TWO EXTENSIONS OF LUBINSKY'S UNIVERSALITY THEOREM

By

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Abstract. We extend some remarkable recent results of Lubinsky and Levin–Lubinsky from $[-1, 1]$ to allow discrete eigenvalues outside $\sigma_{\rm ess}$ and to allow $\sigma_{\rm ess}$ first to be a finite union of closed intervals and then a fairly general compact set in $\mathbb{R}$ (one which is regular for the Dirichlet problem).

1 Introduction

This paper primarily discusses orthogonal polynomials on the real line (OPRL) [39, 10, 28]. To set notation, μ is a measure of compact support $\sigma(d\mu)$ on $\mathbb{R}$, positive but not necessarily normalized. Its Lebesgue decomposition is

$$d\mu(x) = w(x)\,dx + d\mu_{\rm s}(x), \tag{1.1}$$

where $w \in L^1(\mathbb{R}, dx)$ and $\mu_{\rm s}$ is Lebesgue singular. Let $\sigma_{\rm ess}(d\mu)$ denote $\sigma(d\mu)$ with isolated points removed and $\sigma_{\rm s}(d\mu) = \sigma(d\mu_{\rm s})$.

We denote the monic orthogonal polynomials by $P_n(x, d\mu)$ and the orthonormal polynomials by $p_n(x, d\mu)$. The Jacobi parameters $\{a_n, b_n\}_{n=1}^\infty$ are defined by the recursion relation

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_{n+1}p_n(x) + a_n p_{n-1}(x). \tag{1.2}$$

We note for later use that

$$p_0(x) = \Big(\frac{1}{\mu(\mathbb{R})}\Big)^{1/2} \tag{1.3}$$

and that ($\|\cdot\|$ means $L^2(\mathbb{R}, d\mu)$ norm)

$$\|P_n\| = a_1 \cdots a_n \mu(\mathbb{R})^{1/2}. \tag{1.4}$$

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The main focus of this paper is the CD (for Christoffel–Darboux) kernel (for $x, y \in \mathbb{R}$)

$$K_n(x, y; d\mu) = \sum_{j=0}^{n} p_n(x, d\mu) p_n(y, d\mu). \tag{1.5}$$

We often drop $d\mu$ or consider several measures, say $\mu, \mu^\sharp$, and use $K_n(x, y), K_n^\sharp(x, y)$. Then K_n is the integral kernel of the orthogonal projection in $L^2(\mathbb{R}, d\mu)$ onto the polynomials of degree at most n. So if $Q_n(x)$ is such a polynomial, then (the reproducing property)

$$Q_n(x) = \int K_n(x, y) Q_n(y)\, d\mu(y) \tag{1.6}$$

and, in particular (an expression that K_n is the kernel of a projection),

$$\int K_n(x, y) K_n(y, z)\, d\mu(y) = K_n(x, z). \tag{1.7}$$

Going back to Faber [7], Fekete [8], and Szegő [38], it has been known that there are deep connections between potential theory and asymptotics of polynomials; see Stahl–Totik [37] and Simon [33]. We are especially interested in the potential theory associated to $E = \sigma_{\text{ess}}(d\mu)$. We call $E \subset \mathbb{R}$ a **regular set** if it is compact, regular for the Dirichlet problem on $\mathbb{C}$ and with an equilibrium measure $d\rho_E$ of the form $\rho_E(x)\, dx$. Thus, writing $C(E)$ for the logarithmic capacity of E, we have

$$\int_E \rho_E(y)\, dy = 1 \tag{1.8}$$

and

$$G_E(x) = \int_E \log|x - y| \rho_E(y)\, dy - \log C(E) \tag{1.9}$$

is continuous on $\mathbb{C}$ with

$$G_E(y) = 0 \ \text{ if } y \in E \qquad G_E(z) > 0 \ \text{ if } z \notin E. \tag{1.10}$$

Stahl–Totik introduce the important notion of regular measure on E (following its use by Ullman [44] for the special case $E = [-1, 1]$): a measure μ of compact support is called regular if and only if

$$\lim_{n \to \infty} (a_1 \dots a_n)^{1/n} = C(E), \tag{1.11}$$

where $E = \sigma_{\text{ess}}(d\mu)$. (They use $E = \sigma(d\mu)$; but since $\sigma(d\mu) \setminus \sigma_{\text{ess}}(d\mu)$ is countable, it has zero capacity, and so there is no difference.) One reason this is natural is that it is always true that

$$\limsup_{n \to \infty} (a_1 \dots a_n)^{1/n} \le C(E). \tag{1.12}$$

An elegant way to see this (c.f. Widom [47] or [33]) is to note that when $\mu(\mathbb{R}) = 1$, $a_1 \cdots a_n = \|P_n\|_2 \leq \|T_n\|_\infty$ with T_n the Chebyshev polynomial for E and use Szegő's theorem [38] that $\lim \|T_n\|^{1/n} = C(E)$.

More generally than (1.12), one has the results of Stahl–Totik [37] (see also [33]) that

Theorem 1.1 ([37]). *If $E = \sigma_{\text{ess}}(d\mu)$ and μ is regular, then*

$$\limsup_{n\to\infty} |p_n(z, d\mu)|^{1/n} \leq e^{G_E(z)}$$

uniformly on compact subsets of $\mathbb{C}$. In particular, if E is regular, for any ε, there exists δ and C such that

$$\sup_{\text{dist}(y,E)\leq\delta} |p_n(y, d\mu)| \leq Ce^{\varepsilon n}. \tag{1.13}$$

One connection between K and ρ_E is (an analogue of Theorem 8.2.6 of [28]; see Simon [35])

Theorem 1.2 ([35]). *For any regular measure μ,*

$$\frac{1}{n} K_n(x,x)\, d\mu \to d\rho_E, \tag{1.14}$$

the equilibrium measure for $E = \sigma_{\text{ess}}(d\mu)$, in the sense of weak convergence of probability measures on $\text{supp}(d\mu)$.

If E is regular, if $d\mu$ given by (1.1) is regular, and if $\frac{1}{n}K_n(x,x)$ has a uniform limit as $n \to \infty$ for $x \in I$ some open interval, then by (1.14), that limit must be $\rho_E(x)/w(x)$ (and $w(x)$ must be continuous and nonvanishing on I). This motivates

Definition. We say that μ has **normal limits** on a closed interval $I = [a,b]$ if and only if for any $x_n \to x \in I$,

$$\frac{1}{n} K_n(x_n, x_n) \to \frac{\rho_E(x)}{w(x)} \tag{1.15}$$

with convergence which is uniform in the sense that for any ε, there exist N and δ such that when $n \geq N$ and $|x_n - x| < \delta$, the difference between the right and left hand sides of (1.15) is less than ε.

Normal limits for $x_n \equiv x$ have a long history for orthogonal polynomials on the unit circle (OPUC) and for $E = [-1,1]$, going back to Szegő, with important contributions by Erdős, Turán and Freud. This history is discussed in the fundamental paper by Máté–Nevai–Totik [23], who obtained very strong results on pointwise convergence for μ's supported on $[-1,1]$ or OPUC supported on $\partial\mathbb{D}$. The refinement of allowing $x_n \to x$ is one critical idea in a recent paper of Lubinsky [19], who provides a result on off-diagonal behavior of K_n also:

Theorem 1.3 ([19]). *Let $I = [a,b]$ be a closed interval in $(-1,1)$ and $d\mu$ a regular measure with support $[-1,1]$ such that* $\operatorname{supp}(d\mu_{\rm s}) \cap I = \emptyset$. *Suppose that $d\mu(x) = w(x)\,dx$ on I, where w is continuous and nonvanishing. Then μ has normal limits on I, and for $x_0 \in I$ and $\alpha, \beta \in \mathbb{R}$,*

$$\lim_{n\to\infty} \frac{K_n(x_0 + \frac{\alpha}{n}, x_0 + \frac{\beta}{n})}{K_n(x_0, x_0)} = \frac{\sin(\pi\rho_{[-1,1]}(x_0)(\beta-\alpha))}{\pi\rho_{[-1,1]}(x_0)(\beta-\alpha)} \tag{1.16}$$

uniformly if $|\alpha|, |\beta| \le A$, $x_0 \in I$ for any $A > 0$.

Remarks. 1. Continuity "on I" here means continuous at each point in I as a function on $[-1,1]$; that is, continuity at a, b involves values of w outside but near $[a,b]$. Thus, the continuity hypothesis is nonvacuous if $a = b$, and the theorem is interesting in that case.

2. The earliest results of the form (1.16) come from the random matrix and Riemann–Hilbert literatures; see [15].

3. Lubinsky [19] does not use $\rho_{[-1,1]}(x_0) = (\pi\sqrt{1-x_0^2})^{-1}$ for (1.16), but scales using $w(x)K_n(x,x) \sim n\rho_{[-1,1]}(x)$. This gives a form that makes contact with the Riemann–Hilbert literature and is also suitable for end points and Freud weights.

4. As explained in Section 4, Levin–Lubinsky [18] use (1.16) to control the asymptotics of zeros of p_n.

Our goal in this paper is to extend Theorem 1.3 in two ways.

(a) Instead of requiring $\sigma(d\mu) = [-1,1]$, we want to allow $\sigma_{\rm ess}(d\mu) = [-1,1]$, as is natural if one makes assumptions on $\{a_n, b_n\}_{n=1}^\infty$ rather than directly on $d\mu$.

(b) We want to replace $[-1,1]$ by a general finite gap set.

A third important extension involves (1.16) pointwise for a.e. $x_0 \in I$ for situations where $d\mu$ obeys a local Szegő condition on I. I had intended to combine Lubinsky's strategy with ideas of Máté–Nevai–Totik [23] and especially Totik [40], but I was informed by Totik that Findley [9] and he [42] have results along this line. So I decided to focus here only on (a) and (b).

While important, neither of these extensions is especially difficult. Because of Lubinsky's clever inequality (see (4.1)), it is necessary only to find a suitable universal model for E and to control the diagonal kernel.

A key point is to relate $K_n(x,x)$ to the Christoffel function,

$$\lambda_n(x_0) = \min\{\|Q\|^2_{L^2(\mathbb{R},d\mu)} : Q(x_0) = 1,\ \deg Q \le n\}. \tag{1.17}$$

The minimizer is

$$Q_n(x, x_0) = K(x_0, x_0)^{-1} \sum_{j=0}^n p_j(x)p_j(x_0), \tag{1.18}$$

for which

$$\lambda_n(x_0) = K_n(x_0, x_0)^{-1}. \tag{1.19}$$

To handle extension (a) is easy. One can eliminate the point masses distant from E by adding explicit zeros to a trial polynomial and control the point masses near E with some exponential decay.

The key to (b) is to construct a suitable model that is well-behaved; following Lubinsky's strategy (he uses Legendre polynomials as his model), it is easy to extend Theorem 1.3. Our model is the measure associated to a point in the isospectral torus associated to E, where the analysis depends on results of Widom [48], Sodin–Yuditskii [36], Peherstorfer–Yuditskii [26], and Christiansen–Simon–Zinchenko [5].

The most subtle part of the model is establishing (1.16), which follows from Jost asymptotics. Jost asymptotics are the key to proving clock behavior for zeros in [30, 31, 17]. In a sense, using the Levin–Lubinsky strategy, we can regard (1.16) as a kind of infinitesimal Jost asymptotics.

To obtain control of the diagonal CD kernel, all we need is a single model $\mu^\sharp$, obeying

(i) $\sigma_{\text{ess}}(\mu^\sharp) = E$;

(ii) $w^\sharp$ is continuous and nonvanishing in E;

(iii) for any closed interval $I \subset E^{\text{int}}$, and $\varepsilon > 0$,

$$\sup_{x \in I} e^{-\varepsilon n} K_n^\sharp(x, x) \to 0; \tag{1.20}$$

(iv) for any closed interval $I \subset E^{\text{int}}$,

$$\limsup_{\varepsilon \downarrow 0} \left[\limsup_{n \to \infty} \frac{K^\sharp_{(1+\varepsilon)n}(x, x)}{K_n^\sharp(x, x)} \right] = 1; \tag{1.21}$$

(v) for $x_n \to x_\infty$ in E^{int},

$$\lim_{n \to \infty} \frac{K_n(x_n, x_n)}{K_n(x_\infty, x_\infty)} = 1 \tag{1.22}$$

and this is uniform in that for any closed interval $I \subset E^{\text{int}}$ and any ε, there exist δ and N such that the ratio in (1.22) is within ε of 1 if $|x_n - x_\infty| < \delta$ and $n > N$.

It is known [47, 45, 37, 33] that (ii) implies that $\mu^\sharp$ is regular. Of course, we have

$$\frac{1}{n} K_n^\sharp(x, x) \to \frac{\rho(x)}{w^\sharp(x)}, \tag{1.23}$$

from which (1.20) and (1.21) follow. We state the result in this form to allow for future work in which the model focuses on a single point in $E^{\rm int}$, where w vanishes or blows up at some rate.

In Section 3, we prove

Theorem 1.4. *Suppose that E is a regular set and there exists a model, $\mu^\sharp$, obeying* (i)–(v). *Let μ be a measure with $\sigma_{\rm ess}(\mu)=E$, μ regular, and w continuous and nonvanishing on $I=[a,b]\subset E^{\rm int}$. Suppose that $\sigma_{\rm s}(\mu)\cap I=\emptyset$. Then for any $x_n\to x\in I$,*

$$\frac{K_n(x_n,x_n)}{K_n^\sharp(x_n,x_n)}\to\frac{w^\sharp(x)}{w(x)} \tag{1.24}$$

uniformly in the sense discussed after (1.15).

Remark. By (1.23) and (1.24), we have normal limits on I.

In Section 4, we prove

Theorem 1.5. *Suppose that E is a regular set and there exists a model, $\mu^\sharp$, obeying* (i)–(iv) *so that $K^\sharp$ obeys* (1.16) *uniformly for x in compacts of $E^{\rm int}$, and $|\alpha|,|\beta|<A$. Let μ be a measure with $\sigma_{\rm ess}(\mu)=E$, μ regular, and w continuous and nonvanishing on $I=[a,b]\subset E^{\rm int}$. Suppose that $\sigma_{\rm s}(\mu)\cap I=\emptyset$. Then K obeys* (1.16) *uniformly on I and $|\alpha|<A$, $|\beta|<A$.*

Given Theorem 1.4, we obtain Theorem 1.5 by following Lubinsky's argument virtually word for word. Following Levin–Lubinsky, it also implies uniform clock behavior of the zeros in I in the sense of Last–Simon [17] (if $a<b$).

In Section 2, we obtain $\mu^\sharp$ obeying (i)–(v) when E is a finite union of intervals and (1.16), therefore accomplishing extensions (a) and (b).

All our arguments extend with little change to finite gap OPUC and to zeros of paraorthogonal polynomials [3, 4, 11, 14, 32, 49].

During the preparation of this manuscript, I learned that Totik [42] was also working on extending Lubinsky universality to general sets. After I finished the above and Sections 2–4 below, Totik and I exchanged manuscripts. His technical methods are different from what I do in Section 2. After I got his manuscript, I realized that Lubinsky's inequality ((4.1) below) is so strong that it is easy to go from finite gap to general compact sets and prove

Theorem 1.6. *Let $E\subset\mathbb{R}$ be an arbitrary regular compact set such that $I=[a,b]\subset E^{\rm int}$. Let μ be a measure regular in the sense of Stahl–Totik [37] such that $\sigma_{ess}(\mu)=E$ and $\mu\restriction[a-\varepsilon,b+\varepsilon]$ is purely absolutely continuous with*

$w = \frac{d\mu}{dx}$ *continuous and nonvanishing on* $[a,b]$. *Let* $\rho_E(x)\,dx$ *be the density for the equilibrium measure for* E *restricted to* I. (*It is not hard to see* ρ_E *is purely a.c. on* I*; see [33].*) *Then uniformly for* $x \in [a,b]$,

$$K_n(x,x) \to \frac{\rho_E(x)}{w(x)}; \tag{1.25}$$

and, uniformly for $x_0 \in [a,b]$, $|\alpha|, |\beta| \le A$, *one has* (1.16), *and so* (*as in Section 4 following [18]*), *clock behavior for the zeros.*

Remark. (1.25) is not new. It is essentially in Totik [40].

The proof of this theorem is sketched in Section 5. We also note there that it suffices to prove the results in Section 2 when each interval has rational harmonic measure, so that one can use Floquet theory in place of the more subtle analysis of [48, 36, 26, 5].

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2 Models

Let E be a finite gap set, that is,

$$E = \bigcup_{j=1}^{k+1} [\alpha_j, \beta_j], \tag{2.1}$$

where $\alpha_1 < \beta_1 < \alpha_2 < \cdots < \beta_{k+1}$ are reals. Associated with any such E is an isospectral torus of Jacobi matrices defined by the fact that their m-functions are Herglotz functions extendable to minimal degree meromorphic functions on the two-sheeted Riemann surface associated to $[\prod_{j=1}^{k+1}(z-\alpha_j)(z-\beta_j)]^{1/2}$. For OPUC, this is discussed, for example, in [29], and for OPRL in [34].

The spectral measure $\mu^\sharp$ for any such Jacobi matrix has the form (see [36, 26, 5])

$$d\mu^\sharp = w^\sharp(x)\,dx + d\mu_{\rm s}, \tag{2.2}$$

where $d\mu_{\rm s}$ is a pure point measure with at most one pure point in each gap of E and none in E and

$$w^\sharp > 0 \tag{2.3}$$

and real analytic on $E^{\rm int}$.

Our goal in this section is the prove the following

Theorem 2.1. *Let $\mu^\sharp$ be the probability measure associated to a Jacobi matrix in the isospectral torus for E. Then, uniformly for x in any interval $[a,b]$ in E^{int}, we have*

(i)

$$K_n^\sharp(x,x) = \frac{\rho_E(x)}{w(x)}\,n + O(1). \tag{2.4}$$

(ii) *Uniformly in $|\alpha|, |\beta| < L$,*

$$\frac{w(x)}{n}\,K_n^\sharp\Big(x+\frac{\alpha}{n}, x+\frac{\beta}{n}\Big) = \frac{\sin(\pi\rho_E(x)(\beta-\alpha))}{\pi(\beta-\alpha)} + O\Big(\frac{1}{n}\Big). \tag{2.5}$$

We obtain this from results [48, 36, 26, 5] on Jost solutions, that is, solutions of

$$a_{n+1}u_{n+2} + (b_{n+1}-x)u_{n+1} + a_n u_n = 0, \tag{2.6}$$

an equation solved by

$$u_n = p_{n-1}(x).$$

Jost solutions, $u_n(x)$, solve (2.6) for $x \in E$ and obey (see [48, Theorem 7.3])

(i)

$$u_n(x) = e^{in\theta(x)} f_n(x); \tag{2.7}$$

(ii) on any $I \subset E^{\text{int}}$, f_n is analytic in x and its derivatives are uniformly bounded in n and $x \in I$;

(iii)

$$\theta'(x) = \pi\rho_E(x); \tag{2.8}$$

(iv)

$$u_0(x) = 1; \tag{2.9}$$

(v)

$$\overline{u_1(x)} \neq u_1(x); \tag{2.10}$$

(vi) the Wronskian

$$a_n(u_{n+1}\bar u_n - \bar u_{n+1}u_n) \tag{2.11}$$

is n-independent (but x-dependent).

(vii) While we do not need it, we also note that $f_n(x)$ is almost periodic in n.

Because of (2.9)/(2.10), u_n and $\bar{u}_n$ are linearly independent, so p_n is a linear combination of u_{n+1} and $\bar{u}_{n+1}$. Thus (since (2.9) implies equality at $n = 1$, and equality holds at $n = 0$),

$$p_n(x) = \frac{u_{n+1} - \overline{u_{n+1}(x)}}{u_1(x) - \bar{u}_1(x)}. \tag{2.12}$$

Define

$$g_n(x) = \frac{e^{i\theta(x)} f_{n+1}(x)}{[u_1(x) - \bar{u}_1(x)]}, \tag{2.13}$$

so (2.12) becomes

$$p_n(x) = g_n(x)e^{in\theta(x)} + \overline{g_n(x)}\, e^{-in\theta(x)}, \tag{2.14}$$

and the constancy of the Wronskian becomes

$$a_{n+1}[g_{n+1}(x)\bar{g}_n e^{i\theta(x)} - g_n(x)\,\overline{g_{n+1}(x)}\, e^{-i\theta(x)}] = c(x), \tag{2.15}$$

x-dependent but not n-dependent.

We also need the CD formula

$$K_n^\sharp(x,y) = a_{n+1}\left[\frac{p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)}{x-y}\right] \tag{2.16}$$

for $x \neq y$ and its limit at $x = y$,

$$K_n^\sharp(x,x) = a_{n+1}[p'_{n+1}(x)p_n(x) - p_{n+1}(x)p'_n(x)]. \tag{2.17}$$

Proof of Theorem 2.1. By (ii), (iii), and (v), and (2.14)

$$p'_n(x) = in\theta'(x)[g_n(x)e^{in\theta(x)} - \overline{g_n(x)}\, e^{-in\theta(x)}] + O(1) \tag{2.18}$$

where $O(1)$ is bounded uniformly in $x \in I$ and in n. Thus, by (2.14), (2.15), (2.17), and (2.18),

$$K_n^\sharp(x,x) = 2in\theta'(x)c(x) + O(1). \tag{2.19}$$

Therefore, $\frac{1}{n}K_n(x,x)$ converges uniformly on I to $2i\theta'(x)c(x)$, so by (1.14),

$$2i\theta'(x)c(x) = \frac{\rho_E(x)}{w(x)} \tag{2.20}$$

and (2.19) is (2.4).

Similarly, by (2.14), (2.8), and (ii),

$$p_n\Big(x + \frac{\alpha}{n}\Big) = g_n(x)e^{in\theta(x)}e^{i\alpha\pi\rho(x)} + \overline{g_n(x)}\, e^{-in\theta(x)}e^{-i\alpha\pi\rho(x)} + O\Big(\frac{1}{n}\Big) \tag{2.21}$$

uniformly in $|\alpha| < A$.

Plugging this into (2.16) shows that

$$K_n\left(x + \frac{\alpha}{n}, x + \frac{\beta}{n}\right) = 2in\, \frac{\sin((\beta - \alpha)\pi\rho_E(x))}{\beta - \alpha}\, c(x) + O(1). \tag{2.22}$$

By (2.20) and (2.8), $c(x) = 1/2\pi i w(x)$, so (2.22) is (2.5). □

3 Asymptotics of the Diagonal CD Kernel

Our goal here is to prove Theorem 1.4. The key is an idea of Nevai [24], which lets one exponentially localize CD minimizers, augmented by the regularity ideas of Máté–Nevai–Totik [23] and a simple extension that accommodates discrete spectrum.

Lemma 3.1. *Let E be a regular subset of $\mathbb{R}$. Let $\mu, \mu^\sharp$ be two measures, both regular for E. Let I be a closed interval in E^{int} with $I \cap [\sigma_{\text{s}}(\mu) \cup \sigma_{\text{s}}(\mu^\sharp)] = \emptyset$. Fix C_1 larger than the diameter of E. Then for all sufficiently small δ and each $\varepsilon > 0$, there is a constant C_2 and positive integer J depending on μ, $\mu^\sharp$, and ε such that for all m and ℓ,*

$$\lambda_n(x_0, \mu^\sharp) \le \sup_{|y - x_0| \le \delta} \left(\frac{w^\sharp(y)}{w(y)}\right) \lambda_m(x_0, \mu) + C_2 e^{\varepsilon m}\left(1 - \frac{\delta^2}{C_1^2}\right)^{2\ell} \tag{3.1}$$

for all $x_0 \in I$, where $n = m + 2\ell + 2J$.

Remarks. 1. While we apply this to w positive and continuous near x_0, it is stated in a way that should be applicable to situations where w and $w^\sharp$ vanish or blow up in the same way (the $\sup$ in (3.1) is then interpreted as an essential $\sup$).

2. One should be able, as in Lubinsky [20], to accommodate end points with these methods.

Proof. Let $Q_m(x, x_0; \mu)$ be the optimal trial function for the CD problem at x_0, that is,

$$Q_m(x, x_0; \mu) = K_m(x_0, x_0)^{-1} \sum_{n=0}^{m} p_m(x, \mu) p_m(x_0, \mu). \tag{3.2}$$

By (1.13), there exist δ_1 and C_3 such that

$$\sup_{\substack{\text{dist}(y, E) \le \delta_1 \\ x_0 \in I}} |Q_m(x, x_0; \mu)| \le C_3\, e^{\varepsilon m/2}. \tag{3.3}$$

We use here the fact that by (1.13), for $\mathrm{dist}(y, E) \leq \delta_1$ with δ_1 suitable,

$$|p_m(y)| \leq ce^{m/6}$$

and $m \leq 6e^{m/6}$. By shrinking δ_1, we can also ensure that

$$x_0 \in I,\ \mathrm{dist}(y, E) \leq \delta_1 \Rightarrow |x_0 - y| \leq C_1. \tag{3.4}$$

We now define J to be the number of points x in $\mathrm{supp}(\mu^\sharp)$ with $\mathrm{dist}(x, E) \geq \delta_1$, which is finite since $\sigma_{\mathrm{ess}}(d\mu^\sharp) = E$, and let $\{x_j\}_{j=1}^J$ be those points. We take for our trial polynomials for $\lambda_n(x_0, \mu^\sharp)$ with $n = m + 2\ell + 2J$,

$$Q(x) = Q_m(x, x_0; \mu)\biggl(1 - \frac{(x - x_0)^2}{C_1^2}\biggr)^\ell \prod_{j=1}^J \biggl(1 - \biggl[\frac{(x - x_0)}{(x_j - x_0)}\biggr]^2\biggr). \tag{3.5}$$

Pick δ so small that $\{x : \mathrm{dist}(x, I) \leq \delta\}$ is disjoint from $\sigma_{\mathrm{s}}(\mu) \cup \sigma_{\mathrm{s}}(\mu_0)$. If $y \in \mathrm{supp}(d\mu^\sharp)$ and $|y - x_0| \geq \delta$, we use (3.3) to see that

$$|Q(y)| \leq C_4\, e^{\varepsilon m/2}\biggl(1 - \frac{\delta^2}{C_1^2}\biggr)^\ell, \tag{3.6}$$

where

$$C_4 = C_3 \max_{\substack{y \in \mathrm{supp}(d\mu^\sharp) \\ x_0 \in I}} \biggl|\prod_{j=1}^J \biggl(1 - \biggl[\frac{(y - x_0)}{(x_j - x_0)}\biggr]^2\biggr)\biggr|.$$

While (3.3) does not hold at x_j, the product $\prod_{j=1}^J$ vanishes at such points. Thus, $\int_{|y-x_0|\geq\delta} |Q(y)|^2\, d\mu^\sharp(y)$ is bounded by the second term in (3.1).

On the other hand, since the second two factors in (3.4) are bounded by 1 on $[x_0 - \delta, x_0 + \delta]$, $|Q(y)| \leq |Q_m(y)|$; so, since there is no singular spectrum there,

$$\begin{aligned} \int_{|y-x_0|\leq\delta} |Q(y)|^2\, d\mu^\sharp &\leq \sup_{|x-y|\leq\delta} \biggl[\frac{w^\sharp(y)}{w(y)}\biggr] \int_{|y-x_0|\leq\delta} |Q_m(y)|^2\, d\mu \\ &\leq \sup_{|x-y|\leq\delta} \biggl[\frac{w^\sharp(y)}{w(y)}\biggr] \lambda_m(x_0, \mu), \end{aligned}$$

since adding the part of the integral with $|y - x_0| > \delta$ only makes the integral larger. □

Remark. We emphasize that δ and ε are independent small numbers; δ_1 is ε-dependent. Thus, in the proof below, we fixed δ which determines η and can then take ε as small as we wish. Since η is fixed, $m/n \to 1$ as $\varepsilon \to 0$.

Proof of Theorem 1.4. Let $\mu^* = \mu^\sharp$ be the model obeying (i)–(iv). Once δ is fixed, we can pick η so $(1 - \frac{\delta^2}{C_1^2})^{2\eta} \le e^{-2}$. Then in (3.1), we pick $\ell = \eta\varepsilon m$. The second term in (3.1) is thus $O(e^{-\varepsilon m})$.

Divide by $\lambda_m(x_m, \mu^*)$. By (1.20), the second term in (3.1) goes to zero; and so (3.1) implies

$$\liminf \frac{\lambda_m(x_m,\mu)}{\lambda_m(x_m,\mu^*)} \ge \inf_{|x_0-y|\le 2\delta} \left(\frac{w(y)}{w^*(y)}\right) \liminf \left(\frac{\lambda_n^*}{\lambda_m^*}\right). \tag{3.7}$$

Here we have used $x_m - x_0 \to 0$, so that $|x_0 - y| \le 2\delta$ for m large.

As $\varepsilon \downarrow 0$, the $\liminf$ on the right goes to 1 by (1.21). Thus, by continuity, we have as $\delta \downarrow 0$,

$$\liminf \frac{\lambda_m(x_m,\mu)}{\lambda_m(x_m,\mu^*)} \ge \frac{w(x_0)}{w^*(x_0)}. \tag{3.8}$$

Now interchange μ and μ^* in (3.1), divide by $\lambda_n(x_n, \mu^*)$ and use the same arguments to get

$$\limsup \frac{\lambda_n(x_n,\mu)}{\lambda_n(x_n,\mu^*)} \le \frac{w(x_0)}{w^*(x_0)}. \tag{3.9}$$

Taken together, (3.8) and (3.9) complete the proof. All the arguments are uniform in $x_0 \in I$. □

4 Off-Diagonal CD Asymptotics and Clock Behavior

In this section, we prove Theorem 1.5 and note its consequences for zeros of the OPRL. Given Theorem 1.4, this is essentially a straightforward translation of [19] and [18]. We note that earlier Freud [10] had noted that the universality result, (1.16), (which he only had under very restrictive assumptions) implies clock behavior of zeros.

Proof of Theorem 1.5. Let $\mu \le \mu^*$. Then, as noted by Lubinsky [19, eqn. (3.5)], for any x, y,

$$|K_n(x,y) - K_n^*(x,y)|^2 \le K_n^*(y,y)[K_n(x,x) - K_n^*(x,x)]. \tag{4.1}$$

This critical result—which we dub Lubinsky's inequality—is proven in a few lines in [19].

Given μ and x_0, let $\widetilde{\mu}$ be that multiple of the model μ^* with $\widetilde{w}(x_0) = w(x_0)$. Let $\mu^\sharp = \sup(\mu, \widetilde{\mu})$. By the lemma below, $\mu^\sharp$ is regular. Thus, by Theorem 1.4, we see that

$$\frac{K_n(x_0 + \frac{\alpha}{n}, x_0 + \frac{\alpha}{n})}{K_n^*(x_0 + \frac{\beta}{n}, x_0 + \frac{\beta}{n})} \to 1 \tag{4.2}$$

and

$$\frac{\widetilde{K}_n(x_0 + \frac{\alpha}{n}, x_0 + \frac{\alpha}{n})}{K_n^*(x_0 + \frac{\beta}{n}, x_0 + \frac{\beta}{n})} \to 1, \tag{4.3}$$

uniformly if $|\alpha|, |\beta| < A$ and $x_0 \in I$.

From this and (4.1), we find (dividing by $K_n^*(y,y)$) that

$$\frac{K_n(x_0 + \frac{\alpha}{n}, x_0 + \frac{\beta}{n})}{\widetilde{K}_n(x_0 + \frac{\alpha}{n}, x_0 + \frac{\beta}{n})} \to 1 \tag{4.4}$$

also uniformly in $|\alpha|, |\beta| < A$, $x_0 \in I$. By (1.16) for $\widetilde{K}_n$ and (4.2)–(4.4), we have (1.16) for K_n. □

Remark. The sup of two measures $\nu = \sup(\mu, \tilde{\mu})$, (see, e.g., Section IX.3 of Doob [6] or Chapter IX of Jacobs [13] or Luxemburg–Zaanen [21]) is the smallest measure larger than $\mu, \tilde{\mu}$. It can be constructed as follows: if $d\mu = f(d\mu + d\tilde{\mu})$ and $d\tilde{\mu} = g(d\mu + d\tilde{\mu})$, then $d\nu = \max(f,g)(d\mu + d\tilde{\mu})$. It can also be defined via the vector dual lattice construction: if f is continuous and nonnegative, then

$$\nu(f) = \sup\{\mu(g) + \tilde{\mu}(h) : g \geq 0,\ h \geq 0,\ g + h = f\},$$

where g, h are also continuous.

Lemma 4.1. *Suppose μ, μ^* are two measures with $\sigma_{\text{ess}}(\mu) = \sigma_{\text{ess}}(\mu^*) \equiv E$ and $\mu \leq \mu^*$. Then μ regular implies μ^* is regular.*

Proof. Regularity means

$$\lim_{n\to\infty} \|P_n(\,\cdot\,, d\mu^*)\|^{1/n} = C(E). \tag{4.5}$$

By (1.12),

$$\limsup \|P_n(\,\cdot\,, d\mu^*)\|^{1/n} \leq C(E). \tag{4.6}$$

Since $\mu \leq \mu^*$ and $\|P_n(\,\cdot\,, d\nu)\| = \min\{\|Q_n\|_{L^2(d\nu)} : \deg Q_n = n,\ Q_n(x) = x^n + \text{lower order}\}$,

$$\|P_n(\,\cdot\,, d\mu)\| \leq \|P_n(\,\cdot\,, d\mu^*)\|. \tag{4.7}$$

Now (4.6), (4.7), and

$$\lim_{n\to\infty} \|P_n(\,\cdot\,, d\mu)\|^{1/n} = C(E) \tag{4.8}$$

imply (4.5). □

Last–Simon [17] define clock behavior and uniform clock behavior. Theorem 1.5 implies

Theorem 4.2. *Let μ be a measure obeying the hypothesis of Theorem 1.5 (so, in particular, E must have a suitable model). If $a = b$, there is clock behavior for the zeros of $p_n(x, d\mu)$ at a. If $a < b$, there is uniform clock behavior on I. The density of zeros in the clock behavior is $\rho_E(x)$.*

Remark. In particular, E can be a finite gap set by Theorem 2.1. Thus, we recover and vastly generalize the results of [31].

Proof. We need only follow the ideas of [18]. We fix a point x_0 in I.

Step 1. By the CD formula, $\frac{p_{n+1}(y)}{p_n(y)} = \frac{p_{n+1}(x)}{p_n(x)}$ for $y \neq x$ if and only if $K_n(x, y) = 0$. Thus, with $\gamma \equiv \frac{p_{n+1}(x_0)}{p_n(x_0)}$, we see by (1.16) that there is a zero of $p_{n+1}(y) - \gamma p_n(y)$ within $\frac{1}{n\rho(x_0)} + o(\frac{1}{n})$ of x_0. Since zeros of $p_{n+1} - \gamma p_n$ and of p_n interlace, we conclude that there are zeros of $p_n(x)$ within $\frac{1}{n\rho(x_0)} + o(\frac{1}{n})$ of $x_0 \in I$.

Step 2. By the CD formula, for $x \neq y$, if $p_n(x) = 0$, then $p_n(y) = 0$ if and only if $K_n(x, y) = 0$. Thus, by (1.16), there are no two zeros of $p_n(x)$ within $(1 - \varepsilon)\frac{1}{n\rho(x_0)}$ for any $\varepsilon > 0$; that is, we have an $O(\frac{1}{n})$ lower bound.

Step 3. By the CD formula and (1.16), if $p_n(x_0 + \alpha/n) = 0$, there exist zeros which are at $x_0 + \frac{\alpha}{n} + \frac{k}{n\rho(x_0)} + o(\frac{1}{n})$ for $|k| \leq K$; and by Step 2, they are unique.

All these arguments are uniform in x_0, so we have uniform clock behavior. □

5 General Sets

As explained in the Introduction, this section was written after I saw [42] and realized that my results plus Lubinsky's inequality easily allowed one to obtain universality for intervals with continuous a.c. weight in arbitrary compact sets and also allowed an alternative to Section 2 that only requires Floquet theory.

Proof of Theorem 1.6. As an open set, $\mathbb{R} \setminus E$ is a union of countably many maximal open intervals, whose total size, after the two semi-infinite ones are removed, is finite. Thus, for any $q > 0$, only finitely many have size larger than $2/q$; so for any positive integer q, $E_q = \{x : \mathrm{dist}(x, E) \leq 1/q\}$ is a finite gap compact set.

Let $\rho(x)$ be the equilibrium density for E (restricted to I) and $\rho_q(x)$ for E_q. By potential theory comparison theorem ideas, $d\rho_q \restriction I$ is nondecreasing; and by some potential theory using the real analyticity of ρ_q and ρ on I,

$$\rho_q(x) \uparrow \rho(x) \qquad (5.1)$$

uniformly on I (see [2, 12, 16, 27, 33, 37, 43] for background on the needed potential theory; in particular, see Theorems A.15 and A.16 of [33]).

For each q, pick a multiple $c_q\rho_q(x)$ of the equilibrium measure such that $\sup_I c_q\rho_q < \inf_I w(x)$, and let $\mu_q = \mu \vee c_q\rho_q$, the measure theoretic max. This is regular for E_q by Lemma 4.1. Thus, if $x_m \to x_0 \in I$,

$$K_m^{(q)}(x_m, x_m) \to \frac{\rho_q(x)}{w(x)},$$

where $w = \frac{d\mu}{dx}$. The trial functions for μ_q used in proving (3.8) can be used for μ; that is, we can get upper bounds directly and see that

$$\limsup n\lambda_n(x_n, \mu) \leq \frac{w(x)}{\rho_q(x)} \tag{5.2}$$

for each q. Since (5.1) is uniform on I, we have

$$\limsup n\lambda_n(x_n, \mu) \leq \frac{w(x)}{\rho(x)} \tag{5.3}$$

uniformly for $x \in I$ and $x_n \to x$ (as in Section 3, this means that for any ε, there are N and δ such that if $n \geq N$ and $|x_n - x| \leq \delta$, then $n\lambda_n(x_n, \mu) \leq \frac{w(x)}{\rho(x)} + \varepsilon$).

We can use the polynomials $p_n(x, \mu)$ as trial functions for μ_q and so still get (3.1); but, unlike in Section 3, we cannot take ε to zero for q fixed and so do not take $\ell/n \to 0$. Instead for a fixed q, there is $\eta(q)$, and we have to take $\ell \geq n\eta(q)$. But as $q \to \infty$, $\eta(q) \to 0$. We obtain

$$(1 + \eta(q)) \liminf n\lambda_n(x_n, \mu) \geq \frac{w(x)}{\rho_q(x)}. \tag{5.4}$$

Since $\eta(q) \to 0$ and $\rho_q(x) \to \rho(x)$, we therefore obtain

$$\lim n\lambda_n(x_n, \mu) = \frac{w(x)}{\rho(x)}. \tag{5.5}$$

This limit argument is essentially one used several years ago by Totik [40], the only difference being that we make uniform assumptions (i.e., continuity) on w and conclude uniformity in (5.5) with variable points.

Now use (4.1) with $K^* = K^q$, the CD kernel, for $\mu_q \geq \mu$ by construction. Replace (x, y) by $(x_0 + \alpha/n, x_0 + \beta/n)$. Divide by $\frac{n\rho(x_0)}{w(x_0)}$ and take $n \to \infty$, using (1.15) and (1.16) for μ_q and (5.4) to get

$$\limsup_{n\to\infty} \left| \frac{K_n(x_0 + \frac{\alpha}{n}, x_0 + \frac{\beta}{n})}{K_n(x_0, x_0)} - \frac{\sin \pi\rho_q(x_0)(\beta - \alpha)}{\pi\rho(x_0)(\beta - \alpha)} \right|^2 \leq \frac{\rho_q(x_0)}{\rho(x_0)} \left[1 - \frac{\rho_q(x_0)}{\rho(x_0)}\right]. \tag{5.6}$$

Taking $q \to \infty$ yields the desired limit result. □

As a final remark, we note that we can use the same approximation idea to go from finite gap E's with all rational harmonic measures to general finite gap E's. For it is a result of Bogatyrëv [1], Peherstorfer [25], and Totik [41] that any finite gap E with ℓ gaps can be approximated by rational harmonic measure sets $E_q \supset E$ such that $|E_q \setminus E| \to 0$. The arguments above can get results for general E from the E_q. The point of this remark is that the construction in Section 2 relies on Jost solutions. For E's with rational harmonic measures, the Jacobi parameters are periodic; and Jost solutions can be constructed with Floquet theory rather than the more elaborate methods of [48, 36, 26, 5].

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