Commun. math. Phys. 29, 233—247 (1973)

The Vacuum Energy for $P(\phi)_2$: Infinite Volume Limit and Coupling Constant Dependence*

F. Guerra**, L. Rosen, and B. Simon*

Joseph Henry Laboratory of Physics, Princeton University, Princeton, New Jersey, USA

Received September 29, 1972

Abstract. Let $E(\lambda g)$ be the vacuum energy for the $P(\phi)_2$ Hamiltonian with space cutoff $g(x) \geqq 0$ and coupling constant $\lambda \geqq 0$. For suitable families of cutoffs $g \to 1$, the vacuum energy per unit volume converges; i.e., $-E(\lambda g)/\int g(x)\,dx \to \alpha_\infty(\lambda)$. We obtain bounds on the λ dependence of $\alpha_\infty(\lambda)$ for large and small λ. These lead to estimates for $E(\lambda g)$ as a functional of g that permit a weakening of the standard regularity conditions for g. Typical of such estimates is the "linear lower bound", $-E(g) \leqq \text{const} \int g(x)^2\,dx$, valid for all $g \geqq 0$ provided that P is normalized so that $P(0)=0$. Finally we show that the perturbation series for $\alpha_\infty(\lambda)$ is asymptotic to second order.

Section 0: Introduction

This paper is a continuation of our previous investigations [6, 7] on the infinite volume behavior of the vacuum in $P(\phi)_2$. We are mainly concerned with $E(\lambda g)$, the ground state energy of the $P(\phi)_2$ Hamiltonian,

$$H(g) = H_0 + \lambda \int g(x) : P(\phi(x)) : dx\,. \tag{0.1}$$

The polynomial P is semibounded and normalized, i.e.

$$P(X) = \sum_{r=1}^{2n} b_r X^r,\ b_{2n} > 0\,,$$

the coupling constant $\lambda \geqq 0$, and the cutoff $g(x) \geqq 0$. In [6, 7] we restricted our attention to sharp space cutoffs: $g = \chi_\ell$, the characteristic function of the interval $[-\ell/2, \ell/2]$. In particular it was shown that the vacuum energy per unit volume, $\alpha_\ell = -E(\chi_\ell)/\ell$, converges to a finite constant $\alpha_\infty > 0$ as $\ell \to \infty$ [6] and that this convergence is monotone [7].

In the present paper we extend the results of [6] and [7] to more general cutoffs g. In Section 2, for example, we extend the convergence of

---

* Research partially supported by AFOSR under Contract No. F 44620-71-C-0108.

** Postal address after September 30, 1972: via A. Falcone 70, 80127, Napoli, Italy.

* A. Sloan Foundation Fellow.

the energy per unit volume to a class of non-sharp cutoffs. In Section 1 we obtain bounds on $E(\lambda g)$ for both small and large values of λ. These lead to improved estimates for $E(g)$ as a functional of g and permit a weakening of the standard regularity conditions on g (this is outlined in the Appendix).

Finally, in Section 3, we prove that the perturbation series for the *infinite volume* quantity α_∞ is asymptotic to second order.

We remind the reader that Nelson's symmetry states that

$$\langle \Omega_0, e^{-tH_\ell}\Omega_0\rangle = \langle \Omega_0, e^{-\ell H_t}\Omega_0\rangle \tag{0.2}$$

where $H_\ell = H(\chi_\ell)$ and Ω_0 is the free vacuum.

Section 1: Coupling Constant Dependence of the Energy

For the theory with Hamiltonian (0.1) we introduce the notation $\alpha_\ell(\lambda)$ or $\alpha_\ell(\lambda P)$ for $-E(\lambda\chi_\ell)/\ell$. Our goal in this section is to obtain estimates for the λ dependence of $\alpha_\infty(\lambda) = \lim_{\ell\to\infty} \alpha_\ell(\lambda)$. We begin by investigating the λ dependence of $E(\lambda g)$. Throughout this section we make the standard assumption that $g \in L^1 \cap L^2$, $g \geqq 0$.

According to the NGS proof of semiboundedness [8, 1, 16, 5]

$$-E(\lambda g) \leqq \frac{m}{2} \log \|e^{-2\lambda V/m}\|_1 \tag{1.1}$$

where $V = \int g(x) : P(\phi(x)) : dx$. (Here we have used Nelson's best possible hypercontractive estimates [10] to determine the constants in (1.1).) The NGS proof of the finiteness of the L^1 norm on the right of (1.1) consists of showing that the region in Q space where V is large negative has small measure. Explicitly for large K [5, p. 21],

$$\mu\{q \in Q \mid V \leqq -b(\log K)^n\} \leqq e^{-cK^\alpha}, \tag{1.2}$$

where $2n$ is the degree of P, $0 < \alpha < 1/2n$, and b and c are positive constants that depend on g. The estimate (1.2) clearly implies that $e^{-V} \in L^p$ for all $p < \infty$.

It is interesting that the estimates (1.1) and (1.2) display the correct dependence of $E(\lambda g)$ on λ but not on g. For instance we know [3] that $E(\lambda g)$ grows linearly in $L = |supp\, g|$, whereas from (1.1, 1.2) alone we can deduce at best an $L(\log L)^n$ behavior. On the other hand the fact that V is not bounded below implies at once that $E(\lambda g)$ is not linearly bounded in λ for large λ [18, p. 175]. We expect therefore that the following lemma provides the correct λ dependence of $E(\lambda g)$.

Lemma 1.1. *For large* λ,

$$\|e^{-\lambda V}\|_1 = e^{O(\lambda(\log\lambda)^n)}. \tag{1.3}$$

Proof. By (1.2) it is sufficient to show that

$$\int_1^\infty \exp[b\lambda(\log K)^n - cK^\alpha]\, dK = e^{O(\lambda(\log\lambda)^n)}.$$

After the change of variables $K = \lambda^\beta x$, where $\beta > \alpha^{-1}$, the integral becomes

$$\lambda^\beta \int_{\lambda^{-\beta}}^\infty \exp[b\lambda(\beta\log\lambda + \log x)^n - c\lambda^{\alpha\beta}x^\alpha]\, dx.$$

The integral from $\lambda^{-\beta}$ to 1 can be estimated by

$$\lambda^\beta \int_{\lambda^{-\beta}}^1 \exp[b\lambda(\beta\log\lambda)^n]\, dx = e^{O(\lambda(\log\lambda)^n)}$$

whereas the integral from 1 to ∞ is

$$O\left(\lambda^\beta \int_1^\infty \exp(-\mathrm{const}\,\lambda^{\alpha\beta}x^\alpha)\, dx\right) = O(e^{(1-\alpha)\beta\log\lambda}). \quad \blacksquare$$

Corollary 1.2. *There is a constant c (dependent on g) such that for large* λ,

$$-E(\lambda g) \leqq c\lambda(\log\lambda)^n. \tag{1.4}$$

As for the small λ behavior, the observation that the first order term in the perturbation series for $E(\lambda g)$ vanishes leads to this estimate:

Lemma 1.3. *There is a constant a (dependent on g) such that for small* λ

$$-E(\lambda g) \leqq a\lambda^2. \tag{1.5}$$

Remark. This estimate is an immediate consequence of the fact that the perturbation series for $E(\lambda g)$ is asymptotic in λ [18, 15] but we give the more elementary argument below. The asymptotic nature of the perturbation series (which is non-zero in second order) does show that (1.5) is the best possible estimate for small λ.

Proof. Since V and e^{-V} are in L^q for all $q < \infty$, it follows from Hölder's inequality that $f(\lambda) = \|e^{-\lambda V}\|_1 = \langle \Omega_0, e^{-\lambda V}\Omega_0\rangle$ is analytic in λ in the right half plane and $f^{(n)}(\lambda) = \langle \Omega_0, (-V)^n e^{-\lambda V}\Omega_0\rangle$. In particular $f'(0) = 0$ and, writing $f(\lambda) = 1 + \lambda^2 f^{(2)}(\xi)/2, 0 < \xi < \lambda$, we conclude that, for

$d = \sup_{0<\xi<1} f^{(2)}(\xi)/2$, $f(\lambda) \leqq 1 + d\lambda^2$ when $\lambda < 1$. Therefore by (1.1)

$$\begin{aligned} -E(\lambda g) &\leqq \frac{m}{2} \log \|e^{-2\lambda V/m}\|_1 \\ &\leqq \frac{m}{2} \log(1 + d(2\lambda/m)^2) \\ &\leqq 2d\lambda^2/m \end{aligned}$$

for small λ. ∎

These estimates for $E(\lambda g)$ extend at once to $\alpha_\infty(\lambda)$:

Theorem 1.4. *For small* λ, $\alpha_\infty(\lambda) = O(\lambda^2)$; *for large* λ, $\alpha_\infty(\lambda) = O(\lambda(\log \lambda)^n)$.

Proof. We know that for some $T > 0$ [7, p. 16]

$$\alpha_\ell(\lambda) \leqq -E(4\lambda \chi_T)/4T.$$

(For an improved version of this result see Theorem 3.3 below.) Therefore by (1.4) and (1.5) $\alpha_\ell(\lambda)$ is bounded as claimed with bounds independent of ℓ. Accordingly these bounds hold for α_∞. ∎

Remark. In Section 3 we prove that the limit $\lim_{\lambda \to 0} \alpha_\infty(\lambda)/\lambda^2$ exists and takes precisely the (non-zero) value given by perturbation theory. As for the question of whether our large λ estimate can be improved, we note that if the estimate (1.4) for $E(\lambda g)$ is best possible, then by monotonicity ($\alpha_\infty(\lambda) \geqq -E(\lambda \chi_\ell)/\ell$) so is the estimate for $\alpha_\infty(\lambda)$.

Using Theorem 1.4 we can obtain bounds on $E(g)$ as a functional of g. But before doing so we note that $E(g)$ is concave in g:

Lemma 1.5. *For* $0 \leqq \lambda \leqq 1$

$$E(\lambda g_1 + (1-\lambda) g_2) \geqq \lambda E(g_1) + (1-\lambda) E(g_2).$$

Proof. Let $g = \lambda g_1 + (1-\lambda) g_2$ and let Ω_g be the vacuum vector for $H(g)$. Then

$$\begin{aligned} E(g) &= (\Omega_g, H(g)\Omega_g) \\ &= \lambda(\Omega_g, H(g_1)\Omega_g) + (1-\lambda)(\Omega_g, H(g_2)\Omega_g) \\ &\geqq \lambda E(g_1) + (1-\lambda) E(g_2). \end{aligned}$$

Corollary 1.6. $\alpha_\ell(\lambda)$ *and* $\alpha_\infty(\lambda)$ *are convex and hence continuous functions of* λ. *In particular, if* $\mu \leqq \lambda$, *then*

$$\alpha_\infty(\mu) \leqq \mu \alpha_\infty(\lambda)/\lambda.$$

The reason that bounds on α_∞ lead to bounds on $E(g)$ is this:

Theorem 1.7. *For* $g \in L^1 \cap L^2$, $g \geqq 0$,

$$-E(g) \leqq \int \alpha_\infty(g(x))\,dx\,. \tag{1.6}$$

Proof. We first prove the lemma for step functions $g = \sum_{r=1}^{N} \gamma_r \chi_{(a_{r-1}, a_r]}$ where $\chi_{(a,b]}$ is the characteristic function of the interval $(a,b]$, $\gamma_r \geqq 0$, $-\infty < a_0 < a_1 < \cdots < a_N < \infty$. By Nelson's symmetry (0.2) and Eq. (8) of [7]:

$$-E(g) = \lim_{T\to\infty} \frac{1}{T} \log\left\langle \Omega_0, \prod_{r=1}^{N} e^{-(a_r - a_{r-1})H_r}\Omega_0\right\rangle$$

where $H_r = H(\gamma_r \chi_T)$.

But $\|e^{-aH_r}\| = e^{-aE(\gamma_r\chi_T)} = e^{aT\alpha_T(\gamma_r)} \leqq e^{aT\alpha_\infty(\gamma_r)}$. Therefore

$$\begin{aligned}-E(g) &\leqq \lim_{T\to\infty} \frac{1}{T} \log \prod_{r=1}^{N} e^{(a_r - a_{r-1})T\alpha_\infty(\gamma_r)}\\ &= \int \alpha_\infty(g(x))\,dx\,.\end{aligned}$$

Now consider a general function $g \in L^1 \cap L^2$, $g \geqq 0$. We can certainly approximate g by non-negative step functions g_j as above such that $g_j \xrightarrow[L^2]{} g$ and the norms $\|g_j\|_{L^1}$ are uniformly bounded. By a standard argument using Duhamel's formula [18] it follows that $e^{-H(g_j)} \to e^{-H(g)}$ in norm and hence $E(g_j) \to E(g)$. On the other hand, since $\alpha_\infty(\lambda)$ is continuous in λ, $\alpha_\infty(g_j(x)) \to \alpha_\infty(g(x))$ a.e. in x. Therefore by the bounds of Theorem 1.4 and by the Lebesgue dominated convergence and monotone convergence theorems, (1.6) extends to general g. ∎

Corollary 1.8. *If* $0 \leqq g(x) \leqq 1$,

$$-E(g) \leqq \alpha_\infty \|g\|_1\,.$$

Proof. Follows from the theorem and Corollary 1.6. ∎

Corollary 1.9. *Let* $\varepsilon > 0$. *There are constants* a, b *and* c_ε *such that*

$$-E(g) \leqq a\left[\int_{g(x)\leqq 2} g(x)^2\,dx + \int_{g(x)>2} g(x)\,(\log g(x))^n\,dx\right], \tag{1.7}$$

$$-E(g) \leqq b \int g(x)^2\,dx\,, \tag{1.8}$$

and

$$-E(g) \leqq c_\varepsilon \int g(x)^{1+\varepsilon}\,dx\,. \tag{1.9}$$

Proof. (1.8) and (1.9) are clearly consequences of (1.7) which itself follows directly from the theorem and Theorem 1.4. ∎

The inequality (1.7) is highly suggestive. It indicates that the condition $g \in L^1 \cap L^2$ is unnecessarily strong in order that $H(g)$ be well-defined as a self-adjoint semi-bounded operator. It should be sufficient to require that the right side of (1.7) is finite. This line of thought is pursued in the Appendix.

We remark that similar arguments permit a sharpening of the proof [7, Theorem 2] of the Glimm-Jaffe estimate on local perturbations of the Hamiltonian [4] in the sense that the constant c can be determined more precisely:

Lemma 1.10. *Let* $W = \int g(x) : Q(\phi(x)) : dx$ *where the polynomial* $P + Q$ *is semibounded and* $0 \leqq g \leqq 1$, $\operatorname{supp} g \subset [-a, a] \subset [-\ell/2, \ell/2]$. *Then*

$$-W \leqq H(\chi_\ell) - E(\chi_\ell) + c$$

where $c = \|g\|_1 \cdot [\alpha_\infty(P+Q) - \alpha_\infty(P)] + 2a[\alpha_\infty(P) - \alpha_\ell(P)]$.

Since the second term in the definition of c goes to zero when $\ell \to \infty$ we obtain this estimate on the physical Hilbert space:

Corollary 1.11. *Let* $W = \int g(x) : Q(\phi(x)) : dx$ *where* $\deg Q < \deg P$, $g \in L^1$, *and* $|g| \leqq 1$. *Then for any* $\varepsilon > 0$, *there is a constant* d *independent of* g *such that*

$$\pm W < \varepsilon H_{\text{ren}} + d \, \|g\|_1 \, .$$

When Q is semibounded, we can also drop the restriction $g \leqq 1$ in Lemma 1.10 and obtain bounds of the form (1.7).

Section 2: Convergence of the Vacuum Energy per Unit Volume

In [6] it was proved that for the $P(\phi)_2$ theory with a *sharp* space cutoff the energy per unit volume converged as the cutoff went to infinity. Here we extend this result to a more general class of space cutoffs. Consider an increasing sequence of intervals (not necessarily of the form $[-n/2, n/2]$ as in the previous section) with characteristic functions χ_n such that the lengths $\|\chi_n\|_1 \to \infty$. We shall prove:

Theorem 2.1. *Let* $\{g_n\}$ *be a sequence of functions such that*

$$0 \leqq g_n \leqq c\chi_n \tag{2.1}$$

for some constant c, *and, as* $n \to \infty$,

$$\|g_n - \chi_n\|_2 / \|\chi_n\|_2 \to 0 \, . \tag{2.2}$$

Then $\lim_{n\to\infty} E(g_n)/\|g_n\|_2^2 = -\alpha_\infty$.

Remarks 1. For g_n satisfying (2.1), the hypothesis (2.2) may be equivalently written as $\|g_n - \chi_n\|_1 / \|\chi_n\|_1 \to 0$; moreover the volume factors $\|g_n\|_2^2$ in the conclusion may be replaced by $\|g_n\|_1$ or $\|\chi_n\|_1 = \|\chi_n\|_2^2$, since $\|g_n - \chi_n\|_p / \|\chi_n\|_p \to 0$ implies $\|g_n\|_p / \|\chi_n\|_p \to 1$.

2. In particular, conditions (2.1) and (2.2) hold for a sequence going to 1 in the sense of Osterwalder-Schrader [11].

3. It would be interesting to prove convergence for some class of cutoff functions which are not necessarily of compact support or uniformly bounded. If the monotonicity of $E(g)$ in g were known, such a result would follow easily from the methods below.

The basic idea in the proof of the theorem is to exploit the known convergence when $g_n = \chi_n$ and to estimate the error arising from the "discrepancy" $(g_n - \chi_n)$ by means of the concavity of the energy. We prove a series of simple lemmas which serve to establish the theorem first when g_n satisfies (2.1) with $c=1$, then when $\chi_n \leqq g_n \leqq c\chi_n$, and finally when g_n satisfies (2.1) itself.

Lemma 2.2. *If* $\|f_n\|_2^2 / c_n \to 0$, *then* $E(f_n)/c_n \to 0$.

Proof. By the estimate (1.8), $-a\|f_n\|_2^2 \leqq E(f_n) \leqq 0$, and the lemma follows immediately. ∎

We denote the maximum (resp. minimum) of two functions f and g by $f \vee g$ (resp. $f \wedge g$).

Lemma 2.3. *Take* $0 \leqq \lambda < 1$. *If* $g_n \geqq 0$ *satisfies* (2.2), *then*

$$\overline{\lim}\, \lambda E(g_n)/\|g_n\|_2^2 \leqq \overline{\lim}\, E(\lambda(g_n \vee \chi_n))/\|\chi_n\|_2^2 \tag{2.3}$$

and

$$\underline{\lim}\, E(\lambda g_n)/\|g_n\|_2^2 \geqq \underline{\lim}\, \lambda E(g_n \wedge \chi_n)/\|\chi_n\|_2^2 . \tag{2.4}$$

Proof. To prove (2.3) we write $\lambda(g_n \vee \chi_n) = \lambda g_n + (1-\lambda) f_n$ where $f_n = \lambda(g_n \vee \chi_n - g_n)/(1-\lambda) \geqq 0$. By the concavity of the energy (Lemma 1.5),

$$E(\lambda(g_n \vee \chi_n)) \geqq \lambda E(g_n) + (1-\lambda)\, E(f_n) .$$

Since by (2.2) $\|f_n\|_2^2 / \|\chi_n\|_2^2 \to 0$ we obtain (2.3) by dividing by $\|\chi_n\|_2^2$ and applying Lemma 2.2. The decomposition $\lambda g_n \equiv \lambda(g_n \wedge \chi_n) + (1-\lambda) h_n$ similarly yields (2.4). ∎

Lemma 2.4. *If* $0 \leqq g_n \leqq \chi_n$ *and* g_n *satisfies* (2.2), *then*

$$\lim E(g_n)/\|g_n\|_2^2 = -\alpha_\infty .$$

Proof. By Corollary 1.8 and Remark 1 above, $\underline{\lim}\, E(g_n)/\|g_n\|_2^2 \geqq -\alpha_\infty$. On the other hand, by (2.3),

$$\overline{\lim}\, \lambda E(g_n)/\|g_n\|_2^2 \leqq \overline{\lim}\, E(\lambda \chi_n)/\|\chi_n\|_2^2 = -\alpha_\infty(\lambda) \tag{2.5}$$

by the known convergence for sharp cutoffs. Since $\alpha_\infty(\lambda)$ is continuous in λ (Corollary 1.6), we may put $\lambda=1$ in (2.5) to conclude the proof. ∎

Corollary 2.5. *If* $g_n \geqq 0$ *satisfies* (2.2), *then* $\underline{\lim}\, E(g_n)/\|g_n\|_2^2 \geqq -\alpha_\infty$.

Proof. By the lemma and the estimate (2.4)

$$\underline{\lim}\, E(\lambda g_n)/\|g_n\|_2^2 \geqq -\lambda\alpha_\infty\,,$$

for any $0 \leqq \lambda < 1$. But the left side, as a $\underline{\lim}$ of concave functions in λ, is concave and hence continuous in λ so that we may take $\lambda=1$. ∎

Lemma 2.6. *If* $\chi_n \leqq g_n \leqq c\chi_n$ *and* g_n *satisfies* (2.2), *then*

$$\lim E(g_n)/\|g_n\|_2^2 = -\alpha_\infty\,.$$

Proof. By the previous corollary it remains only to show that $\overline{\lim}\, E(g_n)/\|g_n\|_2^2 \leqq -\alpha_\infty$. We write $\chi_n = c^{-1} g_n + (1-c^{-1})\tilde{g}_n$ where $\tilde{g}_n = (c\chi_n - g_n)/(c-1)$. It is easy to see that $\tilde{g}_n$ satisfies the hypotheses of Lemma 2.4 so that by concavity

$$\begin{aligned}
-\alpha_\infty &= \lim E(\chi_n)/\|\chi_n\|_2^2 \\
&\geqq c^{-1}\,\overline{\lim}\, E(g_n)/\|g_n\|_2^2 + (1-c^{-1}) \lim E(\tilde{g}_n)/\|\tilde{g}_n\|_2^2 \\
&= c^{-1}\,\overline{\lim}\, E(g_n)/\|g_n\|_2^2 - (1-c^{-1})\alpha_\infty\,,
\end{aligned}$$

and the lemma follows. ∎

Corollary 2.7. *If* g_n *satisfies* (2.1) *and* (2.2), *then* $\overline{\lim}\, E(g_n)/\|g_n\|_2^2 \leqq -\alpha_\infty$.

Proof. Use (2.3), Lemma 2.6, and the continuity of $\alpha_\infty(\lambda)$. ∎

Proof of Theorem 2.1. The theorem is a consequence of Corollaries 2.5 and 2.7. ∎

Section 3: Asymptotic Nature of Second Order Perturbation Theory for $\alpha_\infty(\lambda)$

For a normalized interaction polynomial P, the first non-trivial term of the formal perturbation expansion in λ for $\alpha_\ell(\lambda P) = -E(\lambda\chi_\ell)/\ell$ is the second order term $\lambda^2 a_\ell^{(2)}(P)$. Since the series for $E(\lambda\chi_\ell)$ is known [18, 15] to be asymptotic we have

$$\lim_{\lambda\downarrow 0} \frac{\alpha_\ell(\lambda P)}{\lambda^2} = a_\ell^{(2)}(P)\,. \tag{3.1}$$

An explicit computation (see [7], § 6) shows that $a_\ell^{(2)}$ is an increasing function of ℓ which converges as $\ell\to\infty$. Defining $a_\infty^{(2)} = a_\infty^{(2)}(P) = \lim_{\ell\to\infty} a_\ell^{(2)}(P)$, we have the following theorem:

Theorem 3.1. *The perturbation series for* $\alpha_\infty(\lambda)$ *is asymptotic at least up to the second order; i.e.,*

$$\lim_{\lambda \downarrow 0} \alpha_\infty(\lambda P)/\lambda^2 = a_\infty^{(2)}(P). \tag{3.2}$$

In order to prove this result we need a stronger version of the hypercontractive bounds of Section 4 of [7]. First we note the estimate:

Lemma 3.2. *If the bounded self-adjoint operator* A *is positivity-preserving, then for all* $0 \leqq \lambda \leqq 1$,

$$\|A\|_{\frac{2}{\lambda}, \frac{2}{1+\lambda}} \leqq \langle \Omega_0, A^2 \Omega_0 \rangle^{1/2}. \tag{3.3}$$

Remark. We recall that $\|A\|_{p,q}$ is the norm of A as a map from L^p to L^q.

Proof. We have $\|Af\|_1 \leqq \|A|f|\|_1 = \langle \Omega_0, A|f|\rangle = \langle A\Omega_0, |f|\rangle \leqq \|A\Omega_0\|_2 \|f\|_2$ so that $\|A\|_{2,1} \leqq \langle \Omega_0, A^2\Omega_0\rangle^{1/2}$. By duality $\|A\|_{\infty,2} = \|A\|_{2,1}$, and interpolation yields (3.3). ∎

Then we can establish the following general theorem concerning the perturbation of hypercontractive semigroups e^{-tH_0}. We take the perturbation V in the standard class [18]

$$\mathscr{W} = \left\{ V \,|\, V \text{ real; } V \in L^p \text{ for some } p > 2; \; e^{-V} \in \bigcap_{q < \infty} L^q \right\}$$

so that $H_0 + V$ is essentially self-adjoint on $D(H_0) \cap D(V)$.

Theorem 3.3. *Let* e^{-tH_0} *be a positivity-preserving hypercontractive semigroup with* $\|e^{-tH_0}\|_{2,4} \leqq 1$ *for* $t \geqq T$. *Then for every* $\alpha > 1$ *and* $V \in \mathscr{W}$,

$$\|e^{-t(H_0+V)}\|_{2,2} \leqq \langle \Omega_0, e^{-t(H_0+\alpha V)} \Omega_0 \rangle^{1/\alpha} \tag{3.4}$$

provided that $t \geqq 4T/(\alpha - 1)$.

Proof. By the Spectral Theorem

$$\begin{aligned}
\|e^{-t(H_0+V)}\|_{2,2}^{\alpha/2} &= \left\| e^{-\frac{\alpha t}{2}(H_0+V)} \right\|_{2,2} \\
&= \left\| e^{-\frac{\alpha t}{2}[(1-\alpha^{-1})H_0 + \alpha^{-1}H_0 + V]} \right\|_{2,2} \\
&\leqq \left\| e^{-\frac{t}{4}(\alpha-1)H_0} e^{-\frac{\alpha t}{2}(\alpha^{-1}H_0+V)} e^{-\frac{t}{4}(\alpha-1)H_0} \right\|_{2,2}
\end{aligned}$$

since by a lemma of Segal [16] (see also [18]),

$$\|e^{-(A+B)}\| \leqq \|e^{-A/2} e^{-B} e^{-A/2}\|$$

for any semibounded operators A and B with $A + B$ essentially self-adjoint on $D(A) \cap D(B)$. Consequently,

$$\|e^{-t(H_0+V)}\|_{2,2}^{\alpha/2} \leqq \left\| e^{-\frac{t}{4}(\alpha-1)H_0} \right\|_{2,4} \left\| e^{-\frac{t}{2}(H_0+\alpha V)} \right\|_{4,\frac{4}{3}} \left\| e^{-\frac{t}{4}(\alpha-1)H_0} \right\|_{\frac{4}{3},2}.$$

Since $\left\| e^{-\frac{t}{4}(\alpha-1)H_0} \right\|_{2,4} = \left\| e^{-\frac{t}{4}(\alpha-1)H_0} \right\|_{\frac{4}{3},2} \leqq 1$ for $t \geqq 4T/(\alpha-1)$, the theorem follows from the estimate (3.3) with $A = e^{-\frac{t}{2}(H_0+\alpha V)}$ and $\lambda = 1/2$. ∎

Proof of Theorem 3.1. By monotonicity in ℓ [7] we have

$$\alpha_\ell(\lambda)/\lambda^2 \leqq \alpha_\infty(\lambda)/\lambda^2 . \tag{3.5}$$

Therefore, taking first the limit $\lambda \to 0$ and then $\ell \to \infty$, we conclude by (3.1) that

$$\varliminf_{\lambda \to 0} \alpha_\infty(\lambda)/\lambda^2 \geqq a_\infty^{(2)} . \tag{3.6}$$

On the other hand, for arbitrary $\varepsilon > 0$, we have by Theorem 3.3 with $\alpha = 1 + \varepsilon$, $V = \lambda V_\ell \equiv \lambda \int_{-\ell/2}^{\ell/2} : P(\phi(x)) : dx$, and $t \geqq 4T/\varepsilon$,

$$\begin{aligned} e^{-tE(\lambda\chi_\ell)} &= \| e^{-t(H_0 + \lambda V_\ell)} \|_{2,2} \\ &\leqq \langle \Omega_0, e^{-t[H_0 + (1+\varepsilon)\lambda V_\ell]} \Omega_0 \rangle^{1/(1+\varepsilon)} \\ &= \langle \Omega_0, e^{-\ell[H_0 + (1+\varepsilon)\lambda V_t]} \Omega_0 \rangle^{1/(1+\varepsilon)} \\ &\leqq e^{-\ell E((1+\varepsilon)\lambda\chi_t)/(1+\varepsilon)} \end{aligned}$$

by Nelson's symmetry (0.2). Therefore,

$$-\frac{E(\lambda\chi_\ell)}{\ell} \leqq -\frac{E((1+\varepsilon)\lambda\chi_t)}{t(1+\varepsilon)}$$

so that dividing by λ^2 and taking $\ell \to \infty$

$$\alpha_\infty(\lambda)/\lambda^2 \leqq \alpha_t((1+\varepsilon)\lambda)/(1+\varepsilon)\lambda^2 .$$

Taking first $\lambda \to 0$ and then $t \to \infty$ we conclude by (3.1) that

$$\varlimsup_{\lambda \to 0} \alpha_\infty(\lambda)/\lambda^2 \leqq (1+\varepsilon) a_\infty^{(2)} . \tag{3.7}$$

Since this relation is valid for any $\varepsilon > 0$ we see that (3.6) and (3.7) establish the theorem. ∎

Appendix: Minimal Conditions on the Space Cutoff

Since the main purpose for studying the spatially cutoff Hamiltonian $H(g)$ is to obtain information about the infinite volume limit $g \to 1$, the only relevant conditions on g are those required by this purpose. For instance the smoothness condition $g \in C_0^\infty$ is convenient for methods involving or yielding "higher order estimates" (see the original self-adjointness proof of Glimm and Jaffe [2] or [13]); in our work [6, 7] the sharp cutoff $g=\chi_\ell$ is useful. Both of these cases are covered by the standard regularity condition $g\in L^1\cap L^2$ [18]. Nevertheless it is an interesting mathematical question to isolate the minimal condition on g needed to establish that $H(g)$ is a semibounded operator essentially self-adjoint on $C^\infty(H_0)$. (Of course we always require that $g(x)\geqq 0$.) In this Appendix we prove a theorem with "almost" minimal conditions on g and we discuss our conjecture of a necessary and sufficient condition for self-adjointness.

As we suggested following the estimate (1.7), the condition $g\in L^1\cap L^2$ is much stronger than necessary for self-adjointness and semiboundedness. In particular $g\in L^2$ is used in the standard proof only to conclude that $|\hat g|_{2n}$ is finite; here $2n$ is the degree of P and

$$|\hat g|_j\equiv\left[\int\frac{|\hat g(k_1+\cdots+k_j)|^2}{\mu(k_1)\dots\mu(k_j)}dk_1\dots dk_j\right]^{1/2} \tag{A.1}$$

where $\mu(k)=(k^2+m^2)^{1/2}$. This norm enters because $\|:\phi^{2n}(g):\Omega_0\|_2=\text{const}\,|\hat g|_{2n}$. As observed in [12] $|\hat g|_j\leqq\text{const}\,|\hat g|'_j$ where

$$|\hat g|'_j\equiv\left[\int|\hat g(k)|^2\frac{(\log\mu(k))^{j-1}}{\mu(k)}dk\right]^{1/2}. \tag{A.2}$$

In fact:

Lemma A.1. *The norms $|\hat g|_j$ and $|\hat g|'_j$ are equivalent, $j=1,2,\dots$.*

Proof. Let $c_j(k)$ be the j-fold convolution $\mu^{-1}*\cdots*\mu^{-1}(k)$. Since $|\hat g|_j^2=\int|\hat g(k)|^2c_j(k)\,dk$, we need only show that for large k the ratio of $c_j(k)$ to $(\log k)^{j-1}/k$ is bounded away from 0 and ∞.

Proceeding by induction, we suppose that

$$c_j(k)\geqq\text{const}\,(\log k)^{j-1}/k$$

for large k. (The inductive assumption is clearly true for $j=1$.) Then

$$\begin{aligned}c_{j+1}(k)&\geqq\text{const}\int_1^k\frac{(\log p)^{j-1}\,dp}{\mu(k-p)\,p}\\&\geqq\frac{\text{const}}{\mu(k)}\int_1^k\frac{(\log p)^{j-1}}{p}dp\\&=\frac{\text{const}}{\mu(k)}(\log k)^j.\end{aligned}$$

For the reverse inequality we note, following [12, Lemma 4.1], that the Fourier transform of μ^{-1} is the modified Bessel function $K_0(mx)$ so that

$$c_j(k)=\text{const}\int e^{ikx}K_0^j(mx)\,dx.$$

Since K_0 vanishes exponentially at infinity and has a logarithmic singularity at the origin the dominant behavior of $c_j(k)$ for large k is given by that of the function

$$f_j(k) = \int_0^1 e^{ikx} (\log x)^j \, dx \, .$$

Using the recursion relation $f_{j+1}(k) = -\frac{j+1}{k} \int_0^k f_j(p)\, dp$, we prove by induction that for large $|k|$ (say k positive)

$$|f_j(k)| \leqq \text{const} (\log k)^{j-1}/k \, . \tag{A.3}$$

Assuming (A.3) for j and using the fact that f_j is bounded, we have for large k

$$\begin{aligned} |f_{j+1}(k)| &\leqq \frac{\text{const}}{k} \int_0^k |f_j(p)|\, dp \\ &\leqq \frac{\text{const}}{k} \int_1^k (\log p)^{j-1}/p \, dp + O(k^{-1}) \\ &\leqq \text{const} (\log k)^j/k \, . \end{aligned}$$

The validity of (A.3) for $j=1$ follows from the recursion relation and the fact that $f_0 \in L^1$. ∎

Thus the condition $g \in L^2$ could be replaced by either $|\hat{g}|_{2n} < \infty$ or $|\hat{g}|'_{2n} < \infty$. However, since these norms are somewhat inconvenient we introduce a slightly stronger condition, namely $g \in L^{1+\varepsilon}$:

Lemma A.2. *For any* $\varepsilon > 0$ *and positive integer* j *there is a constant* c *such that*

$$|\hat{g}|_j, \; |\hat{g}|'_j \leqq c \|g\|_{1+\varepsilon} \, . \tag{A.4}$$

Proof. By the previous lemma it is sufficient to prove (A.4) for $|\hat{g}|'_j$. We take $\varepsilon < 1$; otherwise (A.4) is trivial. Since $(\log \mu)^{j-1}/\mu \in L^p$ for all $p > 1$, we apply Hölder's inequality with $p = (1+\varepsilon)/(1-\varepsilon)$: $|\hat{g}|'_j \leqq \text{const} \|\hat{g}^2\|_{p'}^{1/2} = \text{const} \|\hat{g}\|_{2p'} \leqq \text{const} \|g\|_{(2p')'} = \text{const} \|g\|_{1+\varepsilon}$, by the Hausdorff-Young inequality [19]. ∎

For the applications below, it is enough that $|\hat{g}|_j \leqq c \|g\|_{1+\varepsilon}$. This can be proved without reference to the norm $|\hat{g}|'_j$ by means of the Hausdorff-Young and Young inequalities [16].

Corollary A.3. *Suppose* P *is semibounded and* $g \in L^1 \cap L^{1+\varepsilon}$, $g \geqq 0$. *Then* $H(g)$ *is semibounded and essentially self-adjoint on* $C^\infty(H_0)$.

Proof. According to the preceding discussion the hypercontractive proof [18] goes through with $g \in L^2$ replaced by $g \in L^{1+\varepsilon}$. That $H(g)$ is essentially self-adjoint on $C^\infty(H_0)$ and not merely on $D(H_0) \cap D(V(g))$

is proved in [17]. Here

$$V(g) = \int :P(\phi(x)):g(x)\,dx\,. \quad \blacksquare$$

Note that in Corollary A.3 it has not been necessary to require that P is normalized (i.e., $P(0)=0$). However we now wish to weaken the hypothesis $g \in L^1$ and so we assume that P is normalized. An examination of the NGS proof of semiboundedness (see (1.1)) shows that the assumption $g \in L^1$ enters in a rather crude estimate that establishes that the ultraviolet-cutoff interaction is semibounded. As seen from the estimate (1.7) a more appropriate condition on the small g behavior is that $g \in L^2$. We prove:

Theorem A.4. *Suppose that P is semibounded and normalized and that $g \in L^2 + L^{1+\varepsilon}$, $g \geqq 0$. Then $H(g)$ is semibounded and essentially self-adjoint on $C^\infty(H_0)$.*

Remarks. 1. The notation $g \in L^2 + L^{1+\varepsilon}$ means that $g = f + h$ where $f \in L^2$ and $h \in L^{1+\varepsilon}$. Clearly there is no loss in generality in assuming that $f = g$ when $g \leqq 1$ and $f = 0$ otherwise.

2. Since g is not necessarily in L^1 we will not prove that $e^{-tV(g)}$ is in $L^1(Q)$ as in the NGS proof. In essence our proof involves the fact that

$$\exp\left(-\int_0^t g(x) : P(\phi(x,s)) : dx\,ds\right)$$

is integrable in the Q-space associated with the free Markov field [10].

Sketch of Proof. The idea of the proof is standard (see [18]): We approximate g by a sequence g_j in $L^1 \cap L^{1+\varepsilon}$ and prove that the semigroups $e^{-tH(g_j)}$ converge strongly to a continuous semigroup whose generator can be identified as $(H(g) \,/\, C^\infty(H_0))^-$. Since h is already in $L^1 \cap L^{1+\varepsilon}$ we take $g_j = f_j + h$ where $f_j \in L^1 \cap L^{1+\varepsilon}$, $f_j \geqq 0$, $f_j \to f$ in L^2, and $\|f_j\|_2 \leqq \|f\|_2$. For instance $f_j = f\chi_{(-j,j)}$. The two key facts in the proof are that for any $1 < p < \infty$: (i) $\|V(g_j) - V(g_k)\|_p \to 0$ in $L^p(Q)$ and (ii) $\|e^{-tH(g_j)}\|_{p,p}$ is bounded uniformly in j. The first fact follows from the convergence $\|g_j - g_k\|_2 \to 0$ and the second from Lemma A.5 below since

$$\begin{aligned}\|e^{-tH(g_j)}\|_{p,p} &\leqq \exp\left[\frac{2t}{p}\int \alpha_\infty(p g_j(x)/2)\,dx\right] \\ &\leqq \exp\left[\mathrm{const}\left(\int f^2 + \int h^{1+\varepsilon}\right)\right]\end{aligned}$$

by the estimates of Theorem 1.4. $\blacksquare$

Lemma A.5. *Let $g \in L^1 \cap L^{1+\varepsilon}$, $g \geqq 0$. Then for all $2 \leqq p < \infty$ and $t \geqq 0$*

$$\|e^{-tH(g)}\|_{p,p} \leqq \exp\left[\frac{2t}{p}\int \alpha_\infty(p g(x)/2)\,dx\right]. \tag{A.5}$$

Remarks. 1. By duality $\|e^{-tH}\|_{p,p} = \|e^{-tH}\|_{p',p'}$ so that we have the bound when $1 < p < 2$.

2. By a slight alteration of the proof one can prove a similar bound for $\|e^{-tH}\|_{p,q}$, $p < q$, provided that t is sufficiently large.

3. By the method of Theorem A.4, the lemma can be extended to the more general case $g \in L^2 + L^{1+\varepsilon}$, when P is normalized.

Proof. Let $f(z) = \exp[-t(H_0 + zV(g))]$. For any p, q [7], $\|f(z)\|_{p,q} \leqq \|f(\mathrm{Re}\, z)\|_{p,q}$. Thus when $\mathrm{Re}\, z = 0$, $\|f(z)\|_{\infty,\infty} \leqq 1$, and when $\mathrm{Re}\, z = p/2$,

$$\|f(z)\|_{2,2} \leqq \exp[t \textstyle\int \alpha_\infty(p g(x)/2)\, dx]$$

by Theorem 1.7. The lemma now follows from the Stein Interpolation Theorem. ∎

The condition $g \in L^2 + L^{1+\varepsilon}$ can probably be weakened further; for example, we expect that those g for which the right side of (1.7) is finite will suffice. Thus our condition on g is not minimal but it has an almost minimal character as we now explain. In order for $H(g)$ even to be defined on finite particle vectors it is necessary that $\|:\phi^{2n}(g):\Omega_0\|_2 = \mathrm{const}\,|\hat{g}|_{2n}$ be finite. Since there are $g \in L^{2+\varepsilon}$ and $g \in L^1$ for which $|\hat{g}|_{2n} = \infty$ our conditions on g are only slightly weaker than the necessary condition $|\hat{g}|_{2n} < \infty$. This leads to:

Conjecture. *Suppose that* $P(X) = \sum_{r=1}^{2n} b_r X^r$, $b_{2n} > 0$. *Then* $|\hat{g}|_{2n} < \infty$ *(or* $|\hat{g}|'_{2n} < \infty$*) is a sufficient as well as a necessary condition that* $H(g)$ *be semibounded and essentially self-adjoint on* $C^\infty(H_0)$.

The key to proving this conjecture is to show that if $|\hat{g}|_{2n}$ is finite then so is $E(g)$. In support of the conjecture we note that $|\hat{g}|_{2n} < \infty$ implies that $E(g)$ is finite order by order in perturbation theory. Moreover in the one case where $E(g)$ can be explicitly computed ($P(X) = X^2$) we have the bound [14] for any $\varepsilon > 0$,

$$\begin{aligned} -E(g) &\leqq \mathrm{const} \int \frac{|\hat{g}(k_1 + k_2)|^2}{\mu(k_1)^{3/2-\varepsilon}\,\mu(k_2)^{3/2-\varepsilon}}\, dk_1\, dk_2 \\ &\leqq \mathrm{const}\,|\hat{g}|_2\,. \end{aligned}$$

References

1. Glimm, J.: Boson fields with nonlinear self-interaction in two dimensions. Commun. math. Phys. **8**, 12 (1968).
2. Glimm, J., Jaffe, A.: A $\lambda\phi^4$ quantum field theory without cutoffs. I. Phys. Rev. **176**, 1945 (1968).
3. Glimm, J., Jaffe, A.: The $\lambda(\phi^4)_2$ quantum field theory without cutoffs. III., The Physical Vacuum. Acta Math. **125**, 203 (1970).

4. Glimm, J., Jaffe, A.: The $\lambda(\phi^4)_2$ quantum field theory without cutoffs. IV., Perturbations of the Hamiltonian (to appear).
5. Glimm, J., Jaffe, A.: Quantum field theory models. In: 1970 Les Houches Lectures, De Witt, C., Stora, R. (Ed.), New York: Gordon and Breach, 1972.
6. Guerra, F.: Uniqueness of the vacuum energy density and Van Hove phenomenon in the infinite volume limit for two-dimensional self-coupled bose fields. Phys. Rev. Lett. **28**, 1213 (1972).
7. Guerra, F., Rosen, L., Simon, B.: Nelson's Symmetry and the infinite volume behaviour of the vacuum in $P(\phi)_2$. Commun. math. Phys. **27**, 10 (1972).
8. Nelson, E.: A quartic interaction in two dimensions. In: Mathematical Theory of Elementary Particles, R. Goodman and I. Segal, Eds., Cambridge: M.I.T. Press 1966.
9. Nelson, E.: Quantum fields and Markoff fields. Proc. 1971 Amer. Math. Soc. Summer Conference (Berkeley).
10. Nelson, E.: The free Markoff field. Princeton Univ. preprint.
11. Osterwalder, K., Schrader, R.: On the uniqueness of the energy density in the infinite volume limit for quantum field models. Helv. Phys. Acta, to appear.
12. Rosen, L.: A $\lambda\phi^{2n}$ field theory without cutoffs. Commun. math. Phys. **16**, 157 (1970).
13. Rosen, L.: The $(\phi^{2n})_2$ quantum field theory: higher order estimates. Comm. Pure Appl. Math. **24**, 417 (1971).
14. Rosen, L.: Renormalization of the Hilbert space in the mass shift model. J. Math. Phys. **13**, 918 (1972).
15. Rosen, L., Simon, B.: The $(\phi^{2n})_2$ field Hamiltonian for complex coupling constant. Trans. Amer. Math. Soc. **165**, 365 (1972).
16. Segal, I.: Construction of nonlinear local quantum processes. I. Ann. Math. **92**, 462 (1970).
17. Simon, B.: Essential self-adjointness of Schrödinger operators with positive potentials. Math. Ann. (to appear).
18. Simon, B., Hoegh-Krohn, R.: Hypercontractive semigroups and two dimensional self-coupled bose fields. J. Funct. Anal. **9**, 121 (1972).
19. Stein, E., Weiss, G.: Introduction to Fourier analysis on Euclidean spaces. Princeton: Univ. Press 1971.

Dr. L. Rosen
Princeton University
Jadwin Hall
Princeton, N.J. 08540, USA