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Jost functions and Jost solutions for Jacobi matrices, I. A necessary and sufficient condition for Szegő asymptotics

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Abstract. We provide necessary and sufficient conditions for a Jacobi matrix to produce orthogonal polynomials with Szegő asymptotics off the real axis. A key idea is to prove the equivalence of Szegő asymptotics and of Jost asymptotics for the Weyl solution. We also prove L^2 convergence of Szegő asymptotics on the spectrum.

1. Introduction

In 1922, Szegő [50] proved one of the most celebrated results in classical analysis: his asymptotic theorem for orthogonal polynomials. In modern language, he considered measures, $d\rho$, on $[-2, 2]$ of the form

$$d\rho(x) = f(x)\,dx + d\rho_s(x) \tag{1.1}$$

with orthonormal polynomials

$$p_n(x) = \gamma_n x^n + \text{ lower order} \tag{1.2}$$

obeying $\gamma_n > 0$ and

$$\int p_n(x) p_m(x)\,d\rho(x) = \delta_{nm}. \tag{1.3}$$

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What Szegő proved is that for $z \in \mathbb{D} = \{z \in \mathbb{C} \mid |z| < 1\}$, one has Szegő asymptotics as $n \to \infty$

$$z^n p_n\left(z + \frac{1}{z}\right) \to \frac{D(z)^{-1}}{\sqrt{2}} \tag{1.4}$$

so long as the following, known as the Szegő condition, holds

$$\int_{-2}^{2} \log f(x)(4 - x^2)^{-1/2}\, dx > -\infty. \tag{1.5}$$

(Actually, Szegő, using the still standard convention of the orthogonal polynomial community, took $d\rho$ on $[-1, 1]$ and he did not allow a singular component – that is a later refinement. Also, instead of $z \mapsto z + z^{-1}$ which maps $\mathbb{D} \to \mathbb{C}\setminus[-2, 2]$, he used the inverse map and stated his results in terms of limits of

$$\left(\frac{x}{2} + \frac{\sqrt{4 - x^2}}{2}\right)^n p_n(x) \tag{1.6}$$

rather than (1.4).

Szegő also found an explicit formula for $D(z)$, namely,

$$D(z) = \exp\left[\int \frac{e^{i\theta} + z}{e^{i\theta} - z} \log(f(\cos\theta)) \frac{d\theta}{4\pi}\right]. \tag{1.7}$$

Moreover, if (1.5) fails, so does (1.4).

From the point of view of measures, the restriction to $\operatorname{supp}(d\rho) \subset [-2, 2]$ is natural, but this is less so with respect to the recursion coefficients (aka Jacobi parameters) for the orthonormal polynomials, $p_n(x)$, defined by

$$x p_n(x) = a_{n+1} p_{n+1}(x) + b_{n+1} p_n(x) + a_n p_{n-1}(x) \tag{1.8}$$

for $\{a_n, b_n\}_{n=1}^{\infty}$. From this point of view, the natural condition is

$$a_n \to 1 \qquad b_n \to 0. \tag{1.9}$$

This is associated to, indeed implies that, $\operatorname{ess\,supp}(d\rho) = [-2, 2]$, that is, $\operatorname{supp}(d\rho) = [-2, 2] \cup P$, where P is a bounded set whose only possible limit points are ± 2. Our main goal in this paper is to answer the question of for which $\{a_n, b_n\}_{n=1}^{\infty}$ does one have Szegő asymptotics; we will find (see Theorem 5.1)

Theorem 1.1. *Let $p_n(x)$ be orthonormal polynomials associated to Jacobi parameters $\{a_n, b_n\}_{n=1}^{\infty}$ obeying* (1.9). *Then* $\lim z^n p_n(z + \frac{1}{z})$ *exists for all $z \in \mathbb{D}$, is nonzero for $z \in \mathbb{D}\setminus\mathbb{R}$ with convergence uniform on compacts if and only if*

$$(\alpha) \qquad \sum_{n=1}^{\infty} |a_n - 1|^2 + |b_n|^2 < \infty \tag{1.10}$$

$$(\beta) \qquad \lim_{n\to\infty} a_n a_{n-1} \dots a_1 \textit{ exists and is nonzero}$$

$$(\gamma) \qquad \lim_{n\to\infty} \sum_{j=1}^{n} b_j \textit{ exists}$$

– thereby closing a chapter opened 83 years ago.

There has, of course, been prior literature on these issues, although with considerably stronger hypotheses than (α)–(γ). The initial results relating Jacobi parameters to Szegő asymptotics illustrated how strong $\operatorname{supp}(d\rho) \subset [-2, 2]$ is and include

Theorem 1.2. *Let* $\operatorname{supp}(d\rho) \subset [-2, 2]$. *Then the following are equivalent:*

(a) (β) *holds.*
(b) (α), (β), *and* (γ) *hold.*
(c) *The Szegő condition* (1.5) *holds.*

This theorem combines results of Shohat [39] and Nevai [32]; see also [26] and [46]. Of course, once one drops the restriction on $\operatorname{supp}(d\rho)$, the a's and b's become almost independent, and any subset of (α)–(γ) can hold.

To continue our discussion of earlier results on extending Szegő asymptotics, we need some notation. Since P can only have ± 2 as limit points,

$$P \cap (-\infty, -2) = \{E_j^-\}_{j=1}^{N_-} \tag{1.11}$$

where $N_- = 0$ (i.e., the set is empty), $1, 2, \dots$ or ∞, and $E_1^- < E_2^- < \cdots$. Similarly,

$$P \cap (2, \infty) = \{E_j^+\}_{j=1}^{N_+} \tag{1.12}$$

with $E_1^+ > E_2^+ > \cdots$. The earliest results extending Szegő asymptotics beyond $\operatorname{supp}(d\rho) \subset [-2, 2]$ are due to Gonchar [18], Nevai [32], and Nikishin [33], who noted that the result still holds if $N_+ + N_- < \infty$. More recently,

Theorem 1.3 (Peherstorfer-Yuditskii [34]). *Suppose* $a_n \to 1$, $b_n \to 0$, *and*

$$\sum_{j,\pm} \left(|E_j^\pm| - 2\right)^{1/2} < \infty \tag{1.13}$$

and that (1.5) *holds. Then* (1.4) *holds where the function* $D(z)^{-1}$ *vanishes if and only if* $z + z^{-1}$ *is some* $E_j^\pm$.

Remark. The $D(z)^{-1}$ we use here is not the same as the D^{-1} used in [34], but is a Blaschke product times their D^{-1}.

Related to this is

Theorem 1.4 (Killip-Simon [26]). *If*

$$\sum_{n=1}^{\infty} |a_n - 1| + |b_n| < \infty \tag{1.14}$$

then (1.13) *and* (1.5) *hold.*

From one point of view, (1.13) is quite natural. If $z_j^{\pm}$ is defined by

$$z_j^{\pm} \in (-1, 1) \qquad z_j^{\pm} + \left(z_j^{\pm}\right)^{-1} = E_j^{\pm} \tag{1.15}$$

then (1.13) is equivalent to

$$\sum_{j,\pm} \left(1 - \left|z_j^{\pm}\right|\right) < \infty \tag{1.16}$$

which is exactly what is needed to define a Blaschke product of zeros and obtain $D(z)^{-1}$ as a Nevanlinna function (see [34, 26, 44]). Theorems 1.3 and 1.4 are the strongest prior results on when Szegő asymptotics holds.

Both as input and motivation, the next element of background for our work concerns sum rules. Szegő proved his results for orthogonal polynomials on the real line (OPRL) by mapping the problem to one on orthogonal polynomials on the unit circle (OPUC). For OPUC, he earlier [48] proved asymptotic formulae. He began at $z = 0$ where the limit formula was equivalent to his leading limit theorem for Toeplitz determinants (see [47]) and deduced the general formula from that.

Verblunsky [51] rewrote the $z = 0$ limit theorem as a sum rule, namely, if

$$d\mu(\theta) = w(\theta)\, \frac{d\theta}{2\pi} + d\mu_{\mathrm{s}} \tag{1.17}$$

is a probability measure on $\partial\mathbb{D}$ and α_n are its Verblunsky coefficients (see [43, 44] for definition), then

$$\prod_{j=0}^{\infty} \left(1 - |\alpha_j|^2\right) = \exp\!\left(\int \log(w(\theta))\, \frac{d\theta}{2\pi}\right) \tag{1.18}$$

(which includes the fact that both sides are 0 simultaneously, i.e., $\sum_{j=0}^{\infty} |\alpha_j|^2 = \infty \Leftrightarrow \int \log(w(\theta)) \frac{d\theta}{2\pi} = -\infty$). Without knowing of Verblunsky's work, Case [3, 4], motivated by KdV sum rules, wrote some sum rules for Jacobi matrices with sufficiently nice a's and b's – he was not explicit about the needed conditions, but his arguments at least require

$$\sum_{n=1}^{\infty} n(|a_n - 1| + |b_n|) < \infty. \tag{1.19}$$

It was Killip-Simon [26] who realized the right combination of sum rules and proved

Theorem 1.5 (Killip-Simon [26]). *Let* $a_n \to 1$ *and* $b_n \to 0$. *Then*

$$\sum_{n=1}^{\infty} |a_n - 1|^2 + |b_n|^2 < \infty \tag{1.20}$$

holds if and only if

$$\sum_{j,\pm} \left(|E_j^{\pm}| - 2\right)^{3/2} < \infty \tag{1.21}$$

and

$$\int_{-2}^{2} \log(f(x))(4 - x^2)^{1/2}\, dx < \infty. \tag{1.22}$$

Note that (1.22), which [26] calls the quasi-Szegő condition, is distinct from (1.5) ($(4 - x^2)^{1/2}$ rather than $(4 - x^2)^{-1/2}$). Further developments of sum rules include [27–29, 31, 42, 46]. In particular, one has

Theorem 1.6 (Simon-Zlatoš [46]). *Consider the three assertions:*
(β) $\lim_{n\to\infty} a_n \dots a_1$ *exists and is nonzero.*
(σ) (1.13) *holds.*
(τ) (1.5) *holds.*
If (β) *holds, then* $(\sigma) \Leftrightarrow (\tau)$*, and if* (σ) *and* (τ) *hold, then* (β) *holds.*

The next element in our analysis is to link Szegő asymptotics to a different asymptotic result associated with work of Jost [22]. Jost studied certain solutions of $-u'' + Vu = Eu$, which is the analog of

$$a_n f_{n+1} + \big(b_n - (z + z^{-1})\big) f_n + a_{n-1} f_{n-1} = 0 \qquad n = 2, 3, \dots \tag{1.23}$$

one of whose solutions is

$$f_n(z) = p_{n-1}\left(z + \frac{1}{z}\right). \tag{1.24}$$

As realized by Case [3, 4, 16], the analog of the Jost solution is a solution of (1.23), which is asymptotic to z^n in the sense that

$$z^{-n} u_n(z) \to 1. \tag{1.25}$$

Case showed such solutions exist if $|z| < 1$ and (1.19) holds. In distinction, Szegő asymptotics says $p_{n-1}(z + \frac{1}{z}) \sim C z^{-n}$.

There may or may not be a solution of (1.23) which obeys (1.25) if one only knows $a_n \to 1$, $b_n \to 0$, but from either the discrete version of Weyl's analysis (see, e.g., [35, 40]) or by the Poincaré-Perron theorem (see, e.g., [44, Sect. 9.6]), there is a solution for $z \in \mathbb{D}$ obeying $f_n \to 0$ – indeed, obeying $f_{n+1}/f_n \to z$. From Weyl's point of view, this is given by the Green's function, that is, we can take it to be, for $z \in \mathbb{D}\setminus\{z_j^\pm\}_{j=1}^{N_\pm}$,

$$w_n(z) = \left\langle \delta_n, (z + z^{-1} - J)\delta_1 \right\rangle \tag{1.26}$$

where J is the infinite Jacobi matrix

$$J = \begin{pmatrix} b_1 & a_1 & 0 & \dots \\ a_1 & b_2 & a_2 & \dots \\ 0 & a_2 & b_3 & \dots \\ \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots \end{pmatrix} \tag{1.27}$$

viewed as a bounded selfadjoint operator on $\ell^2(\mathbb{Z}_+)$.

We will say that Jost asymptotics occurs if for $z \in \mathbb{D}\setminus\{z_j^\pm\}_{j=1}^{N_\pm}$, $z^{-n} w_n(z)$ has a nonzero finite limit as $n \to \infty$. A key to our understanding of when Szegő asymptotics holds for general a's and b's (i.e., to Theorem 1.1) is the following result we prove in Sect. 2:

Theorem 1.7. *Fix $z_0 \in \mathbb{D}$ so that $z_0 + z_0^{-1}$ is not an eigenvalue of J. Then Szegő asymptotics (i.e., $z^n p_n(z + \frac{1}{z})$ has a nonzero limit) holds at z_0 if and only if Jost asymptotics holds at z_0.*

We can now turn more closely to our proof of Theorem 1.1. That Szegő asymptotics implies (α)–(γ) will be easy (and done in Sect. 5) once we have Theorem 1.7. Basically, $\tilde{w}_n(z) \equiv z^{-n} w_n(z)$ are analytic near $z = 0$ and Jost asymptotics (uniformly on $|z| = \varepsilon$) implies convergence of derivatives at $z = 0$. The first two Taylor terms at 0 yield (β)–(γ), and as in [26], a suitable combination of the first and third Taylor coefficients is positive and yields (α).

The hard direction is that (α)–(γ) implies Szegő or Jost asymptotics. We will provide three distinct proofs. The first, in Sect. 5, is a relative of Szegő's original proof and of the Peherstorfer-Yuditskii arguments relying on the study of analytic functions on the disk. Szegő just used (1.7) to define D, and Peherstorfer-Yuditskii multiplied D^{-1} by a Blaschke product. We do not have either luxury here. For (1.7) to work, one needs

$$\int \log(f(\cos\theta)) \frac{d\theta}{2\pi} > -\infty \tag{1.28}$$

which is equivalent to (1.5), while all we have is

$$\int \log(f(\cos\theta)) \sin^2(\theta) \frac{d\theta}{2\pi} > -\infty \tag{1.29}$$

which is equivalent to (1.22). Moreover, in place of (1.16), we only have

$$\sum_{j,\pm} (1 - |z_j|)^3 < \infty \tag{1.30}$$

so we cannot define a Blaschke product. The solution will be to define renormalized Blaschke products when (1.30) holds, which we do in Sect. 3, and a renormalized Poisson integral when (1.29) holds, which we do in Sect. 4. This will allow us to define a candidate for the Jost function and prove Jost asymptotics in Sect. 5 and so provide our first proof that (α)–(γ) imply Jost asymptotics. This proof provides bounds we will need in Sect. 8 to handle L^2 convergence on $\partial\mathbb{D}$.

Our second proof in Sect. 6 relies on an idea going back to Jost-Pais [23] that the Jost function is a Fredholm determinant. For OPRL, this is discussed in Killip-Simon [26]. We will use the theory of renormalized determinants for Hilbert-Schmidt operators to construct a candidate Jost function and use it to prove Jost asymptotics.

Our final proof, in Sect. 7, is connected to classical results on the construction of asymptotic solutions of ODE's associated with work of Levinson [30] and Hartman-Wintner [20]; see the book of Eastham [14]. We will use results of Coffman [6] on the difference equation analogs to construct Jost solutions when (α)–(γ) hold. This construction shows that the "hard" part of Theorem 1.1 is related to known results on ODE's with L^2 perturbations. From this point of view, our contribution here is the realization that Jost solutions imply Szegő asymptotics and that the conditions are not only sufficient but necessary.

In Sect. 8, we discuss L^2 convergence on $\partial\mathbb{D}$, following the original scheme of Szegő [48] but with some severe technical complications because the Jost function is not Nevanlinna. This is the hardest argument in the paper. We will show the following result (see Theorem 8.1).

Theorem 1.8. *Let $d\rho$ have the form* (1.1) *and suppose $\{a_n, b_n\}_{n=1}^\infty$ obey* (α)–(γ). *Then*

$$\lim_{n\to\infty} \int_{-2}^{2} \left| p_n(x) - (\sin\theta)^{-1} \operatorname{Im}(\bar{u}(e^{i\theta})e^{i(n+1)\theta})\right|^2 f(x)\,dx = 0$$

with $\theta = \arccos(\frac{x}{2})$, *and*

$$\lim_{n\to\infty} \int |p_n(x)|^2\, d\rho_{\mathrm{s}}(x) = 0.$$

In Sect. 9, we provide examples for each $p < \frac{3}{2}$ of Jacobi matrices with Szegő asymptotics, but with $\sum_{j,\pm}(|E_j^\pm| - 2)^p = \infty$. In Sect. 10, we make some remarks about Schrödinger operators with L^2 potentials.

We announced our results in [8] written in September of 2003 and mentioned our L^2 results but not their proof to Serguei Denisov. In May of 2004, Denisov-Kupin [12] released a preprint discussing modified Szegő asymptotics for certain OPUC when the Szegő condition fails but a condition like (1.29) holds. Their results are quite distinct from ours although, via (1.29), there is some overlap. Many of the methods are similar – in particular, like we do in Sect. 4, they use renormalized Poisson representations. There is also some overlap in the L^2 control of the boundary values which we consider in Sect. 8. In particular, by using some of their ideas, it is likely we could streamline the proof of and slightly strengthen our estimate, Proposition 8.2. We have kept our original proof. We would emphasize that our work on these methods is independent and roughly simultaneous.

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2. Szegő asymptotics and Jost asymptotics

As explained in the introduction, for any Jacobi matrix with $a_n \to 1$, $b_n \to 0$, and $z \in \mathbb{D}$, and not such that $z + z^{-1}$ is an eigenvalue of J, there are two natural solutions of

$$a_n u_{n+1} + \big(b_n - (z + z^{-1})\big)u_n + a_{n-1}u_{n-1} = 0 \qquad n = 2, 3, \dots . \tag{2.1}$$

One is the orthogonal polynomial solution, $u_n = p_{n-1}(z + \frac{1}{z})$, and the other is the Weyl solution,

$$w_n(z) = \big\langle \delta_n, (z + z^{-1} - J)^{-1}\delta_1 \big\rangle. \tag{2.2}$$

In this section, our purpose is to show that for each such z, one has Jost asymptotics at that z, that is

$$\tilde{w}_n(z) \equiv z^{-n} w_n(z) \to \tilde{w}_\infty(z) \tag{2.3}$$

for $\tilde{w}_\infty \neq 0$ if and only if one has Szegő asymptotics for that z, that is,

$$c_n(z) \equiv z^n p_n\left(z + \frac{1}{z}\right) \to c_\infty(z) \tag{2.4}$$

for $c_\infty \neq 0$, and moreover,

$$(1 - z^2)c_\infty(z)\tilde{w}_\infty(z) = 1 \tag{2.5}$$

(as we will see, $\tilde{w}_\infty(z) = 1/u(z)$, where u is the Jost function, so (2.5) is usually written $c_\infty(z) = u(z)/(1 - z^2)$).

Of course, $p_{\cdot\,-1}$ obeys (2.1) also at $n = 1$ if we define $p_{-1} \equiv u_0 = 0$ and $a_0 = 1$. Since

$$(J - z - z^{-1})(z + z^{-1} - J)\delta_1 = -\delta_1$$

w_n also obeys (2.1) if we set $a_0 = 1$ and

$$w_0(z) = 1. \tag{2.6}$$

The constancy of the Wronskian thus implies

$$a_n\left(p_n\left(z+\frac{1}{z}\right)w_n(z)-w_{n+1}(z)p_{n-1}\left(z+\frac{1}{z}\right)\right)=1 \tag{2.7}$$

where we get 1 since

$$a_0(p_0w_0-w_1p_{-1})=1.$$

Using the definitions (2.3)/(2.4) of c and $\tilde{w}$, (2.7) becomes

$$a_n\big(c_n(z)\tilde{w}_n(z)-z^2\tilde{w}_{n+1}(z)c_{n-1}(z)\big)=1. \tag{2.8}$$

Thus, the following lemma is of relevance:

Lemma 2.1. *Let x_n, y_n be sequences of nonzero complex numbers and let λ_n be nonzero positive numbers with*

$$\lambda_n\to 1 \tag{2.9}$$

and so, for some $z\in\mathbb{D}$,

$$x_{n+1}y_n-z^2x_ny_{n+1}=\lambda_n. \tag{2.10}$$

Then

(i) *If $y_n\to y_\infty\neq 0$, then $x_n\to 1/y_\infty(1-z^2)$.*
(ii) *If $x_n\to x_\infty\neq 0$ and $z^{2n}y_n\to 0$, then $y_n\to 1/x_\infty(1-z^2)$.*

Proof. (i) Rewrite (2.10) as

$$x_{n+1}=\lambda_ny_n^{-1}+z^2\,\frac{y_{n+1}}{y_n}\,x_n$$

and iterate $\ell+1$ times to get

$$x_{n+1}=\sum_{j=0}^{\ell}\lambda_{n-j}\,\frac{y_{n+1}}{y_{n+1-j}y_{n-j}}\,z^{2j}+z^{2\ell+2}\,\frac{y_{n+1}}{y_{n-\ell}}\,x_{n-\ell}. \tag{2.11}$$

Set $\ell=n-1$ and see that since $\sum_{j=0}^{n-1}z^{2j}=(1-z^{2n})/(1-z^2)$,

$$\left|x_{n+1}-y_\infty^{-1}(1-z^{2n})(1-z^2)^{-1}\right|\leq\sum_{j=0}^{n-1}e_{n,j}z^{2j}+e_{n,n}z^{2n} \tag{2.12}$$

where

$$e_{n,j}=\lambda_{n-j}\,\frac{y_{n+1}}{y_{n+1-j}y_{n-j}}-\frac{1}{y_\infty}\qquad j=0,\dots,n-1$$
$$e_{n,n}=y_{n+1}x_1y_1^{-1}.$$

Since $y_\ell \to y_\infty \neq 0$, $\sup_{n,j} e_{n,j} < \infty$ and moreover, $\lim_{n\to\infty} e_{n,j} = 0$ for all fixed j. Thus, since $e_{n,j} \to 0$ for j fixed, we have for ℓ fixed,

$$\begin{aligned}\limsup_{n\to\infty}\left|\sum_{j=0}^{n} e_{n,j} z^{2j}\right| &\leq \limsup\left|\sum_{j=\ell}^{n} e_{n,j} z^{2j}\right| \\ &\leq |z|^{2\ell}(1-|z|^2)^{-1}\sup_{n,j}|e_{n,j}|\end{aligned}$$

$\to 0$ as $\ell \to \infty$. Thus (2.12) implies

$$x_{n+1} \to y_\infty^{-1}(1-z^2)^{-1}. \tag{2.13}$$

(ii) Rewrite (2.10) as

$$y_n = \lambda_n x_{n+1}^{-1} + z^2 x_n x_{n-1}^{-1} y_{n+1}$$

and iterate upwards. Since $z^{2n} y_n \to 0$, the remainder after ℓ iterations goes to zero as $\ell \to \infty$, so

$$y_n = \sum_{j=0}^{\infty} \lambda_{n+j} z^{2j} x_n x_{n+j+1}^{-1} x_{n+j}^{-1}.$$

As in the argument in (i), this implies that $y_n \to x_\infty^{-1}(1-z^2)^{-1}$. □

Theorem 2.2 (Szegő asymptotics = Jost asymptotics). *Let J be a Jacobi matrix with $a_n \to 1$, $b_n \to 0$, and let $z \in \mathbb{D}$ be such that $z + z^{-1}$ is not an eigenvalue of J. Then $\tilde{w}_n(z)$ has a nonzero limit if and only if $c_n(z)$ has a nonzero limit, and if either happens,*

$$\lim_{n\to\infty} c_n(z) = \frac{u(z)}{1-z^2} \tag{2.14}$$

where

$$u(z)^{-1} \equiv \lim_{n\to\infty} \tilde{w}_n(z). \tag{2.15}$$

Proof. By (2.8), if $\lambda_n = a_n^{-1}$, $x_n = c_{n-1}(z)$, $y_n = \tilde{w}_n(z)$, then Lemma 2.1 implies this result so long as

$$\lim_{n\to\infty} z^{2n}\tilde{w}_n(z) = 0.$$

But

$$z^{2n}\tilde{w}_n(z) = z^n w_n(z)$$

goes to zero since both $w_n \to 0$ and $z^n \to 0$. □

3. Renormalized Blaschke products

As explained in the introduction, we need a renormalized Blaschke product that works for real zeros that only obey $\sum_n (1-|z_n|)^3 < \infty$ rather than the usual Blaschke condition $\sum_n (1-|z_n|) < \infty$.

One can make a case that the first renormalization in science was the Weierstrass product formula – to get an analytic function vanishing at $\{z_j\}_{j=1}^\infty$ with $|z_j| \to \infty$, one modifies one's first guess

$$F(z) = \prod_{j=1}^{\infty} \left(1 - \frac{z}{z_j}\right)$$

to

$$F(z) = \prod_{j=1}^{\infty} W_{n_j}\left(\frac{z}{z_j}\right) \tag{3.1}$$

where

$$W_n(z) = (1-z)\exp\left(\sum_{k=1}^{n} \frac{z^k}{k}\right) \tag{3.2}$$

picking the argument to be the truncation of the power series for $-\log(1-z)$. It is well known, of course, that (3.1) converges if n_j is chosen so that $\sum |r/z_j|^{n_j+1} < \infty$ for all $r > 0$. Similarly, if our only goal were to get a function with zeros in the right place, things would be easy – for one can show that if $z_j \in \mathbb{D}$, $|z_j| \to 1$ as $j \to \infty$ and $w_j = z_j/|z_j|$, and if n_j is chosen so that $\sum_{j=1}^\infty (\frac{1-|z_j|}{\varepsilon})^{n_j+1} < \infty$ for all $\varepsilon > 0$, then

$$F(z) = \prod_{j=1}^{\infty} W_{n_j}\left(\frac{w_j - z_j}{w_j - z}\right) \tag{3.3}$$

is a product converging absolutely to a nonzero function analytic in $\mathbb{D}$ with zeros at $\{z_j\}$.

We want our Blaschke products to have magnitude one on $\partial\mathbb{D}$ and we will want that for our renormalized Blaschke products.

For $p = 0$, $b(z, p) = z$: if $p \in \mathbb{D}$, $p \neq 0$,

$$b(z, p) = \frac{|p|}{p} \, \frac{p - z}{1 - \bar{p}z} \tag{3.4}$$

so $b(0, p) = |p| > 0$. The key, of course, is that $b(z, p) = 0$ if and only if $z = p$ and

$$|b(e^{i\theta}, p)| = 1. \tag{3.5}$$

If $p = (1-x)\omega$ with $|\omega| = 1$ and $x \in (0,1)$,

$$b(z,(1-x)\omega) = \frac{1-x-z\omega^{-1}}{1-(1-x)\omega^{-1}z}$$

$$= \frac{1-\frac{x}{1-\omega^{-1}z}}{1+\frac{x\omega^{-1}z}{1-\omega^{-1}z}}. \tag{3.6}$$

(3.6) shows immediately if $|z| < 1$ and $\sum |x_j| < \infty$, then $\prod_{j=1}^{\infty} b(z,(1-x_j)\omega_j)$ converges absolutely (and uniformly on $|z| < 1-\delta$) since the numerators and denominators in (3.5) separately do.

(3.6) suggests what to do if $\sum |x_j|^{n+1} < \infty$. Define

$$b_n(z,(1-x)\omega) = \frac{W_n\left(\frac{x}{1-\omega^{-1}z}\right)}{W_n\left(\frac{-x\omega^{-1}z}{1-\omega^{-1}z}\right)}. \tag{3.7}$$

Here is the key fact:

Proposition 3.1. (a) *Let $\delta > 0$. Then for $|z| < 1-\delta$ and $|x| < \delta/2$,*

$$|b_n(z,(1-x)\omega) - 1| \leq 4\delta^{-n-1}x^{n+1}. \tag{3.8}$$

(b) *For $e^{i\theta} \neq \omega$,*

$$\left|b_n(e^{i\theta},(1-x)\omega)\right| = 1. \tag{3.9}$$

Warning. One cannot use the maximum principle and (3.9) to conclude that $|b_n(z,(1-x)\omega)| \leq 1$. Indeed, for $n \geq 1$,

$$\lim_{r \uparrow 1} |b_n(r\omega,(1-x)\omega)| = \infty.$$

This is where b_n's differ from ordinary Blaschke factors. They have very singular inner factors (indeed, for $n \geq 3$, ones whose boundary values are not even signed measures).

Proof. (a) It is known, (e.g., Rudin [36, p. 301]) that

$$|z| < 1 \Rightarrow |W_n(z) - 1| \leq |z|^{n+1}. \tag{3.10}$$

If $|x| < \delta/2$ and $|z| < 1-\delta$, then $|x/(1-\omega^{-1}z)| \leq |x/\delta| < \frac{1}{2}$ so (3.10) can be used, and if N and D are the numerator and denominator in (3.7), $|D| > \frac{1}{2}$. Since

$$\left|\frac{N}{D} - 1\right| \leq \frac{1}{|D|}(|N-1| + |D-1|)$$

(3.10) implies that

$$|b_n(z,(1-x)\omega)| \leq 2\left[\left|\frac{x}{(1-\omega^{-1}z)}\right|^{n+1} + \left|\frac{x\omega^{-1}z}{(1-\omega^{-1}z)}\right|^{n+1}\right]$$

which yields (3.8).

(b) By (3.6), if $e^{i\theta} \neq \omega$, $b(e^{i\theta}, (1-x)\omega)$ can be defined as a limit and (3.6) still holds and $b(e^{i\theta}, \omega) = 1$. Thus for x small, $\log b(e^{i\theta}, (1-x)\omega)$ is analytic in x. By (3.4),

$$|b(e^{i\theta}, p)| = \left| \frac{p - e^{i\theta}}{1 - \bar{p}e^{i\theta}} \right| = \left| \frac{p - e^{i\theta}}{\bar{p} - e^{-i\theta}} \right| = 1$$

so for x positive, $e^{i\theta} \neq \omega$,

$$\operatorname{Re} \log(b(e^{i\theta}, (1-x)\omega)) = 0.$$

It follows that its Taylor coefficients,

$$\log(b(e^{i\theta}, (1-x)\omega)) = \sum_{n=1}^{\infty} \gamma_n(e^{i\theta}, \omega) x^n \tag{3.11}$$

have γ_n pure imaginary. Since

$$b_n(e^{i\theta}, (1-x)\omega) \equiv b(e^{i\theta}, (1-x)\omega) \exp\biggl(\sum_{j=1}^{n} \gamma_j(e^{i\theta}, \theta) x^j \biggr) \tag{3.12}$$

and γ_j is pure imaginary, (3.9) holds. □

Because we will be interested not in b_2 but something related to it by a finite correction, we need to look in detail at γ_1 and γ_2. We consider $\gamma_j(z, \omega)$ defined by (3.11) with $e^{i\theta} \to z$. By (3.6),

$$\gamma_1(z, \omega) = -\left(\frac{1 + \omega^{-1}z}{1 - \omega^{-1}z} \right) \tag{3.13}$$

$$\gamma_2(z, \omega) = -\frac{1}{2} \frac{(1 - (\omega^{-1}z)^2)}{(1 - \omega^{-1}z)^2} = -\frac{1}{2} \frac{(1 + \omega^{-1}z)}{(1 - \omega^{-1}z)}. \tag{3.14}$$

Remarkably, γ_1/γ_2 is independent of ω and z! For reasons that will be clear below, we want to consider

$$\alpha(z) = \frac{1 + z^2}{1 - z^2} \qquad \beta(z) = \frac{2z}{1 - z^2}. \tag{3.15}$$

Notice that

$$\gamma_1(z, \omega = \pm 1) = 2\gamma_2(z, \omega = \pm 1) = -(\alpha(z) \pm \beta(z)). \tag{3.16}$$

Definition. For $p \in (-1, 1)$, $p \neq 0$, and $z \in \mathbb{D}$, we define

$$q(z, p) = b(z, p) \exp\Bigl(-\alpha(z) \log(|p|) - \tfrac{1}{2} \beta(z) \Bigl(p - \tfrac{1}{p} \Bigr) \Bigr). \tag{3.17}$$

Theorem 3.2. (a) *For z near zero and $p \neq 0$, p real,*

$$\log q(z,p) = \log b(z,p) - \alpha(z)\log b(0,p) - \frac{1}{2}\,\beta(z)\,\frac{d}{dz}\,\log b(z,p)\bigg|_{z=0}. \tag{3.18}$$

(b)

$$q(z,p) = b_2(z,p)\exp\Big(-\alpha(z)A(p) - \tfrac{1}{2}\,\beta(z)B(p)\Big) \tag{3.19}$$

where

$$A(p) = \log|p| - (1-|p|) - \frac{(1-|p|)^2}{2} \tag{3.20}$$

$$B(p) = -\frac{(1-|p|)^3}{p}. \tag{3.21}$$

(c) *If $p \in (-1,1)$ with $1-|p| < \delta/2$ and $|z| < 1-\delta$, then*

$$|q(z,p)-1| \leq \Big[4\delta^{-3} + \tfrac{5}{3}(1+4\delta^{-3})\delta^{-1}|p|^{-1}\exp\big(\tfrac{5}{3}\,\delta^{-1}|p|^{-1}(1-|p|)^3\big)\Big](1-|p|)^3. \tag{3.22}$$

Proof. (a) Writing

$$b(z,p) = |p|\,\frac{1-\frac{z}{p}}{1-zp}$$

we see

$$\log b(z,p) = \log|p| + z\left(p - \frac{1}{p}\right) + O(z^2)$$

which, given (3.17), is (3.18).

(b) By (3.12) and (3.16),

$$q(z,p) = b_2(z,x)\exp(C(p,z))$$

where

$$\begin{aligned} C(p,z) = &-\alpha(z)\log(|p|) - \frac{1}{2}\,\beta(z)\left(p - \frac{1}{p}\right) \\ &-\alpha(z)\big((1-|p|) + \tfrac{1}{2}(1-|p|)^2\big) \\ &-\beta(z)\mathrm{sgn}(p)\big((1-|p|) + \tfrac{1}{2}(1-|p|)^2\big). \end{aligned} \tag{3.23}$$

Thus (3.19) follows from

$$p - \frac{1}{p} + \mathrm{sgn}(p)\,[2(1-|p|) + (1-|p|)^2] = -\frac{(1-|p|)^3}{p}. \tag{3.24}$$

(3.24) follows from writing $p = \operatorname{sgn}(p)(1-x)$ and

$$\begin{aligned}\operatorname{sgn}(p)\left[(1-x) - \frac{1}{1-x} + 2x + x^2\right] \\ &= \operatorname{sgn}(p)\left\{(1-x) - \left[1 + x + x^2 + \frac{x^3}{1-x}\right] + 2x + x^2\right\} \\ &= -\operatorname{sgn}(p)\,\frac{x^3}{1-x} = -\frac{(1-|p|)^3}{p}.\end{aligned}$$

(c) In terms of the function C of (3.23),

$$\begin{aligned}|q(z,p)-1| &= |b_2(z,p)\exp(C(z,p))-1| &(3.25)\\ &\le |b_2(z,p)|\,|\exp(C(p,z))-1| + |b_2(z,p)-1|. &(3.26)\end{aligned}$$

We have (3.8) to bound $|b_2(z,p)-1|$. Thus $|b_2| \le 1 + |b_2 - 1| \le 1 + 4\delta^{-3}$. Moreover,

$$|e^c - 1| \le |c| \max(1, |e^c|) \le |c| e^{|c|}$$

so (3.22) follows from

$$|C| \le \tfrac{5}{3}\,\delta^{-1}|p|^{-1}(1-|p|)^3. \tag{3.27}$$

To prove (3.27), note first that $|1-z^2| \ge 1-|z|^2 = (1+|z|)(1-|z|) \ge \delta$. Thus

$$|\alpha(z)| \le \frac{2}{\delta} \qquad |\beta(z)| \le \frac{2}{\delta}. \tag{3.28}$$

Moreover, if $|p| = 1-x$, then

$$\begin{aligned}\left|\log|p| - x - \frac{x^2}{2}\right| &= \left|\sum_{j=3}^{\infty} \frac{x^j}{j}\right| \\ &\le \frac{1}{3}\,\frac{x^3}{1-x} \\ &= \frac{1}{3}\,\frac{(1-|p|)^3}{|p|}.\end{aligned}$$

Thus, by (3.24) and (3.28),

$$|C| \le \frac{2}{\delta}\,\frac{(1-|p|)^3}{p}\left[\frac{1}{3} + \frac{1}{2}\right] = \frac{5}{3\delta}\,\frac{(1-|p|)^3}{|p|}$$

proving (3.27). □

Because each $b_n(z,p)$ is unbounded on $\mathbb{D}$, the usual methods for controlling products on $\partial\mathbb{D}$ do not work; but in the case where the limit points of zeros only are a finite set, they do. Here is what we will need:

Theorem 3.3. *Let* p_n *be a sequence of reals in* $(-1, 1)$ *with* $\lim_{n\to\infty}|p_n| = 1$ *so that*

$$\sum_{n=1}^{\infty}(1 - |p_n|)^3 < \infty. \tag{3.29}$$

Let

$$B_{\text{ren}}(z) = \prod_{n=1}^{\infty} q(z, p_n). \tag{3.30}$$

Then

(i) *The product* (3.30) *converges in* $\mathbb{C}_+ = \{z \in \operatorname{Im} z > 0\}$ *and defines a function analytic in* $\mathbb{D} \cup \mathbb{C}_+ \cup \mathbb{C}_-$ *whose only zeros are at* $\{p_n\}_{n=1}^{\infty}$.

(ii)

$$\left|B_{\text{ren}}(e^{i\theta})\right| = 1 \qquad \theta \in (0, \pi) \cup (\pi, 2\pi). \tag{3.31}$$

(iii) *If*

$$B_{\text{ren}}^{(N)} = \prod_{n=N+1}^{\infty} q(z, p_n) \tag{3.32}$$

then for any $p < \infty$,

$$\int \left|B_{\text{ren}}^{(N)}(e^{i\theta}) - 1\right|^p \frac{d\theta}{2\pi} \to 0. \tag{3.33}$$

Proof. (i) If $z \in \mathbb{C}_+$, we have

$$G(z) \equiv \max\left(\frac{1}{|1 - z|}, \frac{1}{|1 + z|}, \frac{|z|}{|1 - z|}, \frac{|z|}{|1 + z|}\right) < \infty.$$

Thus, if $xG(z) < 1$, the arguments in W_n in (3.7) are less than 1 and the same estimates we used to bound $|q(z, p) - 1|$ still work to see

$$|q(z, p) - 1| \equiv H(z)|1 - p|^3 \tag{3.34}$$

for suitable $H(z)$, and this shows the product converges.

(ii) Since the product converges on $\partial\mathbb{D}\backslash\{\pm 1\}$ and $|q(e^{i\theta}, p_n)| = 1$, (3.31) is immediate.

(iii) Since $|B_{\text{ren}}^{(N)}(e^{i\theta})| = 1$, by (ii), pointwise convergence implies L^p convergence. The estimate (3.34) implies pointwise convergence to 1 since $\sum_{n=N+1}^{\infty}|q(z, p_n) - 1| \to 0$. □

4. Renormalized Poisson representations

Our goal in this section is to start out with a function, $f(z)$, on $\mathbb{D}$, which has a complex Poisson representation

$$F(z) = \int P(z, e^{i\theta}) g(e^{i\theta}) \frac{d\theta}{2\pi} \tag{4.1}$$

where

$$P(z, e^{i\theta}) = \frac{e^{i\theta} + z}{e^{i\theta} - z} \tag{4.2}$$

and $g \in L^1(d\theta/2\pi)$, real-valued, and

$$g(e^{i\theta}) = g(e^{-i\theta}) \tag{4.3}$$

(so $F(z)$ is real on $\mathbb{D} \cap \mathbb{R}$).

We want to define

$$H(z) = F(z) - \alpha(z)F(0) - \beta(z)F'(0) \tag{4.4}$$

and show it has a representation

$$H(z) = \int Q(z, e^{i\theta}) g(e^{i\theta}) \frac{d\theta}{2\pi} \tag{4.5}$$

where Q obeys a bound

$$|Q(z, e^{i\theta})| \leq C(z) \sin^2 \theta. \tag{4.6}$$

This will allow us to extend (4.1) to cases where one only has $\int |g(e^{i\theta})| \sin^2 \theta \frac{d\theta}{2\pi} < \infty$. In (4.4), α and β are the functions in (3.15). For this section, their key property is

$$\alpha(z) \pm \beta(z) = P(z, e^{i\theta} = \pm 1). \tag{4.7}$$

To see why (4.6) should hold, note that, by (4.3), in (4.1) we can replace $P(z, e^{i\theta})$ by

$$S(z, e^{i\theta}) \equiv \tfrac{1}{2} [P(z, e^{i\theta}) + P(z, e^{-i\theta})] \tag{4.8}$$

$$= \frac{1 - z^2}{1 + z^2 - 2z \cos \theta}. \tag{4.9}$$

Since $S(0, e^{i\theta}) = 1$ and $\frac{\partial}{\partial z} S(z, e^{i\theta})\big|_{z=0} = \cos \theta$, (4.5) holds with

$$Q(z, e^{i\theta}) = S(z, e^{i\theta}) - \alpha(z) - \beta(z) \cos \theta. \tag{4.10}$$

Because of (4.7) and $P(z, e^{i\theta} = \pm 1) = S(z, e^{i\theta} = \pm 1)$, Q vanishes at $e^{i\theta} = +1$ and at $e^{i\theta} = -1$. Since α is even under $\theta \to -\theta$ and $\theta \to 2\pi - \theta$, these zeros must be quadratic, which is where (4.6) comes from.

A straightforward calculation shows that, by (4.10),

$$
\begin{aligned}
Q(z,e^{i\theta}) &= S(z,e^{i\theta}) - \frac{1+z^2}{1-z^2} - \frac{2z\cos\theta}{1-z^2} \\
&= \frac{1-z^2}{1+z^2-2z\cos\theta} - \frac{1+z^2+2z\cos\theta}{1-z^2} \\
&= \frac{-4z^2\sin^2\theta}{(1-z^2)(1+z^2-2z\cos\theta)} \qquad (4.11)\\
&= \frac{-4z^2\sin^2\theta}{(1-z)(1+z)(z-e^{i\theta})(z-e^{-i\theta})}. \qquad (4.12)
\end{aligned}
$$

We summarize with

Theorem 4.1. *Let F be given by* (4.1) *with $g \in L^1(d\theta/2\pi)$ satisfying* (4.3) *and let $H(z)$ be given by* (4.4). *Then* (4.5) *holds with Q given by* (4.11). *In particular,*

$$
|Q(z,e^{i\theta})| \le \frac{4\sin^2\theta}{(1-|z|)^3}. \qquad (4.13)
$$

Proof. To get (4.13), note that $|1-z^2| \ge 1-|z|^2 = (1-|z|)(1+|z|) \ge 1-|z|$ and $|z-e^{\pm i\theta}| \ge 1-|z|$. □

As a final result about renormalized Poisson representations, we note that

Theorem 4.2. *Let $g \in L^1(\sin^2\theta\,\frac{d\theta}{2\pi})$ be real-valued with $g(e^{i\theta}) = g(e^{-i\theta})$. Define*

$$
F(z) = \int_0^{2\pi} Q(z,e^{i\theta})g(e^{i\theta})\,\frac{d\theta}{2\pi}. \qquad (4.14)
$$

Then for a.e. θ, $\lim_{r\uparrow 1} F(re^{i\theta}) \equiv F(e^{i\theta})$ exists, and for a.e. θ,

$$
\mathrm{Re} F(e^{i\theta}) = g(e^{i\theta}). \qquad (4.15)
$$

Proof. Given $\theta_0 \in (0,\pi)$, break the integral in (4.14) into two parts: $I_1 \equiv (\theta_0-\delta,\theta_0+\delta) \cup (-\theta_0-\delta,-\theta_0+\delta)$ for $|\delta| < \min(\theta_0,\pi-\theta_0)$ and the complement, I_2. By (4.12), if $\theta \in I_2$,

$$
|Q(re^{i\theta_0},e^{i\theta}| \le C\sin^2\theta \qquad (4.16)
$$

uniformly in r and $\lim_{r\uparrow 1} Q(re^{i\theta_0},e^{i\theta})$ exists and is pure imaginary. Thus the part of the integral in (4.14) for $\theta \in I_2$ has a limit with real part 0; if $z = re^{i\theta_0}$, $r \uparrow 1$.

On I_1, we can rewrite Q as a sum of its four summands ($\frac{1}{2}P(z,e^{i\theta})$, $\frac{1}{2}P(z,e^{-i\theta})$, $\alpha(z)$, and $\beta(z)\cos\theta$). Clearly, $\alpha(re^{i\theta_0})$ and $\beta(re^{i\theta_0})$ have limits which are pure imaginary. By the standard theory of Poisson kernels (Rudin [36], Duren [13]), the P terms have a limit for a.e. θ_0 whose real part is $\frac{1}{2}(g(e^{i\theta_0}) + g(e^{-i\theta_0})) = g(e^{i\theta_0})$ by the assumed symmetry of g. □

5. A necessary and sufficient condition for Jost asymptotics

Our goal in this section is to prove

Theorem 5.1. *Let J be a Jacobi matrix with $a_n \to 1$, $b_n \to 0$. Let $Q = \{z \in \mathbb{D} \mid z + z^{-1}$ is an eigenvalue of $J\}$. Then the following are equivalent:*

(i) *Szegő asymptotics (i.e., $z^n p_n(z + \frac{1}{z})$ converges to a nonzero limit as $n \to \infty$) hold for all $z \in \mathbb{D}\backslash Q$ uniformly on compact subsets of $\mathbb{D}\backslash Q$.*
(ii) *Szegő asymptotics hold for all z with $|z| = \varepsilon$ for some $\varepsilon > 0$ and uniformly in such z.*
(iii) *Jost asymptotics (i.e., $z^{-n} w_n(z)$ has a nonzero limit) hold for all $z \in \mathbb{D}\backslash Q$ uniformly on compact subsets of $\mathbb{D}\backslash Q$.*
(iv) *Jost asymptotics hold for all z with $|z| = \varepsilon$ for some $\varepsilon > 0$ uniformly in such z.*
(v) *The a's and b's obey three conditions:*
(α)

$$\sum_{n=1}^{\infty} |a_n - 1|^2 + \sum_{n=1}^{\infty} |b_n|^2 < \infty. \tag{5.1}$$

(β) $\lim_{n\to\infty} a_1 \dots a_n$ *exists and is not zero.*
(γ) $\lim_{n\to\infty} \sum_{j=1}^{n} b_j$ *exists.*
(vi) *The spectral measure, μ, on $\mathbb{R}$ and orthonormal polynomials obey the following properties:*
(δ) $\int_{-2}^{2} \log(d\mu_{\mathrm{ac}}/dE)\sqrt{4 - E^2}\, dE > -\infty$.
(ε) $\sum_n (|E_n^{\pm}| - 2)^{3/2} < \infty$.
(κ) *If the orthonormal polynomials have the form*

$$p_n(x) = \gamma_n\big(x^n - \lambda_n x^{n-1} + \cdots\big) \tag{5.2}$$

then $\lim_{n\to\infty} \gamma_n$ *exists and is nonzero and* $\lim_{n\to\infty} \lambda_n$ *exists.*

Remarks. 1. We will see shortly that $w_n(z)$ has an nth-order zero at $z = 0$, so $z^{-n} w_n(z)$ has a removable singularity at $z = 0$ – and it is that value we intend when we say the limit exists at $z = 0$.

2. We will discuss below what happens at the z_0's in Q. (Basically, $z^n p_n(z + \frac{1}{z})$ has a zero limit there and, by shifting from Weyl to Jost solutions, we will also have control at z_0's in Q of the other solutions.)

3. We will see that $u(z) \equiv (\lim w_\infty(z))^{-1}$ always has a factorization formula when (v) holds. u will be expressed in terms of "spectral data" and the limits in (β) and (γ).

Define

$$M(z, J) = \big\langle \delta_1, (z + z^{-1} - J)^{-1}\delta_1 \big\rangle = w_1(z). \tag{5.3}$$

Let $J^{(n)}$ be the Jacobi matrix obtained by removing the first n rows and left n columns of J. Let

$$M_n(z, J) = M(z, J^{(n)}) \tag{5.4}$$

so $M_0(z, J) \equiv M(z, J)$. We will often drop the J if it is fixed in some discussion.

Lemma 5.2.

(i) $$M(z) = z + O(z^2) \tag{5.5}$$

(ii) $$M_n(z) = \frac{w_{n+1}(z)}{a_n w_n(z)} \tag{5.6}$$

(iii) $$w_n(z) = M(z)(a_1 M_1(z)) \dots (a_{n-1} M_{n-1}(z)) \tag{5.7}$$

(iv) $$w_n(z) = (a_1 \dots a_n) z^n + O(z^{n+1}) \tag{5.8}$$

(v) $$M_n(z) = \big(z + z^{-1} - b_{n+1} - a_{n+1}^2 M_{n+1}(z)\big)^{-1} \tag{5.9}$$

(vi) $$\log\left(\frac{M_n(z)}{z}\right) = b_{n+1} z + \big(\tfrac{1}{2} b_{n+1}^2 + a_{n+1}^2 - 1\big) z^2 + O(z^3). \tag{5.10}$$

Remark. Some of these equalities are intended in the sense of the field of meromorphic functions. For example, if $\ell < n$ and $w_\ell(z_0) = 0$, then $M_\ell(z)$ has a pole at z_0 and $M_{\ell-1}(z)$ a zero there and they are intended to cancel in (5.7). Alternatively, these formulae hold initially away from $\{z \in \mathbb{D} \mid z + z^{-1} \in \sigma(J^{(\ell)})$ for some $\ell = 0, 1, \dots\}$ and then they have removable singularities in some cases.

Proof. (i) $M(z)/z = \langle \delta_1, (1 + z^2 - zJ)^{-1} \delta_1 \rangle = 1 + O(z)$ as $z \to 0$.

(ii) As noted in Sect. 2, $w_n(z)$ is normalized by (2.6), that is, by

$$a_1 w_2(z) + \big(b_1 - z - z^{-1}\big) w_1(z) = -1$$

and, of course, $M(z) = w_1(z)$. (5.6) thus follows from

$$a_{n+1}\left(\frac{w_{n+2}}{a_n w_n}\right) + (b_{n+1} - z - z^{-1})\left(\frac{w_{n+1}}{a_n w_n}\right) = -1$$

since $w_{n+j}/a_n w_n$ solves the difference equation for $J^{(n)}$.

(iii) follows from (5.6) and $w_1 = M$.

(iv) is immediate from (5.5) for $M_n(z)$ and (5.7).

(v) follows from (5.6) and the difference equation for w.

(vi) From (5.9) for $n = 0$ and (5.5) for M_1,

$$\frac{M(z)}{z} = \big(1 - b_1 z - a_1 z^2 + z^2 + O(z^3)\big)^{-1}$$

so

$$\log\left(\frac{M(z)}{z}\right) = -\log\left(1 - b_1 z - a_1^2 z^2 + z^2 + O(z^3)\right)$$
$$= \left(b_1 z + \tfrac{1}{2} b_1 z^2\right) + \left(a_1^2 - 1\right) z^2 + O(z^3).$$

□

Reduction of Theorem 5.1 to (v) ⇒ (iii). By Theorem 2.2, (i) ⇔ (iii) and (ii) ⇔ (iv). (iii) ⇒ (iv) is trivial. Thus we need to prove (iv) ⇒ (v) and (v) ⇔ (vi) to reduce the proof to (v) ⇒ (iii).

The equivalence of (v) and (vi) is easy, given the result of Killip-Simon [26]. They prove that (α) ⇔ (δ), (ε). The equivalence of (κ) and (β), (γ) is immediate since the recursion relations for p imply that

$$a_{n+1}\gamma_{n+1} = \gamma_n$$
$$-\lambda_{n+1} = -\lambda_n - b_{n+1}$$

so

$$\gamma_n = (a_1 \dots a_n)^{-1} \qquad \lambda_n = \sum_{j=1}^{n} b_j. \tag{5.11}$$

To study (iv) ⇒ (v), define

$$\tilde{w}_n(z) = z^{-n} w_n(z) \qquad \tilde{M}_n(z) = z^{-1} M_n(z) \tag{5.12}$$

so, by (5.7),

$$\log \tilde{w}_n(z) = \sum_{j=1}^{n-1} \log(a_j) + \sum_{j=1}^{n-1} \log \tilde{M}_{j-1}(z). \tag{5.13}$$

Convergence of $\tilde{w}_n(z)$ uniformly on the circle and analyticity of $\tilde{w}_n(z)$ implies the derivatives of $\tilde{w}_n(z)$ at $z = 0$ all converge. By (5.10) and (5.13), the terms of order $1, z, z^2$ yield

$$\lim_{n\to\infty} \sum_{j=1}^{n-1} \log(a_j) = \nu_1 \tag{5.14}$$

$$\lim_{n\to\infty} \sum_{j=1}^{n-1} b_j = \nu_2 \tag{5.15}$$

$$\lim_{n\to\infty} \sum_{j=1}^{n-1} a_j^2 - 1 + \frac{1}{2} b_j^2 = \nu_3 \tag{5.16}$$

all exist. Following Killip-Simon [26], we look at (5.16) $-2\times$(5.14) to see

$$\lim_{n\to\infty}\sum_{j=1}^{n-1} G(a_j)+\tfrac{1}{2}\,b_j^2=\nu_3-2\nu_1 \tag{5.17}$$

where

$$G(a)=a^2-1-2\log(a).$$

Since $G(a)>0$ for $a\in(0,\infty)$, the summand in (5.17) is nonnegative, so (5.17) implies absolute convergence. Since $G(a)\geq(a-1)^2$ (for $G(1)=G'(1)=0$ and $G''(a)\geq 2$), (5.17) implies

$$\sum_{j=1}^{\infty}(a_j-1)^2+\tfrac{1}{2}\,b_j^2<\infty$$

which is (α). (β) is the exponential of (5.14) and (γ) is (5.15). □

We now turn towards proving Jost asymptotics when (α), (β), (γ) hold. We will give three proofs: one in this section using canonical factorization of M-functions, one in Sect. 6 using renormalized determinants, and one in Sect. 7 using Levinson-type asymptotic analysis of difference equations.

Our starting point for the proof in this section will be the "nonlocal" step-by-step sum rule of Simon [42]:

Theorem 5.3. *For any Jacobi matrix with $a_n\to 1$, $b_n\to 0$,*

$$\{\theta\mid \mathrm{Im}M_n(\theta)\neq 0\}=\{\theta\mid \mathrm{Im}M_{n+1}(\theta)\neq 0\} \tag{5.18}$$

(*modulo sets of $d\theta/2\pi$-measure zero*). *For any $p<\infty$,*

$$\log\left(\frac{\mathrm{Im}M_n(\theta)}{\mathrm{Im}M_{n-1}(\theta)}\right)\in L^p\left(\partial\mathbb{D},\frac{\partial\theta}{2\pi}\right) \tag{5.19}$$

and

$$a_{n+1}M_n(z)=zB_n^+B_n^-(z)\exp\left(\frac{1}{4\pi}\int\log\left(\frac{\mathrm{Im}M_n(z)}{\mathrm{Im}M_{n+1}(z)}\right)\left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right)d\theta\right). \tag{5.20}$$

Here $B_n^\pm$ are alternating Blaschke products (B^+ for $0<p_{1,+}^{(n)}<z_{1,+}^{(n)}<\cdots<p_{\ell,+}^{(n)}<z_{\ell,+}^{(n)}$ and B^- for $0>p_{1,-}^{(n)}>z_{1,-}^{(n)}>\cdots$) with $p_{j,\pm}^{(n)}+p_{j,\pm}^{(n)-1}$ the eigenvalues of $J^{(n)}$ and $z_{j,\pm}^{(n)}+z_{j,\pm}^{(n)-1}$ the eigenvalues of $J^{(n+1)}$.

Remarks. 1. (5.20) is a special case of a general factorization theorem for meromorphic Herglotz functions, f, of $\mathbb{D}$. The general theorem has $\frac{1}{2\pi}\log(|f(e^{i\theta})|)$. (5.20) then follows from $|a_{n+1}M_n|^2 = \mathrm{Im}M_n/\mathrm{Im}M_{n+1}$, which is a consequence of (5.9).

2. In our applications, the set in (5.18) is all of $\partial\mathbb{D}$. When this is false, $\mathrm{Im}M_n/\mathrm{Im}M_{n+1}$ in (5.19) and (5.20) have to be suitably defined on the complement of the set in (5.18); see [42] for details.

We define

$$L_n(z) = \log\left(\frac{a_{n+1}M_n(z)}{z}\right) \tag{5.21}$$

$$N_n(z) = L_n(z) - \alpha(z)L_n(0) - \tfrac{1}{2}\,\beta(z)L_n'(0) \tag{5.22}$$

where α, β are given by (3.15). If $p_{1,\pm}^{(n)}$ are the poles of M_n closest to $z=0$, we define $L_n(z)$ unambiguously on $\mathbb{D}\backslash[p_{1,+}^{(n)}, 1) \cup (-1, p_{1,-}^{(n)}]$ by requiring $L_n(z)$ analytic and $L_n(0)$ real. Since $p_{1,\pm}^{(n)} \to \pm 1$ as $n\to\infty$, the result below exponentiated holds on $\mathbb{D}\backslash\{p_{j,\pm}^{(0)} \mid j=1,2,\dots\}$.

Lemma 5.4. *Suppose that* (α)*, that is,* (5.1) *holds. Then for all* $z \in \mathbb{D}\backslash[p_{1,+}^{(0)}, 1) \cup (-1, p_{1,+}^{(0)}] = S$,

$$\lim_{N\to\infty} \sum_{n=0}^{N} N_n(z) \tag{5.23}$$

exists and the convergence is uniform on compact subsets of S.

Proof. By (5.20),

$$L_n(z) = \log B_n^+(z) + \log B_n^-(z) + \frac{1}{4\pi}\int \left(\frac{e^{i\theta}+z}{e^{i\theta}-z}\right)\log\left(\frac{\mathrm{Im}M_n}{\mathrm{Im}M_{n+1}}\right)d\theta.$$

Using (3.18) and (4.4),

$$\begin{aligned} N_n(z) = \sum_{j,\pm}\bigl[-\log q\bigl(z, p_{j,\pm}^{(n)}\bigr) + \log q\bigl(z, z_{j,\pm}^{(n)}\bigr)\bigr] \\ + \frac{1}{4\pi}\int Q(z, e^{i\theta})\log\left(\frac{\mathrm{Im}M_n}{\mathrm{Im}_{n+1}}\right)d\theta. \end{aligned} \tag{5.24}$$

Since $z_{j,\pm}^{(n)} = p_{j,\pm}^{(n+1)}$, this implies

$$\begin{aligned} \sum_{n=0}^{N} N_n(z) = \sum_{j,\pm}\bigl[-\log q\bigl(z, p_{j,\pm}^{(0)}\bigr) + \log q\bigl(z, p_{j,\pm}^{(N+1)}\bigr)\bigr] \\ + \frac{1}{4\pi}\int Q(z, e^{i\theta})\log\left(\frac{\mathrm{Im}M}{\mathrm{Im}M_{N+1}}\right)d\theta. \end{aligned} \tag{5.25}$$

The Killip-Simon [26] P_2 sum rule implies

(a) $\int \sin^2\theta |\log(\frac{\operatorname{Im} M}{\sin\theta})| \frac{d\theta}{2\pi} < \infty$

(b) $\sum (1 - |p_{j,\pm}^{(0)}|)^3 < \infty$

(c) $\lim_{N\to\infty} \int \sin^2\theta |\log(\frac{\operatorname{Im} M_{N+1}}{\sin\theta})| \frac{d\theta}{2\pi} = 0$

(d) $\lim_{N\to\infty} \sum (1 - |p_{j,\pm}^{(N+1)}|)^3 = 0.$

(a) and (b) and the estimates (3.22) and (4.13) allow us to write (5.25) as a difference of $M^{(0)}/p^{(0)}$ terms and $M^{(N+1)}/p^{(N+1)}$ terms and then (c), (d) show that the error terms go to zero. The result is that $\lim \sum_{n=0}^{N} N_n(z)$ exists and

$$\begin{aligned} &\lim_{n\to\infty} \exp\bigg(\sum_{n=0}^{N} N_n(z)\bigg) \\ &\quad = \prod_{j=\pm 1} q(z, p_{j,\pm}^{(0)})^{-1} \exp\bigg(\frac{1}{4\pi}\int Q(z, e^{i\theta}) \log\bigg(\frac{\operatorname{Im} M}{\sin\theta}\bigg)\, d\theta\bigg). \end{aligned} \tag{5.26}$$

The proof shows the convergence is uniform. □

Proof of Theorem 5.1 (v) ⇒ (iii). By (5.7), with $\tilde{w}_n(z) = z^{-n} w_n(z)$,

$$a_n \tilde{w}_n(z) = \exp\bigg(\sum_{j=0}^{n-1} L_j(z)\bigg). \tag{5.27}$$

Since $a_n \to 1$, $\tilde{w}_n(z)$ has a nonzero limit (i.e., Jost asymptotics hold) if and only if

$$\lim_{N\to\infty} \sum_{j=0}^{N} L_j(z)$$

exists and the convergence of $\tilde{w}_n$ is uniform if and only if the convergence of the sum is. Since

$$L_j(0) = \log(a_{j+1})$$

by (5.5) and

$$L_j'(0) = b_{j+1}$$

by (5.10), we have that

$$L_j(z) = N_j(z) + \alpha(z)\log(a_{j+1}) + \tfrac{1}{2}\beta(z) b_{j+1}. \tag{5.28}$$

By Lemma 5.4, $\sum_{n=0}^{N} N_n(z)$ converges uniformly if (α) holds, (β) and (γ) say that $\sum_{n=0}^{N} \log(a_{n+1})$ and $\sum_{n=0}^{N} b_{j+1}$ converge so $\sum_{n=0}^{N} L_n(z)$ converges uniformly. □

If $\tilde{w}_n(z)$ has a nonzero limit, we define the Jost function by

$$u(z) = \lim_{n\to\infty} \tilde{w}_n(z)^{-1}. \tag{5.29}$$

This agrees with the usual definition if $\sum_{n=1}^{\infty}|a_n - 1| + |b_n| < \infty$. Thus

$$u(z) = \exp\biggl(-\sum_{n=0}^{\infty} L_n(z)\biggr) \tag{5.30}$$

and we have proven (by (5.26)) that

Theorem 5.5. *If* (α), (β), (γ) *hold, then*

$$\begin{aligned} u(z) = {} & \biggl(\prod_{j=1}^{\infty} a_j\biggr)^{-\alpha(z)} e^{-\frac{1}{2}\beta(z)\sum_{j=1}^{\infty} b_j} \\ & \prod_{j=1,\pm}^{\infty} q\bigl(z, p_{j,\pm}^{(0)}\bigr) \exp\biggl(-\frac{1}{4\pi}\int Q(z, e^{i\theta}) \log\biggl(\frac{\operatorname{Im} M}{\sin\theta}\biggr)\, d\theta\biggr). \end{aligned} \tag{5.31}$$

In the above, $\prod_{j=1}^{\infty} a_j$ and $\sum_{j=1}^{\infty} b_j$ refer to the conditional limits. The integral representation (5.31) implies

Theorem 5.6. *Let* (α), (β), (γ) *hold and let*

$$u(z) = \Bigl(\lim_{n\to\infty} z^{-n} w_n(n)\Bigr)^{-1}.$$

Then

(i) *After removing the removable singularities at* $\{p_j^{(0)}\}$, $u(z)$ *is analytic in* $\mathbb{D}$ *and* $u(z_0) = 0$ ($z_0 \in \mathbb{D}$) *if and only if* $z_0 \in \{p_j^{(0)}\}$, *that is, if and only if* $z_0 + z_0^{-1}$ *is an eigenvalue of* J.

(ii) *For a.e.* θ, $\lim_{r\uparrow 1} u(re^{i\theta}) \equiv u(e^{i\theta})$ *exists and*

$$\operatorname{Im} M(e^{i\theta})|u(re^{i\theta})|^2 = \sin\theta \tag{5.32}$$

for a.e. θ.

Proof. (i) is immediate from (5.31) and (ii) follows from the fact that α, β have purely imaginary values on $\partial\mathbb{D}\backslash\{\pm 1\}$ from (3.31) and Theorem 4.2. □

Remark. However, unlike the case where $\sum_{n=1}^{\infty}|a_n - 1| + |b_n| < \infty$, u may not be Nevanlinna. Indeed, if $\sum_{n=1}^{\infty}(|E_n^{\pm}| - 2)^{1/2} = \infty$, u cannot be Nevanlinna. This is the subject of Sect. 8.

6. Renormalized determinants

The idea that Jost functions are given by Fredholm determinants goes back to Jost-Pais [23], and for Jacobi matrices was made explicit by Killip-Simon [26]. They define the perturbation determinant by

$$L(z, J) = \det\big(\mathbf{1} + \delta J\big(J_0 - (z + z^{-1})\big)^{-1}\big) \tag{6.1}$$

where

$$\delta J = J - J_0 \tag{6.2}$$

and J_0 is the Jacobi matrix associated to $a_n \equiv 1$, $b_n \equiv 0$. This definition is used when $z \in \mathbb{D}$ and

$$\sum_{n=1}^{\infty} |a_n - 1| + |b_n| < \infty. \tag{6.3}$$

In this case, δJ is trace class and the det in (6.1) is the standard trace class determinant (see Simon [45] and Goh'berg-Krein [17]).

What Killip-Simon [26] prove in their Theorem 2.16 is

Theorem 6.1. *For $J - J_0$ trace class, $z \in \mathbb{D}$, and $z + z^{-1} \notin \sigma(J)$, we have that the function M given by* (5.4) *obeys*

$$M(z, J) = \frac{zL(z, J^{(1)})}{L(z, J)} \tag{6.4}$$

and with $w_n(z)$, the Weyl solution,

$$z^{-n} w_n(z) \to \frac{\big[\prod_{j=1}^{\infty} a_j\big]}{L(z, j)}. \tag{6.5}$$

Remarks. 1. (6.4) implies (6.5) using (5.7) and $L(z, J^{(n)}) \to 1$ since $\|J^{(n)} - J_0\|_1 \to 0$.

2. (6.5), of course, says that the Jost function is given by

$$u(z; J) = \Big[\prod_{j=1}^{\infty} a_j\Big]^{-1} L(z; J). \tag{6.6}$$

3. Formula (6.4) is an expression of Cramer's rule since, very formally, Cramer's rule says

$$M(z, J) = \frac{\det(z + z^{-1} - J^{(1)})}{\det(z + z^{-1} - J)} \tag{6.7}$$

and

$$z = M(z, J^{(0)}) = \frac{\det\left(z + z^{-1} - J_0^{(1)}\right)}{\det\left(z + z^{-1} - J_0\right)}. \tag{6.8}$$

Moreover,

$$L(z, J) = \frac{\det(J - (z + z^{-1}))}{\det\left(J_0 - (z + z^{-1})\right)}. \tag{6.9}$$

(6.7)–(6.9) manipulate to (6.4). Of course, the det's in (6.7)–(6.9) are all infinite, but one way to prove (6.4) is to prove (6.7)–(6.9) for cutoff finite matrices and take limits.

When (5.1) holds, $J - J_0$ may not any longer be trace class, but it is Hilbert-Schmidt, which suggests that we use the renormalized determinant for such operators. Such determinants go back to Carleman [2]. They are discussed in [45, 17]. Our approach, due to Seiler [37, 38] and used in [45], relies on the fact that if A is Hilbert-Schmidt, then

$$B = (1 + A)e^{-A} - \mathbf{1}$$

is trace class, so we can define

$$\det_2(\mathbf{1} + A) \equiv \det(\mathbf{1} + B) = \det((\mathbf{1} + A)e^{-A}). \tag{6.10}$$

It obeys (see [45, Chap. 9])

$$A \in \mathcal{I}_1 \to \det(\mathbf{1} + A) = \det_2(\mathbf{1} + A)e^{\mathrm{Tr}(A)} \tag{6.11}$$

$$\left|\det_2(\mathbf{1} + A) - \det_2(\mathbf{1} + C)\right| \leq \|A - C\|_2 \exp(\Gamma_2(\|A\|_2 + \|C\|_2 + 1)) \tag{6.12}$$

for a suitable constant, Γ_2.

We note (see [43, eqn. (1.2.24)]) that

$$\left[\left(J_0 - (z + z^{-1})\right)^{-1}\right]_{nm} = -(z^{-1} - z)^{-1}[z^{|m-n|} - z^{m+n}]. \tag{6.13}$$

Thus

$$\begin{aligned}\left[\delta J\left(J_0 - (z + z^{-1})\right)^{-1}\right]_{nn} = &- (z^{-1} - z)^{-1}\left\{(a_{n-1} - 1)(z - z^{2n-1})\right. \\ &\left. + b_n(1 - z^{2n}) + (a_n - 1)(z - z^{2n+1})\right\}\end{aligned} \tag{6.14}$$

which implies:

Lemma 6.2. *If δJ is trace class and $z \in \mathbb{D}$, then*

$$\begin{aligned}&\mathrm{Tr}\left(\delta J\left(J_0 - (z + z^{-1})\right)^{-1}\right) \\ &\quad = -(z^{-1} - z)^{-1}\left\{\sum_{n=1}^{\infty} b_n(1 - z^{2n}) + 2\sum_{n=1}^{\infty}(a_n - 1)(z - z^{2n+1})\right\}.\end{aligned} \tag{6.15}$$

It also explains the relevance of

Proposition 6.3. *Suppose a_n, b_n obey (α)–(γ) of Theorem 5.1. Then*

$$T(z;J)=\lim_{N\to\infty}\left[-(z^{-1}-z)^{-1}\sum_{n=1}^{N}b_n(1-z^{2n})+2\sum_{n=1}^{N}(a_n-1)(z-z^{2n+1})\right] \tag{6.16}$$

exists for all $z\in\mathbb{D}$ and the convergence is uniform for compact subsets of $\mathbb{D}$.

Proof. $z^{2\cdot}\in\ell^2$ for $z\in\mathbb{D}$ uniformly on compact subsets, so (α) implies

$$\sum_{n=1}^{\infty}b_n z^{2n} \qquad \text{and} \qquad \sum_{n=1}^{\infty}(a_n-1)z^{2n+1}$$

converge absolutely to an analytic limit.

(β) plus $\sum_{n=1}^{\infty}|a_n-1|^2<\infty$ implies $\lim_{N\to\infty}\sum_{n=1}^{N}(a_n-1)$ exists, and this plus (γ) implies the existence of the remaining terms. □

Definition. If (α)–(γ) of Theorem 5.1 hold, we define

$$L_{\text{ren}}(z,J)=\det\nolimits_2\bigl(1+\delta J\bigl(J_0-(z+z^{-1})\bigr)^{-1}\bigr)e^{T(z;J)}. \tag{6.17}$$

Proposition 6.4. *Let $z\in\mathbb{D}$. $L_{\text{ren}}(z,J)=0$ if and only if $z+z^{-1}\in\sigma(J)$.*

Proof. $\det_2(1+A)=0$ if and only if $1+A$ is not invertible (see [45, Chap. 9]). Since $1+\delta J(J_0-(z+z^{-1}))^{-1}=(J-(z+z^{-1}))(J_0-(z+z^{-1}))$, this happens if and only if $z+z^{-1}\in\sigma(J)$. □

By (6.1), (6.11), and (6.15), we have that

Proposition 6.5. *If $\sum_{n=1}^{\infty}|a_n-1|+|b_n|<\infty$, then*

$$L_{\text{ren}}(z,J)=L(z,T). \tag{6.18}$$

Theorem 6.6. *If (α)–(γ) of Theorem 5.1 hold, then for all $z\in\mathbb{D}$ so $z+z^{-1}\notin\sigma(J)$,*

$$M(z,J)=\frac{zL_{\text{ren}}(z;J^{(1)})}{L_{\text{ren}}(z;J)} \tag{6.19}$$

and

$$z^{-n}w_n(z)=\left(\prod_{j=1}^{n-1}a_j\right)\frac{L_{\text{ren}}(z;J^{(n)})}{L_{\text{ren}}(z;J)}. \tag{6.20}$$

Proof. (6.20) is implied by (6.19) and (5.7), so we need only prove (6.19). Define J_n to be the Jacobi matrix with

$$a_j^{(n)} = \begin{cases} a_j & j \leq n-1 \\ 1 & j \geq n \end{cases} \tag{6.21}$$

$$b_j^{(n)} = \begin{cases} b_j & j \leq n \\ 0 & j \geq n+1. \end{cases} \tag{6.22}$$

Then, by (α),

$$\| J_n - J \|_2 \to 0 \tag{6.23}$$

so $\|\delta J_n - \delta J\|_2 \to 0$, so by (6.12),

$$\det_2\bigl(1 + \delta J_n\bigl(J_0 - (z+z^{-1})\bigr)^{-1}\bigr) \to \det_2\bigl(1 + \delta J\bigl(J_0 - (z+z^{-1})\bigr)^{-1}\bigr). \tag{6.24}$$

It is easy to see that $T(z, J_n) \to T(z, J)$. Thus, using (6.18), uniformly on compacts of $\mathbb{D}$,

$$\lim_{n\to\infty} L(z, J_n) = \lim_{n\to\infty} L_{\text{ren}}(z, J_n) = L_{\text{ren}}(z, J). \tag{6.25}$$

The same is true for $J_n^{(1)}$ and $J^{(1)}$. Therefore, since $u(z, J) \neq 0$, (6.4) implies (6.19). □

We therefore have the second proof of the hard part of Theorem 5.1:

Theorem 6.7. *Let* (α)–(γ) *of Theorem 5.1 hold. Then uniformly on compact subsets of* $\mathbb{D}\backslash\{z \mid z + z^{-1} \in \sigma(J)\}$,

$$z^{-n} w_n(z) \to u(z)^{-1} \tag{6.26}$$

where

$$u(z) = \left(\lim_{n\to\infty} \prod_{j=1}^{n} a_j \right)^{-1} L_{\text{ren}}(z, J). \tag{6.27}$$

Proof. By (6.20), this is equivalent to

$$\lim_{n\to\infty} L_{\text{ren}}(z, J^{(n)}) = 1 \tag{6.28}$$

uniformly on compacts. It is easy to see that $\lim_{n\to\infty} T(z, J^{(n)}) = 0$. Since $\| J^{(n)} - J_0 \|_2 \to 0$,

$$\det_2\bigl(1 + \delta J_n\bigl(J_0 - (z+z^{-1})\bigr)^{-1}\bigr) \to 1.$$

This proves (6.28). □

7. Geronimo-Case equations

Given a set $\{a_n, b_n\}_{n=1}^\infty$ of real Jacobi parameters, the Geronimo-Case polynomials $c_n(z)$, $g_n(z)$ are defined by the recursion relations:

$$c_{n+1}(z) = a_{n+1}^{-1}\big[\big(z^2 - b_{n+1}z\big)c_n(z) + g_n(z)\big] \tag{7.1}$$

$$g_{n+1}(z) = a_{n+1}^{-1}\big[\big(\big(1 - a_{n+1}^2\big)z^2 - b_{n+1}z\big)c_n(z) + g_n(z)\big] \tag{7.2}$$

with initial conditions

$$c_0(z) = g_0(z) = 1. \tag{7.3}$$

They were introduced in a slightly different form by Geronimo-Case [16] who, under a condition that $\sum n[|a_n - 1| + |b_n|] < \infty$, proved that for $z \in \overline{\mathbb{D}}$, $\lim_{n\to\infty} g_n(z)$ exists and defined it to be the Jost function. In Paper II of our current series [9], we will reexamine these equations to prove convergence in $\mathbb{D}$ if $\sum_n |a_n - 1| + |b_n| < \infty$ and, most importantly, identify what c_n and g_n are, namely,

$$c_n(z) = z^n p_n\left(z + \frac{1}{z}\right) \tag{7.4}$$

where p_n are the orthonormal polynomials. Moreover, if $\tilde{J}_\ell$ is defined like J_ℓ (see (6.21)/(6.22)) but with a different cutoff on a_j, that is,

$$\tilde{a}_j^{(\ell)} = \begin{cases} a_j & j \le \ell \\ 1 & j \ge \ell + 1 \end{cases} \tag{7.5}$$

$$\tilde{b}_j^{(\ell)} = \begin{cases} b_j & j \le \ell \\ 0 & j \ge \ell + 1 \end{cases} \tag{7.6}$$

then

$$g_n(z, J) = u(z, \tilde{J}_n) \tag{7.7}$$

the conventional Jost function for $\tilde{J}_n$, that is, $(\lim_{m\to\infty} z^{-m} w_m(z, \tilde{J}_n))^{-1}$.

Our goal here is to extend Theorem 5.1 by proving:

Theorem 7.1. *The following are equivalent:*

(a) *For some* $\varepsilon \in (0, 1)$, $\lim_{n\to\infty} c_n(z)$ *exists for* $|z| = \varepsilon$ *uniformly in such* z.
(b) *For all* $z \in \mathbb{D}$, $\lim_{n\to\infty} c_n(z)$ *and* $\lim_{n\to\infty} g_n(z)$ *exist uniformly on compacts of* $\mathbb{D}$, *and* $\lim g_n(z)$ *is the Jost function* $u(z; J)$.
(c) *Conditions* (α)–(γ) *of Theorem 5.1 hold.*

Proof. That (a) ⇒ (c) is just (ii) ⇒ (iv) in Theorem 5.1, and (b) ⇒ (a) is trivial. So we only need (c) ⇒ (b). Convergence of c_n is just (iv) ⇒ (i) of Theorem 5.1, so we only need convergence of g_n. To see this, we use (7.7), (6.17), and (6.27). $\|\tilde{J}_n - J\|_2 \to 0$, so

$$\det{}_2\big(1 + \delta\tilde{J}_n\big(J_0 - (z + z^{-1})\big)^{-1}\big) \to \det{}_2\big(1 + \delta J\big(J_0 - (z + z^{-1})\big)^{-1}\big). \tag{7.8}$$

Clearly, $T(z; \tilde{J}_n) \to T(z; J)$. Thus, g_n converges to the Jost function for J. □

The point of this theorem is that we establish the validity of the GC equations for defining u in the general context of Theorem 5.1. There is a second point – we want to turn this analysis around and directly use the GC equations to prove that, when (α)–(γ) hold, $c_n(z)$ and $g_n(z)$ have limits for $z \in \mathbb{D}$, thereby providing a third proof of the hard part of Theorem 5.1. The key is the following theorem of Coffman [6]:

Theorem 7.2 ([6]). *Let J be a $d \times d$ diagonal matrix with entries $\lambda_1, \dots, \lambda_d$ along the diagonal. Let A_n be a sequence of $d \times d$ matrices so*

(i)

$$\sum_{n=1}^{\infty} \|A_n\|^2 < \infty. \tag{7.9}$$

(ii) *J and $\{J + A_n\}_{n=1}^{\infty}$ are all invertible.*

Consider solutions $y_n \in \mathbb{C}^d$ of

$$y_{n+1} = (J + A_n)y_n \tag{7.10}$$

with some initial condition y_1. Suppose λ_j is a simple eigenvalue with $|\lambda_j| \neq |\lambda_\ell|$ for $\ell \neq j$. Let

$$Y(n) = \prod_{m=1}^{n-1} [\lambda_j + (A_m)_{jj}]. \tag{7.11}$$

Then there exists an initial condition y_1 so that

$$\lim_{n\to\infty} \frac{y_{n,j}}{Y(n)} \tag{7.12}$$

exists and is nonzero, while for $\ell \neq j$,

$$\lim_{n\to\infty} \frac{y_{n,\ell}}{Y(n)} = 0. \tag{7.13}$$

Remarks. 1. Coffman's result is a discrete analog of continuum (ODE) results of Hartman-Wintner [20]. Related work includes Ford [15], Benzaid-Lutz [1], and Janas-Moszyński [21].

2. Coffman [6] only requires that λ_j be a simple eigenvalue and allows others can have Jordan blocks. In (7.9), he allows 2 to be replaced by $p \in [1, 2]$, but such assumptions imply (7.9)!

3. A pedagogical presentation of Theorems 7.1 and 7.2 will appear in the second edition of [43]. Until that second edition appears, the section will be available on the web at http://www.math.caltech.edu/opuc.html.

Corollary 7.3. *Let J be a $d \times d$ matrix with simple eigenvalues $\lambda_1 = 1, \lambda_2, \dots, \lambda_d$ with $|\lambda_i| \neq |\lambda_j|$ if $i \neq j$ and $\lambda_2, \dots, \lambda_d \in \mathbb{D}$. Let $y^{(0)}$ be the eigenvector of J with $Jy^{(0)} = y^{(0)}$. Suppose A_n obeys* (7.9) *and*

$$\lim_{N\to\infty} \sum_{n=1}^{N} A_n$$

exists. Then for any initial condition y_1, the solution of (7.10) *obeys*

$$\lim_{n\to\infty} y_n = c(y_1)y^{(0)}. \tag{7.14}$$

As a preliminary, note that if $\sum_{n=1}^{N} a_n$ has a limit and

$$\sum_{n=1}^{\infty} |a_n|^2 < \infty \tag{7.15}$$

then $\prod_{j=1}^{N}(1 + a_n)$ has a finite limit which is nonzero if all $a_n \neq -1$. For $\sum_{n=1}^{\infty} \log(1 + a_n) - a_n$ is absolutely convergent by (7.15).

Proof. By this remark and Theorem 7.2, there are solutions $y_n^{(k)}$ with

$$y_n^{(k)}\lambda_k^{-n} \to \text{ multiple of eigenvector of } J \text{ with eigenvalue } \lambda_k.$$

(7.14) follows since $\lambda_1 = 1$ while $|\lambda_k| < 1$ for $k \neq 1$. □

Remark. By using Perron's theorem, one can show that only $|\lambda_j| < 1$ for $j \geq 2$ is needed, not $|\lambda_j| \neq |\lambda_k|$.

Here is the promised third proof of the hard part of Theorem 5.1:

Theorem 7.4. *Let conditions* (α)–(γ) *of Theorem 5.1 hold. Let c_n, g_n be defined by* (7.1)–(7.3). *Then*

$$\begin{pmatrix} c_n \\ g_n \end{pmatrix}(z) \to f(z)\begin{pmatrix} (1 - z^2)^{-1} \\ 1 \end{pmatrix}. \tag{7.16}$$

Proof. Let $J = \left(\begin{smallmatrix} z^2 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ with $z \in \mathbb{D}$. J has eigenvalues $z^2 \in \mathbb{D}$ and 1, and the eigenvector for eigenvalue 1 is $((1-z^2)^{-1}1)^t$. Let

$$A_n = \begin{pmatrix} -b_{n+1}z & 0 \\ (1-a_{n+1}^2)z^2 - b_{n+1}z & 0 \end{pmatrix}$$

which obeys the hypothesis of Corollary 7.3 by (α)–(γ). This corollary plus existence of the limit $\prod_{j=1}^{\infty} a_j$ imply (7.16). □

Remark. One can also apply Theorem 7.2 directly to the recursion relation (2.1) to see that there exist solutions $\sim Cz^n$, that is, Jost solutions.

8. L^2 Convergence on the boundary

Our goal in this section is to prove:

Theorem 8.1. *Let $d\rho$ have the form* (1.1) *and suppose $\{a_n, b_n\}_{n=1}^{\infty}$ obey (α)–(γ) of Theorem* 5.1. *Then*

$$\lim_{n\to\infty} \int_{-2}^{2} \left| p_n(x) - (\sin\theta)^{-1}\mathrm{Im}(\bar{u}(e^{i\theta})e^{i(n+1)\theta}) \right|^2 f(x)\, dx = 0 \tag{8.1}$$

with $\theta = \arccos(\frac{x}{2})$, *and*

$$\lim_{n\to\infty} \int |p_n(x)|^2 \, d\rho_s(x) = 0. \tag{8.2}$$

Remark. Unfortunately, there are some errors in the analogous formula in [44], namely, (13.3.15) should have

$$\frac{\bar{u}(x)e^{i(n+1)\theta} - u(x)e^{-i(n+1)\theta}}{2i\sin\theta} \tag{8.3}$$

where it has

$$\frac{\bar{u}(x)e^{i(n-1)\theta} - u(x)e^{i(n-1)\theta}}{4\sin\theta} \tag{8.4}$$

and p_n, not P_n. As a check, when $b_n \equiv 0$, $a_n \equiv 1$, $p_n(2\cos\theta) = \frac{\sin(n+1)\theta}{\sin\theta}$ and $u \equiv 1$.

Remark. It is desirable to prove the existence of wave operators under the assumptions of Theorem 8.1. The existence of wave operators for L^2 perturbations was indeed established for the one-dimensional Dirac operator by Denisov [11] and for OPUC by Simon [44, Theorem 10.7.9]. The proofs rely on the fact that, in both cases, the free evolution is especially simple and they do not extend immediately to the context of Jacobi matrices or one-dimensional continuum Schrödinger operators. For the latter case, Christ and Kiselev have shown solution asymptotics pointwise for almost every positive energy and existence of wave operators for potentials in L^p, $p < 2$, and their ideas should allow one to prove analogous results for Jacobi matrices. See [5] and references therein to related earlier work by the same authors. One can use (8.1) to prove that for a.e. θ, the transfer matrix is bounded along a subsequence. The deeper and interesting open question is pointwise convergence for a.e. θ. We do not know how to prove this or how to derive the existence of wave operators from the L^2 estimates given in Theorem 8.1.

Theorem 8.1 is an analog of what Szegő proved in [48] for OPUC and then translated [49] to exactly this form for OPRL with $\text{supp}(d\rho) \subset [-2, 2]$. Peherstorfer-Yudistkii [34] proved precisely this when (1.5) and (1.13) hold. While the underlying core idea behind the proof we use is that of those authors, our technicalities are much more complex.

For all these proofs, the key is to prove what is essentially a weak L^2 convergence that in the current context is

$$\lim_{n\to\infty} \int_0^{2\pi} e^{in\theta} p_n(2\cos\theta) u(e^{i\theta})^{-1}(1 - e^{2i\theta}) \frac{d\theta}{2\pi} = 1. \tag{8.5}$$

In the Szegő case, $u(z)^{-1}$ is an H^2-function, so the left side of (8.5) is just $[z^n p_n(z + \frac{1}{z}) u(z)^{-1}(1 - z^2)]\big|_{z=0}$, which converges to 1 by the asymptotic result inside the circle. If there are finitely many bound states, $u(z)^{-1}$ has finitely many poles. Using the fact that eigenfunctions go to zero, it is easy to accommodate the poles. For the case that Peherstorfer-Yuditskii study, the argument is more subtle but, by cutting off Blaschke products, still involves a contour integral around the whole unit circle.

In contrast, our $u(z)^{-1}$ is so singular at ± 1 that we do not see how to directly deal with the integral in (8.1). Instead, we will deal with arcs by mapping a sector to the unit disk and relating this to distributional convergence of suitable boundary values of analytic functions. As noted in the introduction, this argument has some elements in common with work of Denisov-Kupin [12] which was done subsequently to our work. In turn, we were all motivated by some arguments of Killip [25].

The technical core of our proof is the following:

Proposition 8.2. *Let* (α)–(γ) *of Theorem 5.1 hold. Fix a sector on* $\mathbb{D}$,

$$S = \{z \mid |z| < 1,\ 0 < \theta_0 \le \arg z \le \theta_1 < \pi\}. \tag{8.6}$$

Then there exist N and C so that for all n and all $z \in S$,

$$\left| z^n p_n\left(z + \frac{1}{z}\right)(1 - z^2) u(z)^{-1} \right| \le C(1 - |z|)^{-N}. \tag{8.7}$$

Moreover, C and N are uniformly bounded for S fixed for all $\{a_n, b_n\}_{n=1}^{\infty}$ *with*

$$\sup_N \left(\left| \sum_{n=1}^{N} \log(a_n) \right| + \left| \sum_{n=1}^{N} b_n \right| \right) + \sum_{n=1}^{\infty} |a_n - 1|^2 + |b_n|^2 < K. \tag{8.8}$$

Remarks. 1. What is critical is the C and N are n independent. N is also S independent, but C is S dependent and diverges as S approaches the real axis, that is, as we approach the singularities at $z = \pm 1$.

2. As noted in the introduction, by using ideas of Denisov-Kupin [12], we can likely prove this with $N = 1$; we will have $N = \frac{5}{2}$.

3. We defer the proof until the end of the section.

Definition. If $f(z)$ is a function analytic on $\mathbb{D}$, with

$$f(z) = \sum_{n=0}^{\infty} \hat{f}(n) z^n \tag{8.9}$$

and if f obeys

$$|f(z)| \le C(1 - |z|)^{-N} \tag{8.10}$$

so

$$|\hat{f}(n)| \le 4C(n+1)^N \tag{8.11}$$

(obtained by writing $\hat{f}(n)$ as a contour integral over a circle of radius $1 - (n+1)^{-1}$), we define the distributional boundary values of f by

$$\int f(e^{i\theta}) g(e^{i\theta}) \, \frac{d\theta}{2\pi} \equiv \sum_{n=0}^{\infty} \hat{f}(n) \int e^{in\theta} g(e^{i\theta}) \, \frac{d\theta}{2\pi} \tag{8.12}$$

for C^∞ functions g on $\partial\mathbb{D}$.

Power bounds like (8.7) are important because of

Proposition 8.3. *Let $f_n(z)$ be a sequence of functions analytic on $\mathbb{D}$ so that for some fixed C, N and all n,*

$$|f_n(z)| \le C(1 - |z|)^{-N}. \tag{8.13}$$

Suppose $f_n \to f_\infty$ uniformly on compacts of $\mathbb{D}$. Then the distributional boundary values converge in (weak) distributional sense.

Proof. Let $f \in C^\infty(\partial\mathbb{D})$ and $\hat{g}(-n)$ the integral in (8.12). Write

$$\sum_{k=0}^{\infty} |\hat{f}_n(k) - \hat{f}_\infty(k)| \, |\hat{g}(-k)| \le \boxed{1} + \boxed{2}$$

where $\boxed{2} = \sum_{k=K+1}^{\infty}$ and $\boxed{1} = \sum_{k=1}^{K}$. Then, by (8.11) and (8.13), for g fixed, we can choose K so $|\boxed{2}| < \varepsilon$. By the assumed convergence, $\hat{f}_n(k) \to \hat{f}_\infty(k)$ for each k, so $\boxed{1} \to 0$. □

Proposition 8.4. *For any sector S of the form* (8.6)*, there is an analytic bijection $\varphi : \mathbb{D} \to S$ and constant C so that*

$$(1 - |\varphi(z)|)^{-1} \leq C(1 - |z|)^{-1}. \tag{8.14}$$

Proof. By compactness, we only need to prove this near points where $|\varphi(z)| = 1$. Such points are on the part of $\partial\mathbb{D}$ that maps into $\partial S \cap \partial\mathbb{D}$, that is, an arc. At interior points, φ is locally linear and (8.14) holds, so we only need to worry about neighborhoods, N, of the points that map to corners. In suitable local coordinates, ζ, the corner maps to 0, $N \cap \mathbb{D}$ maps to $\mathbb{C}_+ \cap \{\zeta \mid |\zeta| < \varepsilon\}$ and φ is transformed to $\varphi(z(\zeta)) = \sqrt{\zeta}$ mapping $\mathbb{C}_+$ to a 90° corner. In these local coordinates, $1 - |\varphi(z)| \sim \operatorname{Im}\sqrt{\zeta}$ and $1 - |z| \sim \operatorname{Im}\zeta$, and (8.14) says

$$\frac{1}{\operatorname{Im}\sqrt{\zeta}} = \frac{2\operatorname{Re}\sqrt{\zeta}}{\operatorname{Im}\zeta} \leq \frac{C}{\operatorname{Im}\zeta}$$

which is immediate since $|\zeta|$ is small. Thus, (8.14) holds locally, and so globally. □

Given a sector S of the form (8.6) and $\varphi : \mathbb{D} \to S$, we can define analytic functions on $\mathbb{D}$,

$$f_n(z) = g_n(\varphi(z)) \tag{8.15}$$

where

$$g_n(z) = z^n p_n\left(z + \frac{1}{z}\right)u(z)^{-1}(1 - z^2). \tag{8.16}$$

This allows us to consider boundary values of f_n, and so g_n, as distributions. But g_n, and so f_n, also has pointwise boundary values, and we want to prove that the distributional boundary value is given by the function. We have

Lemma 8.5. *For any sector S of the form* (8.6)*, there is an $\mathbb{H}^2$ function h and a function q analytic in a neighborhood of $\mathbb{D} \cup \bar{S}$ so that*

$$u(z)^{-1} = h(z)q(z). \tag{8.17}$$

Remark. The point is that $q(z)$ is analytic in ∂S, so the boundary values of u^{-1} on ∂S are given by the well-studied theory of boundary values of $\mathbb{H}^2$ functions [36, 13].

Proof. We use the representation (5.31). Define

$$H(\theta) = \begin{cases} \operatorname{Im} M(e^{i\theta}) & e^{i\theta} \in T \cup \bar{T} \\ |\sin\theta| & e^{i\theta} \notin T \cup \bar{T} \end{cases} \tag{8.18}$$

where T is a slightly enlarged $\partial S \cap \partial\mathbb{D}$, but not so enlarged that it includes $+1$ or -1. Let

$$h(z) = \exp\left(\frac{1}{4\pi}\int \frac{e^{i\theta}+z}{e^{i\theta}-z}\log\left(\frac{H(\theta)}{\sin\theta}\right)d\theta\right). \tag{8.19}$$

Since $\frac{H(\theta)}{\sin\theta} > 0$ (by (5.32)),

$$\int \log\left(\frac{H(\theta)}{\sin\theta}\right) > -\infty \tag{8.20}$$

and, by (5.32) again,

$$\int \left(\frac{H(\theta)}{\sin\theta}\right)\frac{d\theta}{2\pi} < \infty \tag{8.21}$$

the function (8.19) is in $\mathbb{H}^2$ by the standard approximant argument of Szegő (see [43, Sect. 2.4]).

Thus, we define $q(z) = u(z)^{-1}h(z)^{-1}$ and need to show that it is analytic in a neighborhood of $\mathbb{D} \cup S$. By (5.31), we can write

$$q(z) = q_1(z)q_2(z)q_3(z)q_4(z) \tag{8.22}$$

where

$$q_1(z) = \left(\prod_{j=1}^{\infty} a_j\right)^{\alpha(a)} \exp\left(\frac{1}{2}\beta(z)\sum_{j=1}^{\infty} b_j\right) \tag{8.23}$$

is clearly analytic away from ± 1. We have that q_2 is the inverse of the renormalized Blaschke product is analytic away from $\mathbb{R}$ by Theorem 3.3. Next,

$$q_3(z) = \exp\left(\frac{1}{4\pi}\int Q(z,e^{i\theta})\log\left(\frac{\operatorname{Im}M}{H(\theta)}\right)d\theta\right) \tag{8.24}$$

is analytic since $\log(\frac{\operatorname{Im}M}{H(\theta)})$ is supported on $\partial\mathbb{D}\backslash(T \cup \bar{T})$ and $Q(z,e^{i\theta})$ has singularities only at $z = \pm 1, \pm e^{i\theta}$. Finally,

$$q_4(z) = \exp\left(\frac{1}{4\pi}\int [Q(z,e^{i\theta}) - P(z,e^{i\theta})]\log\left(\frac{H(\theta)}{\sin\theta}\right)d\theta\right) \tag{8.25}$$

is analytic since $\log(\frac{H(\theta)}{\sin\theta})$ is even, so we can replace $Q - P$ by $Q(z, e^{i\theta}) - \frac{1}{2}P(z, e^{i\theta}) - \frac{1}{2}P(z, e^{i\theta})$ and this kernel is only singular at $z = \pm 1$. □

Given $\theta_0 \in [0, 2\pi)$, let R_{θ_0} be the region

$$R_{\theta_0} = \left\{ z \,\middle|\, 1 > |z| > \frac{1}{2},\ \arg(1 - e^{-i\theta_0} z) < \frac{\pi}{4} \right\} \tag{8.26}$$

a region of nontangential approach to θ_0. Define the maximal function,

$$M(\theta_0) = \sup_{z \in R_{\theta_0}} |u(z)^{-1}|. \tag{8.27}$$

Given Lemma 8.5, standard $\mathbb{H}^2$ theory [36, 13, 24] implies that

Proposition 8.6. *$u(re^{i\theta})^{-1}$ has boundary values as $r \uparrow 1$ for a.e. $\theta \in (0, 2\pi)$. Indeed, for a.e. θ,*

$$\lim_{\substack{|z| \uparrow 1 \\ z \in R_{\theta_0}}} u(z)^{-1} = u(e^{i\theta_0})^{-1}. \tag{8.28}$$

Moreover, for every $\eta > 0$,

$$\int_{\eta}^{\pi-\eta} M(\theta_0)^2 \frac{d\theta}{2\pi} < \infty. \tag{8.29}$$

This implies

Proposition 8.7. *Let S be a sector of the form* (8.6) *and f_n be given by* (8.15)/(8.16). *Let $\tilde{S} \subset \partial\mathbb{D}$ be the image of $\partial S \cap \partial\mathbb{D}$ under φ^{-1} and let T_{f_n} denote the distribution induced by f_n on $\partial\mathbb{D}$. Let $t(\theta)$ be a function in $C^\infty(\tilde{S}^{\mathrm{int}})$. Then*

$$T_{f_n}(t) = \int_{\tilde{S}} t(\theta) g_n(\varphi(e^{i\theta})) \frac{d\theta}{2\pi}. \tag{8.30}$$

where g_n is defined by the pointwise boundary value of u^{-1}.

Proof. By the definition of (8.12) and the absolute convergence of the sum,

$$T_{f_n}(t) = \lim_{r \uparrow 1} \int_{\tilde{S}} t(\theta) g_n(\varphi(re^{i\theta})) \frac{d\theta}{2\pi}. \tag{8.31}$$

By the continuity of $\varphi \restriction \{re^{i\eta} \mid \eta \in \mathrm{supp}(t)\}$, for all r close enough to 1, $\varphi(re^{i\theta}) \in R_{\varphi(e^{i\theta})}$, so $g_n(\varphi(re^{i\theta})) \to g_n(\varphi(e^{i\theta}))$ by (8.28), and by (8.29) and $|tg_n| \le 2|t| M \sup_{|t| \le 1} |z^n p_n(z + \frac{1}{z})|$, we have domination by a function in L^2, and so in L^1. Thus, (8.30) follows from the dominated convergence theorem. □

Proposition 8.8. *For each C^∞ function, let S be a sector of the form* (8.6) *and φ an analytic map of $\mathbb{D}$ to S. Let t be a C^∞ function with support in S^{int}. Then*

$$\lim_{n\to\infty} \int_{\tilde{S}} t(\theta) g_n(\varphi(e^{i\theta})) \frac{d\theta}{2\pi} = \int t(\theta) \frac{d\theta}{2\pi}. \tag{8.32}$$

Proof. By (8.7) and (8.14), plus Theorem 5.1 (which implies $g_n(z) \to 1$ for $z \in \mathbb{D}$ and so for $z \in S$) and Proposition 8.3, $T_{f_n} \to 1$ in distributional sense. This implies (8.32), given Proposition 8.7. □

These lengthy preliminaries imply the key to L^2 convergence on the boundary:

Theorem 8.9. *Let $u(e^{i\theta})^{-1}$ be the boundary values of u^{-1} on $\partial\mathbb{D}$. Let g_n be given by* (8.16) *on $\partial\mathbb{D}$. Then*
(1)

$$\int_0^{2\pi} |g_n(e^{i\theta})|^2 \frac{d\theta}{2\pi} \le 2. \tag{8.33}$$

(2) $g_n \to 1$ *in weak-$L^2(\partial\mathbb{D}, \frac{d\theta}{2\pi})$.*

Proof. (1) We begin with some preliminaries concerning the measure $d\mu_{\text{ac}}$ on $\partial\mathbb{D}$ obtained by using $\theta = \arccos(\frac{x}{2})$ to move the a.c. part of $d\rho$, that is, $f(x)\,dx$ to $\partial\mathbb{D}$. Since $e^{i\theta} \to 2\cos\theta$ is $2-1$ from $\partial\mathbb{D}$ to $[-2, 2]$ (see [44, Sect. 13.15]),

$$d\mu_{\text{ac}}(\theta) = |\sin\theta| f(2\cos\theta)\, d\theta. \tag{8.34}$$

By (5.3) and standard theory of Stieltjes transforms,

$$f(2\cos\theta) = \frac{|\text{Im} M(e^{i\theta})|}{\pi} \tag{8.35}$$

so, by (5.32),

$$d\mu_{\text{ac}} = \frac{\sin^2(\theta)}{|u(e^{i\theta})|^2} \frac{d\theta}{\pi} \tag{8.36}$$

$$= \frac{1}{2} \frac{|1-e^{2i\theta}|^2}{|u(e^{i\theta})|^2} \frac{d\theta}{2\pi}. \tag{8.37}$$

Thus,

$$\begin{aligned} \int_0^{2\pi} |g_n(e^{i\theta})|^2 \frac{d\theta}{2\pi} &= 2\int_0^{2\pi} |p_n(2\cos\theta)|^2\, d\mu_{\text{ac}}(\theta) \\ &= 2\int_{-2}^{2} |p_n(x)|^2\, d\rho_{\text{ac}}(x) \\ &\le 2\int_{-2}^{2} |p_n(x)|^2\, d\rho(x) = 2 \end{aligned}$$

proving (8.33).

(2) By (1), the functions g_n are uniformly bounded in L^2, so it suffices to prove that

$$\int t(e^{i\theta})g_n(e^{i\theta})\,\frac{d\theta}{2\pi} \to \int t(e^{i\theta})\,\frac{d\theta}{2\pi} \tag{8.38}$$

for a total set of t's. If t is C^∞ and supported in some sector S of the form (8.6), (8.38) follows from Proposition 8.8 (there is a Jacobian to go from $d\varphi(\theta)$ to $d\theta$, but it is C^∞ on S^{int} and occurs on both sides of (8.38)). Since such t's are total, (2) is proven. □

Proof of Theorem 8.1. Define in $L^2([-2,2],\, f(x)\,dx)$,

$$j_n^+(x) = (2\sin\theta)^{-1}\,\overline{u(e^{i\theta})}\,e^{i(n+1)\theta} \tag{8.39}$$

$$j_n^-(x) = \overline{j_n^+(x)} \tag{8.40}$$

where $\theta(x) \in (0,\pi)$ is given by $x = 2\cos(\theta(x))$. By (8.35) and (5.32),

$$f(x) = \frac{\sin\theta}{\pi|u(e^{i\theta})|^2}. \tag{8.41}$$

Thus, by a change of variables,

$$\int_{-2}^{2} \left|j_n^+(x)\right|^2 f(x)\,dx = \int_{-2}^{2} \frac{1}{4}\,\sin^{-2}\theta|u|^2\,\frac{\sin\theta}{\pi|u|^2}\,dx$$
$$= \frac{1}{2}\int_0^\pi \frac{d\theta}{\pi} = \frac{1}{2}. \tag{8.42}$$

On the other hand, by the same change of variables,

$$\begin{aligned}
&\left\langle j_n^-, p_n\right\rangle_{L^2(f\,dx)} \\
&\quad= \int_{-2}^{2} (2\sin\theta)^{-1}\,\overline{u(e^{i\theta})}\,e^{in\theta}e^{i\theta}\,p_n(2\cos\theta)\,\frac{\sin\theta}{\pi|u|^2}\,dx \\
&\quad= \int_0^\pi u(e^{i\theta})^{-1}\left[(1-z^2)z^n p_n\left(z+\frac{1}{z}\right)\bigg|_{z=e^{i\theta}}\right]\frac{-1}{2i}\,\frac{d\theta}{\pi} \qquad (8.43)\\
&\quad\to \frac{i}{2} \qquad (8.44)
\end{aligned}$$

by Theorem 8.9. (8.43) uses

$$e^{i\theta}\,\frac{1}{2\sin\theta}\,\sin\theta\,d(2\cos\theta) = e^{i\theta}\sin\theta = \frac{-1}{2i}\,(1-e^{2i\theta}).$$

Similarly, since p_n is real,

$$\left\langle j_n^+, p_n\right\rangle_{L^2(f\,dx)} \to -\frac{i}{2}. \tag{8.45}$$

Finally, by the same change of variables that led to (8.42),

$$\langle j_n^+, j_n^- \rangle_{L^2(f\,dx)} = \frac{1}{2} \int_0^\pi e^{-2i(n+1)\theta} \, \frac{u(e^{i\theta})}{\overline{u(e^{i\theta})}} \, \frac{d\theta}{\pi} \to 0 \tag{8.46}$$

since $\frac{u}{\bar u} \in L^2(\partial\mathbb{D}, \frac{d\theta}{2\pi})$ and $e^{-2i(n+1)\theta)} \to 0$ weakly.

Now,

$$p_n(x) - (\sin\theta)^{-1} \mathrm{Im}(\bar u e^{i(n+1)\theta}) = p_n(x) - i^{-1}\big[j_n^+ - j_n^-\big] \tag{8.47}$$

so, by (8.42), (8.43), (8.45), and (8.44) (all norms in $L^2(f\,dx)$),

$$\begin{aligned}
0 &\le \liminf \|\text{LHS of (8.47)}\|^2 \\
&= \liminf \big(\|p_n\|^2 + \big\|j_n^+\big\|^2 + \big\|j_n^-\big\|^2 - 2\mathrm{Re}\langle j_n^+, j_n^-\rangle \\
&\qquad\qquad + 2\mathrm{Re}\langle p_n, ij_n^+\rangle - 2\mathrm{Re}\langle p_n, ij_n^-\rangle \big) \\
&= \liminf \big(\|p_n\|^2 + \tfrac{1}{2} + \tfrac{1}{2} - 0 - 1 - 1 \big) \\
&= \liminf \|p_n\|^2 - 1.
\end{aligned}$$

Thus, since $\|p_n\|^2_{L^2(f\,dx)} \le 1$, we conclude

$$\lim \|p_n\|^2_{L^2(f\,dx)} = 1$$

so $\|p_n\|_{L^2(d\rho_s)} \to 0$ and, by the above calculation, LHS of (8.47) $\to 0$. □

Thus, Theorem 8.1 is reduced to the proof of Proposition 8.2, to which we now turn. As a preliminary, we want to exploit the proof of Lemma 8.5:

Proposition 8.10. *Let S be a sector of the form* (8.6) *and let K be given. Then there exists C depending only on S and K so that if* (8.8) *holds, then for $z \in S$,*

$$|u(z)^{-1}| \le C(1 - |z|)^{-1/2}. \tag{8.48}$$

Proof. As in Lemma 8.5, we construct the factorization (8.17). The proof shows that $\|h\|_{\mathbb{H}^2}$ and $\|q\|_{S,\infty} \equiv \sup_{z\in S}|q(z)|$ are bounded by constants C_1 and C_2 depending only on K and S.

Let $h(z) = \sum_{j=0}^\infty h_j z^j$, then $\|h\|_{\mathbb{H}^2} = (\sum_{j=0}^\infty |h_j|^2)^{1/2}$, so, by the Schwarz inequality,

$$\begin{aligned}
|h(z)| &\le \|h\|_{\mathbb{H}^2} \Bigg(\sum_{j=0}^\infty |z|^{2j} \Bigg)^{1/2} \\
&= \|h\|_{\mathbb{H}^2} (1 - |z|^2)^{-1/2} \\
&\le C_1 (1 - |z|)^{-1/2}
\end{aligned}$$

so, by (8.17), we have (8.48) with $C = C_1 C_2$. □

Our proof will exploit (2.11) where $y_n = \tilde{w}_n$ and $x_n = c_{n-1}$. We are interested in controlling $u^{-1}x_n$, which means controlling $u^{-1}y_n^{-1}$ and ratios y_n/y_{n-j}, that is, the functions $u^{-1}\tilde{w}_n^{-1}$ and $\tilde{w}_n/\tilde{w}_{n-j}$. So we turn first to $u^{-1}\tilde{w}_n^{-1}$ and then $\tilde{w}_n/\tilde{w}_{n-j}$.

Let $J^{(n)}$ be the Jacobi matrix given after (5.3) and make the J-dependence of w_n explicit. Then:

Proposition 8.11. *Let* $\tilde{w}_n(z, J)$ *be given by* (2.2)/(2.3). *Then*

$$\tilde{w}_n(z, J)^{-1}u(z, J)^{-1} = a_n u(z, J^{(n)})^{-1}. \tag{8.49}$$

In particular, if (8.8) *holds, then for any* S *obeying* (8.6), *there is a* C *so*

$$\sup_n \left|\tilde{w}_n(z, J)^{-1}u(z, J)^{-1}\right| \le C(1 - |z|)^{-1/2}. \tag{8.50}$$

Remarks. 1. In order to get (8.48), one does not need a bound on $\sup_N|\sum_1^N b_n|$ but only on $\lim_N|\sum_1^N b_n|$ (and similarly for $\log(a_n)$). But to get (8.50), we need control on $\sup_N \lim_M(\sum_N^{N+M} b_n)$ – and that is why we state (8.8) in the form we do.

2. One can also prove this result using the fact that uw_n is the unique solution asymptotic to z^n.

Proof. By (5.7),

$$\tilde{w}_n(z, J) = \left(\frac{M(z)}{z}\right)\left(\frac{a_1 M_1(z)}{z}\right)\cdots\left(\frac{a_{n-1}M_{n-1}(z)}{z}\right)$$

from which it follows that

$$\tilde{w}_{n+k}(z, J) = a_n\tilde{w}_n(z, J)\tilde{w}_k(z, J^{(n)}). \tag{8.51}$$

Taking k to infinity using (2.3) and $\tilde{w}_\infty = u^{-1}$, we obtain

$$u(z, J)^{-1} = a_n\tilde{w}_n(z, J)u(z, J^{(n)})^{-1}$$

which is (8.49). □

Proposition 8.12. *For any* J*,*

(i)

$$|\tilde{w}_n(z, J)| \le \frac{\pi}{4}\,|z|^{-n}\varepsilon^{-1}(1 - |z|)^{-1} \tag{8.52}$$

when $\arg z \in (\varepsilon, \pi - \varepsilon)$ *and* $0 < |z| < 1$.

(ii)

$$\left|\frac{\tilde{w}_{n+k}(z, J)}{\tilde{w}_n(z, J)}\right| \le \frac{\pi}{4}\,a_n|z|^{-k}\varepsilon^{-1}(1 - |z|^{-1}) \tag{8.53}$$

when $\arg z \in (\varepsilon, \pi - \varepsilon)$ *and* $0 < |z| < 1$.

Proof. (i) By (2.2),

$$
\begin{aligned}
|\tilde{w}_n(z, J)| &\le \|(z + z^{-1} - J)^{-1}\| \, |z|^{-n} \\
&\le |\mathrm{Im}(z + z^{-1})|^{-1} \, |z|^{-n}
\end{aligned}
$$

since $\sigma(J) \subset \mathbb{R}$ and J is selfadjoint. But if $z = re^{i\varphi}$ with $0 < r < 1$,

$$
|\mathrm{Im}(z + z^{-1})| = (r^{-1} - r)|\sin\varphi|. \tag{8.54}
$$

For $\varphi \in (\varepsilon, \pi - \varepsilon)$,

$$
|\sin\varphi| \ge \frac{2}{\pi}\,\varepsilon \tag{8.55}
$$

and

$$
r^{-1} - r = (1 - r)(r^{-1} + 1) \ge 2(1 - r). \tag{8.56}
$$

Thus (8.52) holds.

(ii) (8.53) follows immediately from (8.52) and (8.51). □

Proof of Proposition 8.2. By (2.11) and the proof of Theorem 2.2,

$$
|c_n(z)| \le \sum_{j=0}^{n-1} a_{n-j}^{-1} \, \frac{\tilde{w}_{n+1}}{\tilde{w}_{n+1-j}\tilde{w}_{n-j}} \, z^{2j} + z^{2n} \, \frac{\tilde{w}_{n+1}}{\tilde{w}_1}. \tag{8.57}
$$

Define

$$
A = \sup_n \left(|a_n|, |a_n|^{-1}\right) < \infty
$$

since $a_n \to 1$. Then, by (8.57),

$$
\begin{aligned}
\sup_n \left|c_n(z)u(z)^{-1}\right| \le A(1 - |z|)^{-1} \sup_{n,j} \left|\frac{\tilde{w}_{n+1}z^j}{\tilde{w}_{n+1-j}}\right| \sup_{n,j} \, \frac{1}{|\tilde{w}_{n-j}u|} \\
+ \sup_n \left|\frac{\tilde{w}_{n+1}z^n}{\tilde{w}_1}\right| |u|^{-1}.
\end{aligned} \tag{8.58}
$$

By (8.53), with ε chosen so $S \subset \{z \mid \arg z \in (\varepsilon, \pi - \varepsilon)\}$, we have

$$
\sup_{n,j} \left|\frac{\tilde{w}_{n+1}z^j}{w_{n+i-j}}\right| \le \frac{\pi A}{4} \, \varepsilon^{-1}(1 - |z|)^{-1}.
$$

By (8.50),

$$
\sup_{n,j} \left|\frac{1}{\tilde{w}_{n-j}u}\right| \le C(1 - |z|)^{-1/2}.
$$

Thus, (8.58) implies that

$$
\left|z^n p_n\left(z + \frac{1}{z}\right)(1 - z^2)u(z)^{-1}\right| \le C(1 - |z|)^{-5/2}
$$

where C depends on S and the constant K in (8.8). □

9. Bound states

One knows that with regard to Szegő asymptotics, sometimes simple-looking assumptions are really quite restrictive: for instance (see, e.g., [44, Chapter 13]), if $\operatorname{supp}(d\mu) \subset [-2, 2]$, then ($\beta$) of Theorem 5.1 implies all of (α)–(γ) and all the other hypotheses of that theorem. Also (see [46]), if (β) holds and f is given by (1.1), then

$$\sum_{j=1,\pm}^{N_\pm} \big(|E_j^\pm| - 2\big)^{1/2} < \infty \tag{9.1}$$

if and only if

$$\int_{-2}^{2} (\log f)(4 - x^2)^{-1/2}\, dx > -\infty. \tag{9.2}$$

Here we want to show that (β), (γ) alone do not imply spectral restrictions. In particular, we want to show that for each $q < \frac{3}{2}$, there is a Jacobi matrix obeying (α)–(γ) where

$$\sum_{j=1,\pm}^{N_\pm} \big(|E_j^\pm| - 2\big)^{q} = \infty. \tag{9.3}$$

Of course, by [26], (α) implies

$$\sum_{j=1,\pm}^{N_\pm} \big(|E_j^\pm| - 2\big)^{3/2} < \infty. \tag{9.4}$$

Our construction will have $a_n \equiv 1$ and b_n nonzero in blocks. In [44, Sect. 13.9], examples with b_n nonzero in a sequence of isolated points are constructed where (α)–(γ) hold and (9.3) holds for p arbitrarily close to 1. So this section improves that result. Our construction is closely related to that in Theorem 5.12 of [7].

Pick α in $(\frac{1}{2}, 1)$ and p so that

$$\frac{\alpha}{1-\alpha} > p > \frac{\alpha}{2-\alpha}. \tag{9.5}$$

We will eventually take α to $\frac{1}{2}$ and $p - \frac{\alpha}{2-\alpha} \to 0$. Pick M_0 and C_1 so for $m \geq M_0$, the distances between the blocks $B_m \equiv [m^{p+1} - C_1 m^p, m^{p+1} + C_1 m^p]$ for $m = M_0, M_0 + 1, \dots$ are each at least 2. This is easy to do if one fixes $C_1 < \frac{1}{2}(p+1)$. We should use $[C_1 m^p]$, but for notational simplicity, we will pretend that $C_1 m^p$ is an integer. We pick b_n by

$$b_n = \begin{cases} n^{-\alpha} & n \in B_{2k},\ 2k \geq M_0 \\ -n^{-\alpha} & n \in B_{2k+1},\ 2k+1 \geq M_0 \\ 0 & \text{otherwise.} \end{cases} \tag{9.6}$$

Lemma 9.1. *$a_n \equiv 1$ and b_n in* (9.6) *obey* (α)–(γ) *of Theorem 5.1.*

Proof. Since $|b_n| \leq n^{-\alpha}$ and $\alpha > \frac{1}{2}$, condition (α) holds and (β) is trivial. So we only need to check (γ). Since $\frac{d}{dn} n^{-\alpha} = -\alpha n^{-\alpha-1}$, if $n \in B_m$, then

$$\begin{aligned}|n^{-\alpha} - m^{-(p+1)\alpha}| &\leq Cm^{-(p+1)(\alpha+1)} m^p \\ &= Cm^{-[(p+1)\alpha+1]}.\end{aligned}$$

Thus,

$$\left| \sum_{n \in B_m} |b_n| - 2C_1 m^{-\alpha(p+1)+p} \right| \leq C_2 m^{-1} m^{-\alpha(p+1)+p}. \tag{9.7}$$

We claim

$$\alpha(p+1) > p \tag{9.8}$$

which implies, first, that the estimate on the right of (9.7) is absolutely summable and, second, that $\sum_m (-1)^m m^{-\alpha(p+1)+p}$ is conditionally summable, proving (γ).

To prove (9.8), note that it is equivalent to $\alpha > p(1-\alpha)$ or $p < \frac{\alpha}{1-\alpha}$, which is true by (9.5). □

For m even, we will pick φ_m to be the trial vector supported in B_m, which is 1 at the center of B_m (i.e., at $n = m^{p+1}$), 0 at the end points, and constant slope in between. For m odd, we do the same construction and then multiply by $(-1)^n$.

Consider m even first. Since $a_n \equiv 1$,

$$\langle \varphi_m, (J_0 - 2)\varphi_m \rangle = -\sum_n |\varphi_m(n+1) - \varphi_m(n)|^2 \tag{9.9}$$

$$\geq -m^p \left[\frac{C}{m^p} \right]^2$$

$$= -C^2 m^{-p} \tag{9.10}$$

since the slope $\sim m^{-p}$ and there are $O(m^p)$ nonzero terms in the sum (9.9). On the other hand, since $b_n > C_3 m^{-\alpha(p+1)}$ on B_m and, on average, $|\varphi_m|^2 \geq \frac{1}{4}$ on B_m,

$$\langle \varphi_m, b\varphi_m \rangle \geq C_3 m^p m^{-\alpha(p+1)}. \tag{9.11}$$

It is easy to see that $p > \frac{\alpha}{2-\alpha}$ is equivalent to $2p > \alpha(p+1)$, so for $m \geq M_1$ for some M_1,

$$\langle \varphi_m (J_0 + b - 2)\varphi_m \rangle \geq \tfrac{1}{2} C_3 m^p m^{-\alpha(p+1)}. \tag{9.12}$$

Since $\|\varphi_m\|^2 \le C_4 m^p$, we see that

$$\frac{\langle \varphi_m, (J_0 + b - 2)\varphi_m \rangle}{\|\varphi_m\|^2} \ge C_4 m^{-\alpha(p+1)} \tag{9.13}$$

for m even. Similarly, for m odd,

$$\frac{\langle \varphi_m, (J_0 + 2 + b)\varphi_m \rangle}{\|\varphi_m\|^2} \le -C_4 m^{-\alpha(p+1)}. \tag{9.14}$$

Since $\langle \varphi_m, \varphi_k \rangle = 0 = \langle \varphi_m, (J_0 + b)\varphi_k \rangle$ for $m \neq k$, a variational argument proves for m large,

$$\big| |E_m^{\pm}| - 2 \big| \ge \tfrac{1}{2}\, C_4 m^{-\alpha(p+1)}. \tag{9.15}$$

Thus, (9.3) holds if $q\alpha(p+1) < 1$. Taking $\alpha \downarrow \frac{1}{2}$, $p \downarrow \frac{1}{3}$, we see $q \uparrow ((\frac{1}{2})(\frac{4}{3}))^{-1} = \frac{3}{2}$. Thus,

Theorem 9.2. *For any $q < \frac{3}{2}$, there is a set of Jacobi parameters for which* (α)–(γ) *of Theorem 5.1 hold, but for which* (9.3) *also holds.*

10. A remark on Schrödinger operators

In this section, we want to show how the ideas of Sect. 6 provide a simple proof of

Theorem 10.1. *Suppose $V \in L^2(0, \infty)$ and*

$$\lim_{x\to\infty} \int_0^x V(y)\, dy \tag{10.1}$$

exists. Then for any κ with $\kappa > 0$, there is a solution of

$$-u'' + Vu = -\kappa^2 u \tag{10.2}$$

so that

$$\lim_{x\to\infty} e^{\kappa x} u(x) = 1. \tag{10.3}$$

This result is not new. It was proven by Hartman-Winter [20] using sophisticated ODE asymptotic methods. Even with the simplification of Harris-Lutz [19], the proof is involved (see Eastham [14] for a particularly clear discussion of this proof). Here, as in Sect. 6, we will use renormalized determinants to construct u. The same argument shows that if (10.1) does not have a finite limit, then there is a solution so $u(x)/\exp[f(x)] \to 1$, where

$$f(x) = -\kappa x + \frac{1}{2\kappa} \int_0^x V(y)\, dy \tag{10.4}$$

also a result of Hartman-Wintner [20].

In the argument below, we will use unfactorized kernels (i.e., VG_0) rather than factorized kernels (i.e., $V^{1/2}G_0|V|^{1/2}$). By using factorized kernels, one should be able to extend this theorem to the case where $V \in L^2$ is replaced by $\sum_n (\int_n^{n+1} |V(x)|\, dx)^2 < \infty$.

The starting point is a formula of Jost-Pais [23] for the Jost function extended to get the Jost solutions.

Proposition 10.2. *Let $G_0(-\kappa^2)$ be the operator $(H_0+\kappa^2)^{-1}$ where H_0 is $-\frac{d^2}{dx^2}$ with $u(0)$ boundary conditions, so G_0 has integral kernel*

$$G_0(x,y;-\kappa^2) = (2\kappa)^{-1}[e^{-\kappa|x-y|} - e^{-\kappa(x+y)}]. \tag{10.5}$$

For any $V \in L^2$ of compact support and any $x_0 > 0$, let $K(x_0;\kappa)$ be the operator with integral kernel

$$K(x,y;x_0;\kappa) = V(x+x_0)G_0(x,y;-\kappa^2). \tag{10.6}$$

Then K is trace class and

$$u(x_0) \equiv e^{-\kappa x_0} \det(1+K(x_0;\kappa)) \tag{10.7}$$

obeys (10.2)*, and for x large,*

$$u(x) = e^{-\kappa x}. \tag{10.8}$$

Remark. (10.8) is trivial since V has compact support, which means $K \equiv 0$ for x_0 large.

Proof. This is essentially Proposition 2.9 of [41]. That paper uses a factorized kernel, but by the Birman-Solomyak theorem (see [45, Chap. 4], K is trace class, and so the determinants are equal. As noted, (10.8) is immediate. □

Proposition 10.3. *If $V \in L^2$ and* (10.1) *holds, then $K(x_0;\kappa)$ is Hilbert-Schmidt and*

$$u(x_0) = e^{-\kappa x_0}\det{}_2(1+K(x_0;\kappa)) \exp\biggl((2\kappa)^{-1}\int_{x_0}^{\infty} V(y)[1-e^{-2\kappa y}]\, dy\biggr). \tag{10.9}$$

Moreover, u obeys (10.2).

Proof. If V has compact support, (10.9) is just (10.7) since $\mathrm{Tr}(K(x_0;\kappa)) = \int_{x_0}^{\infty} V(y)(2\kappa)^{-1}[1-e^{-2\kappa y}]\, dy$ and we have (6.11). Given general V, let $V_L(x)$ be given by

$$V_L(x) = \begin{cases} V(x) & x \le L \\ 0 & x > L \end{cases}$$

and u_L given by (10.1). Since $V \in L^2$, $K_L(x_0,\kappa) \to K(x_0;\kappa)$ in Hilbert-Schmidt norm, so $\det_2$ converges. By (10.1), the exponentials converge. Thus, $u_L \to u$. This means u is a distributional solution of (10.2) and so, by elliptic regularity, a solution L^2 at infinity. □

Proof of Theorem 10.1. $K(x_0;\kappa) \to 0$ in Hilbert-Schmidt norm as $x_0 \to \infty$, so $\det_2(1 + K(x_0;\kappa)) \to 1$. The integral goes to 0 as $x_0 \to \infty$. Thus, $u(x_0)e^{\kappa x_0} \to 1$. □

The point here is that it is natural to try to construct u as a limit of u_L's, and then prove asymptotics of u. The fact that we have an explicit formula in terms of renormalized determinants allows us to control both the limit as $L \to \infty$ and then as $x \to \infty$.

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