^2|\delta^2\alpha - \alpha|^2\|_1 \leqslant \|\,\alpha^2\|_3\,\||\delta^2\alpha - \alpha|^2\|_{3/2}$$

(by Hölder's inequality with $\frac{1}{3}+\frac{2}{3}=1$)

$$\leqslant 4\|\alpha\|_6^2\,\|\partial\alpha\|_3^2<\infty$$

(by first using $\|\delta^2\alpha-\alpha\|_3\leqslant 2\|\partial\alpha\|_3$ and then (iv)). Next,

$$|\alpha_{j+1}^2-\alpha_j^2|^2\leqslant(|\alpha_{j+1}|+|\alpha_{j+1}|)^2|\alpha_{j+1}-\alpha_j|^2$$

can be controlled as the first term was and the final term is controlled in the same way since $|\alpha_{j+1}|^2-|\alpha_j|^2\leqslant|\alpha_{j+1}^2-\alpha_j^2|$. □

Proof of Theorem 1.4. Suppose first that the right-hand side of (1.10) holds, that is, $\alpha\in\ell^6$ and $\partial^2\alpha\in\ell^2$. Iterate n times (4.1)/Lemma 4.2 for the nth Bernstein–Szegő approximation (with weight $w^{(n)}$) to obtain

$$\inf_n\left[\int(1-\cos\theta)^2\log(w^{(n)}(\theta))\,\frac{d\theta}{2\pi}\right]>-\infty$$

since the left-hand side is just

$$\inf_n\left[\tilde{K}_0^{(n)}+\tilde{L}_0^{(n)}+\tilde{L}_1^{(n)}+\sum_{j=0}^{n-1}(\tilde{H}_j^{(n)}+\tilde{I}_j^{(n)}+\tilde{J}_j^{(n)})\right],$$

which is finite by Lemma 4.3 and the hypothesis. Again, we have that $\int(1-\cos\theta)^2\log w(\theta)\,\frac{d\theta}{2\pi}$ is an entropy up to a constant and so upper semicontinuous. Thus RHS of (1.10) $\Rightarrow$ LHS of (1.10).

For the opposite direction, as in the last section, we use iterated (4.1)/Lemma 4.2 plus the fact that $\int(1-\cos\theta)^2\log(w_{2m}(\theta))\frac{d\theta}{2\pi}$ is bounded from above to conclude

$$\sum_{j=0}^{\infty}-(\tilde{H}_j+\tilde{I}_j+\tilde{J}_j)<\infty.$$

Since each is positive, $\sum(-\tilde{H}_j)<\infty$, which implies $\sum|\alpha_j|^6<\infty$ by (ii) of Lemma 4.3, and $\sum_{j=0}^{\infty}(-\tilde{I}_j)<\infty$, which implies $\partial^2\alpha\in\ell^2$. □

5. The general case

Finally, we turn to the general case of Theorem 1.3, and we define

$$\mathcal{I}_m\equiv\int\big[1-\cos(\theta-\theta_1)\big]\big[1-\cos(\theta-\theta_2)\big]\log\left(\frac{w(\theta)}{w_m(\theta)}\right)\frac{d\theta}{2\pi}. \tag{5.1}$$

Using (2.9) and (2.10), we obtain

$$\mathcal{I}_1=\frac{4+e^{i(\theta_1-\theta_2)}+e^{-i(\theta_1-\theta_2)}}{4}\,A_0-\mathrm{Re}\big[(e^{i\theta_1}+e^{i\theta_2})A_1\big]$$
$$+\tfrac{1}{2}\mathrm{Re}\big[e^{i(\theta_1+\theta_2)}A_2\big]. \tag{5.2}$$

The situation is now somewhat more complicated than in the previous sections and it will be more convenient to work with $\mathcal{I}_m$ from the start, only keeping track of the essential components of the sums (analogs of $\sum(B_j + C_j + D_j)$ and $\sum(\tilde{H}_j + \tilde{I}_j + \tilde{J}_j)$ above) and ignore the ones that are always bounded and hence irrelevant for us (analogs of $F_0 - F_1 + G_0 + G_1 - G_m - G_{m+1}$ and $\tilde{K}_0 - \tilde{K}_m + \tilde{L}_0 + \tilde{L}_1 - \tilde{L}_m + \tilde{L}_{m+1}$). Hence substituting (2.6)–(2.8) in (5.2) and iterating, we obtain

$$\mathcal{I}_m = C_{\alpha,m} + \frac{4 + e^{i(\theta_1-\theta_2)} + e^{-i(\theta_1-\theta_2)}}{4} \sum_{j=0}^{m-1} \log(1 - |\alpha_j|^2)$$

$$+ \sum_{j=0}^{m-1} \operatorname{Re}\Big\{ \big(e^{i\theta_1} + e^{i\theta_2}\big)\alpha_{j+1}\bar{\alpha}_j - \tfrac{1}{2} e^{i(\theta_1+\theta_2)}$$

$$\times \big[\alpha_{j+2}\bar{\alpha}_j(1 - |\alpha_{j+1}|^2) - \tfrac{1}{2}\alpha_{j+1}^2\bar{\alpha}_j^2\big]\Big\},$$

where

$$C_{\alpha,m} \equiv -\operatorname{Re}\big[(e^{i\theta_1} + e^{i\theta_2})(\alpha_0 - \alpha_m)\big]$$

$$+ \tfrac{1}{2}\operatorname{Re}\big[e^{i(\theta_1+\theta_2)}\big(\tfrac{1}{2}\alpha_0^2 - \tfrac{1}{2}\alpha_m^2 + \alpha_1 - \alpha_{m+1} - \alpha_1|\alpha_0^2| + \alpha_{m+1}|\alpha_m|^2\big)\big].$$

We let

$$\beta_j \equiv \alpha_j e^{i(\theta_1+\theta_2)j/2}$$

and

$$a \equiv \tfrac{1}{2}\big(e^{i(\theta_1-\theta_2)/2} + e^{-i(\theta_1-\theta_2)/2}\big) \in (-1, 1).$$

We will assume $a \neq 0$ since the case when θ_1 and θ_2 are antipodal follows from Theorem 3.1. With $C_{\beta,m} \equiv C_{\alpha,m}$ and all the sums taken from 0 to $m-1$, the above becomes

$$\mathcal{I}_m = C_{\beta,m} + \big(\tfrac{1}{2} + a^2\big) \sum \log(1 - |\beta_j|^2) + a \sum \big[\beta_{j+1}\bar{\beta}_j + \bar{\beta}_{j+1}\beta_j\big]$$

$$- \tfrac{1}{4} \sum \big[\beta_{j+2}\bar{\beta}_j(1 - |\beta_{j+1}|^2) + \bar{\beta}_{j+2}\beta_j(1 - |\beta_{j+1}|^2)\big]$$

$$+ \tfrac{1}{8} \sum \big[\beta_{j+1}^2\bar{\beta}_j^2 + \bar{\beta}_{j+1}^2\beta_j^2\big]. \tag{5.3}$$

In the following manipulations with the sums, we will use $C_{\beta,m}$ as a general pool/depository of terms that will be added/left over in order to keep all the sums from 0 to $m-1$. Its value will therefore change along the argument, but it will always depend on a few β_j's with j close to 0 or m only (i.e., it will gather all the "irrelevant" terms) and will always be bounded by a universal constant.

Lemma 5.1. *With $C_{\beta,m}$ universally bounded, we have*

$$\mathcal{I}_m = C_{\beta,m} + \big(\tfrac{1}{2} + a^2\big) \sum \big[\log(1 - |\beta_j|^2) + |\beta_j|^2 + \tfrac{1}{2}|\beta_j|^4\big]$$

$$- \tfrac{1}{4} \sum (1 - |\beta_{j+1}|^2)\big|\beta_{j+2} - 2a\beta_{j+1} + \beta_j\big|^2$$

$$-\tfrac{1}{4}\sum|\beta_{j+1}|^2\big|\beta_{j+2}-2a\beta_{j+1}\big|^2-\tfrac{1}{4}\sum|\beta_{j+1}|^2\big|\beta_j-2a\beta_{j+1}\big|^2$$
$$-\tfrac{1}{8}\sum\big|\beta_{j+1}^2-\beta_j^2\big|^2+\tfrac{1}{2}a^2\sum|\beta_j|^4 \tag{5.4}$$

with all the sums taken from 0 *to* $m-1$.

Remarks. 1. This enables us to prove the "⇐" part of (1.8) (even if $a=0$) since

$$\big|\{(\delta-e^{-i\theta_2})(\delta-e^{-i\theta_1})\alpha\}_j\big|=\big|\beta_{j+2}-2a\beta_{j+1}+\beta_j\big|. \tag{5.5}$$

But to prove the other implication, we first need to deal with the last sum in (5.4), which has the "wrong" sign.

2. Note that we actually did not need to exclude the case $a=0$ since then the last sum in (5.4) vanishes and an examination of (5.4) shows that $\lim_{m\to\infty}\mathcal{I}_m>-\infty$ if and only if the RHS of (1.8) holds. An argument from the proofs of Theorems 1.3 and 1.4 then gives the "⇒" part of (1.8).

Proof. Multiplying out the terms in the second, third, and fourth sums of (5.4) and after obvious cancellations, we are left with

$$-\tfrac{1}{4}\sum\Big[|\beta_{j+1}|^2\big(4a^2|\beta_{j+1}|^2-\beta_{j+2}\bar\beta_j-\bar\beta_{j+2}\beta_j\big)+\big|\beta_{j+2}-2a\beta_{j+1}+\beta_j\big|^2\Big].$$

But this is just

$$-\tfrac{1}{4}\sum\big[|\beta_{j+2}|^2+4a^2|\beta_{j+1}|^2+|\beta_j|^2+4a^2|\beta_{j+1}|^4\big] \tag{5.6}$$

plus the second and third sums in (5.3), the latter written as

$$\tfrac{1}{2}a\sum[\beta_{j+2}\bar\beta_{j+1}+\bar\beta_{j+2}\beta_{j+1}+\beta_{j+1}\bar\beta_j+\bar\beta_{j+1}\beta_j]$$

(with $C_{\beta,m}$ keeping the change). Adding the fifth and sixth sums in (5.4) to (5.6) and subtracting the last sum in (5.3), we obtain

$$-\tfrac{1}{4}\sum(2+4a^2)|\beta_j|^2-\tfrac{1}{8}\sum(2+4a^2)|\beta_j|^4$$

(again replacing all $|\beta_{j+1}|$ and $|\beta_{j+2}|$ by $|\beta_j|$ and adding the difference to $C_{\beta,m}$). But this together with the first sum in (5.4) gives exactly the first sum in (5.3). □

If we define

$$\gamma_j\equiv\beta_{j+2}-2a\beta_{j+1}+\beta_j$$

then the second, third, and fourth sums in (5.4) involve $|\gamma_j|$, $|\gamma_j-\beta_j|$ and $|\gamma_j-\beta_{j+2}|$. Using $|x-y|^2\geqslant|x|^2+|y|^2-2|x||y|$ for the last two, we obtain (with a new $C_{\beta,m}$)

$$(-8)\mathcal{I}_m\geqslant C_{\beta,m}+\sum O(|\beta_j|^6)+\sum(2+2|\beta_{j+1}|^2)|\gamma_j|^2$$
$$+4\sum|\beta_{j+1}|^2|\beta_j|^2-4\sum|\beta_{j+1}|^2\big(|\beta_{j+2}|+|\beta_j|\big)|\gamma_j|$$
$$+\sum\big|\beta_{j+1}^2-\beta_j^2\big|^2-4a^2\sum|\beta_{j+1}|^4 \tag{5.7}$$

since

$$\log(1-|\beta_j|^2)+|\beta_j|^2+\tfrac{1}{2}|\beta_j|^4=O(|\beta_j|^6).$$

Next, we use $-4xy \geqslant -8x^2-\frac{1}{2}y^2$ with $x=|\beta_{j+1}|^2(|\beta_{j+2}|+|\beta_j|)$ and $y=|\gamma_j|$ to estimate the fourth sum by $\sum O(|\beta_j|^6)-\frac{1}{2}\sum|\gamma_j|^2$. Also,

$$\begin{aligned}
-4a^2\sum|\beta_{j+1}|^4 &= -\sum|\beta_{j+1}|^2|\beta_{j+2}+\beta_j-\gamma_j|^2\\
&\geqslant -\sum|\beta_{j+1}|^2|\beta_{j+2}+\beta_j|^2-\sum|\beta_{j+1}|^2|\gamma_j|^2\\
&\quad -2\sum|\beta_{j+1}|^2|\beta_{j+2}+\beta_j||\gamma_j|\\
&\geqslant C_{\beta,m}-4\sum|\beta_{j+1}|^2|\beta_j|^2-\sum|\beta_{j+1}|^2|\gamma_j|^2\\
&\quad -\sum O(|\beta_j|^6)-\tfrac{1}{4}\sum|\gamma_j|^2
\end{aligned}$$

again using $-2xy \geqslant -4x^2-\frac{1}{4}y^2$. Plugging these into (5.7), we have

$$(-8)\mathcal{I}_m \geqslant C_{\beta,m}+\sum O(|\beta_j|^6)+\sum\left(\tfrac{5}{4}+|\beta_{j+1}|^2\right)|\gamma_j|^2+\sum\left|\beta_{j+1}^2-\beta_j^2\right|^2.$$

The last sum is just $\sum\frac{1}{2}(|\beta_{j+2}^2-\beta_{j+1}^2|^2+|\beta_{j+1}^2-\beta_j^2|^2)$ plus a piece that goes into $C_{\beta,m}$. Letting $\varepsilon\equiv\frac{1}{3}\min\{2|a|,2-2|a|\}>0$, we obtain

$$|\beta_{j+1}|^2|\gamma_j|^2+\tfrac{1}{2}|\beta_{j+2}^2-\beta_{j+1}^2|^2+\tfrac{1}{2}|\beta_{j+1}^2-\beta_j^2|^2 \geqslant \tfrac{1}{2}\varepsilon^4|\beta_{j+1}|^4.$$

Indeed, if the third term is smaller than $\frac{1}{2}\varepsilon^4|\beta_{j+1}|^4$, then $|\beta_j-\beta_{j+1}|$ or $|\beta_j+\beta_{j+1}|$ is less than $\varepsilon|\beta_{j+1}|$, and similarly for the second term. But then $|\beta_{j+2}+\beta_j|/|\beta_{j+1}|\in[0,2\varepsilon)\cup(2-2\varepsilon,2+2\varepsilon)$ and so $|\gamma_j|/|\beta_{j+1}|\geqslant\min\{2|a|-2\varepsilon,2-2\varepsilon-2|a|\}\geqslant\varepsilon$, meaning that the first term is at least $\varepsilon^2|\beta_{j+1}|^4$. So finally,

$$(-8)\mathcal{I}_m \geqslant C_{\beta,m}+\sum O(|\beta_j|^6)+\sum|\gamma_j|^2+\tfrac{1}{2}\varepsilon^4\sum|\beta_j|^4$$

that is (by (5.5) and the definition of β_j,γ_j),

$$\begin{aligned}
\mathcal{I}_m &\leqslant C_{\alpha,m}+\sum O(|\alpha_j|^6)-\tfrac{1}{8}\sum\left|\left\{(\delta-e^{-i\theta_2})(\delta-e^{-i\theta_1})\alpha\right\}_j\right|^2\\
&\quad -\tfrac{1}{16}\varepsilon^4\sum|\alpha_j|^4.
\end{aligned}\tag{5.8}$$

Proof of Theorem 1.3. If the RHS of (1.8) holds, then the RHS of (5.4) for the nth Bernstein–Szegő approximation (with $m\geqslant n$) is bounded (in n), and so

$$\inf_n\left[\int\left[1-\cos(\theta-\theta_1)\right]\left[1-\cos(\theta-\theta_2)\right]\log(w^{(n)}(\theta))\frac{d\theta}{2\pi}\right]>-\infty.$$

By upper semicontinuity of the above integral (which is again an entropy up to a constant), we obtain the LHS of (1.8).

Conversely, assume the LHS of (1.8) holds. Then the essential support of w is all of $\partial\mathbb{D}$, and so by Rakhmanov's theorem [13], $|\alpha_j| \to 0$. Hence, starting from some j, we have $O(|\alpha_j|^6) \leqslant \frac{1}{32}\varepsilon^4|\alpha_j|^4$ and so

$$\mathcal{I}_m \leqslant D_{\alpha,m} - \tfrac{1}{8}\sum \left|\left\{(\delta - e^{-i\theta_2})(\delta - e^{-i\theta_1})\alpha\right\}_j\right|^2 - \tfrac{1}{32}\varepsilon^4 \sum |\alpha_j|^4 \tag{5.9}$$

for large m and some bounded (in m) $D_{\alpha,m}$. As in the previous sections, $\int \left[1 - \cos(\theta - \theta_1)\right]\left[1 - \cos(\theta - \theta_2)\right] \log(w_m(\theta))\frac{d\theta}{2\pi}$ is bounded above, and so $\mathcal{I}_m$ is bounded below by the hypothesis. (5.9) then shows that the RHS of (1.8) holds. □

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