STATISTICAL MECHANICS RESULTS IN THE $P(\phi)_2$ QUANTUM FIELD THEORY*

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We apply ideas and methods from classical statistical mechanics to study the $P(\phi)_2$ self-coupled two-dimensional Boson field theory in the Euclidean region. In particular, we consider correlation inequalities of Griffiths type; the thermodynamic limit for the pressure, the average interaction and the entropy; and the equilibrium equations for states associated with a given interaction.

Symanzik [1] has considered quantum field theory in the Euclidean region and investigated its close analogy with classical statistical mechanics (CSM). In recent work [2], Nelsen has clarified the connections between Euclidean and Minkowski field theories, thereby introducing powerful new techniques*3 into constructive quantum field theory [5]. Armed with these techniques, we have confronted the $P(\phi)_2$ field theory with the ideas and methods [6] used to study the thermodynamic limit in statistical mechanics.

The correspondence between Euclidean field theory (EFT) and CSM can be understood from a consideration of the Schwinger functions (Euclidean region Green's functions) for the $P(\phi)_2$ theory,

$$S(x_1, \ldots, x_n) = \frac{\langle \phi(x_1) \ldots \phi(x_n) \exp [-\int : P(\phi(y)) : \mathrm{d}^2 y] \rangle}{\langle \exp [-\int : P(\phi(y)) : \mathrm{d}^2 y] \rangle} . \quad (1)$$

Here P is the polynomial, bounded below, which defines the interaction energy of the theory, and $\langle \ldots \rangle$ is the free expectation value *4. We take the configuration space Q to be, roughly speaking, the underlying space of the Schrödinger representation in which the fields $\phi(x)$, x in $\mathbf{R}^2$, form a maximal set of commuting observables, Then the right-hand side of (1) is just the interacting expectation value of the observable $\phi(x_1) \ldots \phi(x_n)$ with Gibbs density $\exp [-\int :P(\phi(y)): \mathrm{d}^2 y]$ *5. We emphasize that the resulting statistical mechanics is *classical* rather than quantum because the Euclidean fields are commutative. Indeed, given that the "time" zero fields commute, then the Euclidean fields at all points must commute because of Euclidean invariance. After the analytic continuation to the Minkowski region, the fields of course commute only at space-like separated points.

It should be pointed out that eq. (1) is formal; at best one can hope to take the integrals in (1) over a finite region Λ of Euclidean space-"time". Thus we define the interaction in Λ as $U(\Lambda) = \int_\Lambda : P(\phi(y)) : \mathrm{d}^2 y$ and consider the cutoff Schwinger functions $S_\Lambda(x_1, \ldots, x_n) = Z_\Lambda^{-1} \langle \phi(x_1) \ldots \phi(x_n) \exp \{-U(\Lambda)\} \rangle$ where $Z_\Lambda = \langle \exp \{-U(\Lambda)\} \rangle$ is the "partition function". The problem then is to prove the convergence of S_Λ as $\Lambda \to \infty$. This problem of adiabatic switching is the precise analogue of the problem of the existence of the thermodynamic limit in statistical mechanics.

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*3 These techniques have been used to prove the existence of the vacuum energy per unit volume in the $P(\phi)_2$ theories [3]. These techniques are also used as one element in the proof of the remaining Wightman axioms for $P(\phi)_2$ with small coupling constant by Dimock et al. [4].

*4 In the space Q the free expectation value is taken with respect to the Gaussian measure in the fields with covariance defined by the free Green's function, i.e. $\langle \phi(f) \phi(g) \rangle = (f, (\Delta + m^2)^{-1} g)$ (see [2]). We remark that Q is configuration space and not phase space; the conjugate variables may be regarded as being included in the measure. Although we are emphasizing the similarities between EFT and CSM we wish to point out one important difference: in our formulation the interaction is local whereas the measure is not. This is just the reverse situation from CSM. However, the measure connects distant points in an exponentially decreasing way (see [7]).

*5 K. Symanzik realizes the correspondence between EFT and CSM in a somewhat different way from us. He writes the Schwinger functions in terms of n-functions which have a particle structure just like the correlation functions of statistical mechanics.

In this letter we wish to report our major results. Detailed proofs will appear elsewhere [7].

(a) *Griffiths inequalities* [8].

Theorem 1. Let P be an *even* polynomial with positive leading coefficient. Then $S(x_1,\ldots,x_n) \geqslant 0$ and

$$S(x_1,\ldots,x_{n+m}) \geqslant S(x_1,\ldots,x_n)\, S(x_{n+1},\ldots,x_{n+m})$$

for all $x_1,\ldots,x_{n+m}$ in $\mathbf{R}^2$.

Strictly speaking, these inequalities hold only when smeared with positive test functions in the x-variables and with S replaced by S_Λ. They then hold automatically in the limit $\Lambda \to \infty$, provided such limits exist. In the case $P(X) = X^4$, the positivity of S is due to Symanzik [1] although our proof is very different from his. Our proof makes use of a general analysis of Griffiths inequalities by Ginibre [9] and of a lattice approximation to the free Gaussian measure that exhibits its "ferromagnetic" nature *6.

When P is not necessarily even but merely bounded below we can prove inequalities of FKG type [10]. An interesting application of these inequalities and of theorem 1 is that when P is even the physical mass is an increasing function of the coefficient of the quadratic term in P [11].

(b) *Pressure*

Theorem 2. If $\Lambda \to \infty$ in the sense of Fisher [6, p. 14] the following limit exists

$$\alpha_\infty = \lim_{\Lambda \to \infty} \frac{1}{|\Lambda|} \log Z_\Lambda$$

($|\Lambda|$ = volume of the region Λ).

The quantity $\alpha_\infty = \alpha_\infty(P)$ can be regarded as the pressure of the system in the thermodynamic limit and is equal to the negative of the energy per unit volume in the Minkowski theory [3]. The proof of theorem 2 is based on the exponential fall-off of the correlation between distant regions as their separation goes to infinity.

(c) *Entropy*. Following the qualitative suggestions of perturbation theory and of simple solvable models, we consider those states f of EFT which can be described by a family of functions $\{f_\Lambda\}$ in the following sense: For each finite region Λ there is associated a positive normalized function f_Λ which is a function of the fields in Λ. $f(A)$, the expectation in the state f of an observable A associated with the region Λ, is given by $f(A) = \langle A f_\Lambda \rangle$ where $\langle\cdot\rangle$ is the free expectation value. The f_Λ satisfy obvious compatibility conditions.

A state f of this type is called weakly tempered if for some $\alpha < \frac{1}{2}$ it satisfies a bound of the form

$$\log \langle f_\Lambda^2 \rangle \leqslant \exp |\Lambda|^\alpha$$

for large $|\Lambda|$. Given such a state f we define the entropy in the region Λ by $S_\Lambda(f) = -\langle f_\Lambda \log f_\Lambda \rangle$. It turns out that $S_\Lambda(f)$ has the usual property of monotonicity in Λ and satisfies a weak form of subadditivity in Λ.*7

Theorem 3. Let f be a translation invariant weakly tempered state. Then the limit, $s(f)$, of the entropy density $S_\Lambda(f)/|\Lambda|$ exists as $\Lambda \to \infty$ in the sense of Fisher.

If we introduce the (Λ independent!) average interaction

$$\rho(f,P) = \langle U(\Lambda) f_\Lambda \rangle / |\Lambda|$$

then the following inequality holds:

$$s(f) - \rho(f,P) \leqslant \alpha_\infty(P). \tag{2}$$

We expect that the physical vacuum state (or states) associated with a given interaction P will be weakly tempered and yield equality in (2) (global variational principle).

(d) *Equilibrium equations*. Among the most impotant recent discoveries in rigorous statistical mechanics are the DLR equations [12]. These give meaning, independently of limiting procedures, to the statement that a state is an equilibrium state for a given interaction at a particular temperature. In general the DLR equations are equivalent to the variational principle.

Relying on a local version of the Gibbs variational principle [7], we can express the general state f, associated with an interaction P, in the form $f_\Lambda = \exp[-U(\Lambda)]\,\Omega_{\partial\Lambda}$. This expression contains a Gibbsian part $\exp[-U(\Lambda)]$ and a "correction" $\Omega_{\partial\Lambda}$ which is concentrated on the boundary $\partial\Lambda$ of the region Λ and which takes into account the part of the interaction occurring outside Λ. Then the compatibility conditions for the family $\{f_\Lambda\}$ immediately give rela-

*6 E. Nelson has recently shown that the Griffiths inequalities imply that the Schwinger functions S_Λ^D with Dirichlet boundary conditions are increasing functions of Λ, thereby solving the problem of the thermodynamic limit when P is even.

*7 That is, if $\Lambda \subset \Lambda'$ then $S_{\Lambda'}(f) \leqslant S_\Lambda(f)$ and if Λ and Λ' are disjoint, then $S_{\Lambda\cup\Lambda'}(f) \leqslant S_\Lambda(f) + S_{\Lambda'}(f) + r(\Lambda,\Lambda')$ where $r(\Lambda,\Lambda')$ vanishes exponentially in the distance between Λ and Λ'.

tions among the boundary terms $\{\Omega_{\partial\Lambda}\}$ for various regions Λ. These relations are the analogues of the DLR equations in statistical mechanics. In this way one can give meaning to the statement that a family of Schwinger functions is associated with a given interaction P. This notion is independent of any limiting procedure and of any equation of motion for fields.

If the Euclidean analogues of the DLR equations are supplemented by the right boundary conditions for $\Omega_{\partial\Lambda}$ as $\Lambda \to \infty$, then they could represent an effective tool for the construction of the relevant physical states without the use of cutoff procedures. We expect that once the global variational principle is established, then the study of its equivalence with the DLR equations will suggest the right boundary conditions for the latter.

We hope that the work described above will be the first step in a much closer working relationship between constructive field theory and the theory of statistical mechanics of infinite systems.

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