

The Nevai Condition

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Abstract We study Nevai's condition that for orthogonal polynomials on the real line, $K_n(x, x_0)^2 K_n(x_0, x_0)^{-1} d\rho(x) \rightarrow \delta_{x_0}$, where K_n is the Christoffel–Darboux kernel. We prove that it holds for the Nevai class of a finite gap set uniformly on the spectrum, and we provide an example of a regular measure on $[-2, 2]$ where it fails on an interval.

Keywords Orthogonal polynomials · Regular measures · CD kernel

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1 Introduction

This paper studies material on the borderline of the theory of orthogonal polynomials on the real line (OPRL) and spectral theory. Let $d\rho$ be a measure on \mathbb{R} of compact but not finite support, and let $P_n(x, d\rho)$, $p_n(x, d\rho)$ be the standard [15, 44, 46] monic and

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normalized orthogonal polynomials for $d\rho$. Let $\{a_n, b_n\}_{n=1}^\infty$ be the Jacobi parameters defined by

$$xp_n(x) = a_{n+1}p_{n+1}(x) + b_{n+1}p_n(x) + a_n p_{n-1}(x). \tag{1.1}$$

The CD (for Christoffel–Darboux) kernel is defined by

$$K_n(x, y) = \sum_{j=0}^n p_j(x)p_j(y) \tag{1.2}$$

for x, y real. The CD formula (see, e.g., [42]) asserts that

$$K_n(x, y) = \frac{a_{n+1}[p_{n+1}(x)p_n(y) - p_n(x)p_{n+1}(y)]}{x - y}. \tag{1.3}$$

The Christoffel variational principle (see [42]) says that if

$$\lambda_n(x_0) = \min \left\{ \int |Q_n(x)|^2 d\rho(x) \mid \deg Q_n \leq n, Q_n(x_0) = 1 \right\}, \tag{1.4}$$

then

$$\lambda_n(x_0) = \frac{1}{K_n(x_0, x_0)}, \tag{1.5}$$

and the minimizer is given by

$$\tilde{Q}_n(x, x_0) = \frac{K_n(x, x_0)}{K_n(x_0, x_0)}. \tag{1.6}$$

It is quite natural to look at the probability measures (called G_n in Nevai [32])

$$d\eta_n^{(x_0)}(x) = \frac{|\tilde{Q}_n(x, x_0)|^2 d\rho(x)}{\int |\tilde{Q}_n(y, x_0)|^2 d\rho(y)}, \tag{1.7}$$

so that, by (1.5) and (1.6),

$$d\eta_n^{(x_0)} = \frac{|K_n(x, x_0)|^2 d\rho(x)}{K_n(x_0, x_0)}. \tag{1.8}$$

Definition We say $d\rho$ obeys a *Nevai condition* at x_0 if and only if

$$\text{w-lim}_{n \rightarrow \infty} d\eta_n^{(x_0)} = \delta_{x_0}, \tag{1.9}$$

the point mass at x_0 .

(Here $\text{w-lim}_{n \rightarrow \infty} d\nu_n = d\nu$ means $\lim_{n \rightarrow \infty} \int f(x) d\nu_n(x) = \int f(x) d\nu(x)$ for any compactly supported continuous f .)

The name comes from the fact that this condition was studied in the seminal work of Nevai [32], who considered the following:

Definition The *Nevai class* for $[-2, 2]$ is the set of all measures $d\rho$ whose Jacobi parameters obey

$$a_n \rightarrow 1, \quad b_n \rightarrow 0. \tag{1.10}$$

The interval $[-2, 2]$ is relevant since, by Weyl’s theorem on the essential spectrum, $[-2, 2]$ is the derived set of $\text{supp}(d\rho)$, that is, the essential spectrum for the Jacobi matrix of $d\rho$.

Nevai proved the following:

Theorem 1.1 (Nevai [32]) *If $d\rho$ is in the Nevai class for $[-2, 2]$, then the Nevai condition holds for all x_0 in $[-2, 2]$. Indeed, the limit is uniform for x_0 in any compact set $K \subset (-2, 2)$.*

The connection of this to spectral theory comes from the relation to the following condition, sometimes called subexponential growth:

$$\lim_{n \rightarrow \infty} \frac{|p_n(x_0)|^2}{\sum_{j=0}^n |p_j(x_0)|^2} = 0, \tag{1.11}$$

which we will show (see Proposition 2.1) is equivalent to

$$\lim_{n \rightarrow \infty} \frac{(|p_{n-1}(x_0)|^2 + |p_n(x_0)|^2)}{\sum_{j=0}^n |p_j(x_0)|^2} = 0 \tag{1.12}$$

and to

$$\lim_{n \rightarrow \infty} \frac{|p_{n+1}(x_0)|^2}{\sum_{j=0}^n |p_j(x_0)|^2} = 0. \tag{1.13}$$

We will sometimes need

$$0 < A_- \equiv \inf_n a_n \leq \sup_n a_n \equiv A_+ < \infty. \tag{1.14}$$

We note that $A_+ < \infty$ follows from the assumption that $\text{supp}(d\rho)$ is compact, and if $A_- = 0$, then by a result of Dombrowski [14], $d\mu$ is purely singular with respect to Lebesgue measure. Obviously, (1.14) holds for the discrete Schrödinger case, $a_n \equiv 1$.

The relation of (1.11) to Nevai’s condition is direct:

Theorem 1.2 *Let (1.14) hold. Nevai’s condition holds at x_0 if and only if (1.11) holds.*

That (1.11) \Rightarrow (1.9) is due to Nevai. The converse is new here and appears as Theorem 2.2.

Equation (1.11) is, of course, the kind of asymptotic eigenfunction result of interest to spectral theorists and is susceptible to the methods of spectral theory. In particular, Theorem 1.2 shows that for compactly supported measures, the Nevai condition

is intimately connected with the existence of certain natural sequences of approximate eigenvectors for the associated Jacobi matrix (for the relevance of approximate eigenvectors to spectral analysis, see, e.g., [5]). Note that (1.1) and the orthogonality relation say that, for ρ -a.e. x , the sequence $(p_0(x), p_1(x), \dots)$ is a generalized eigenfunction at x for the Jacobi matrix, J , defined by the parameters $\{a_n, b_n\}_{n=1}^\infty$ (namely, it is a polynomially bounded solution of the corresponding eigenvalue equation). Truncations of generalized eigenfunctions are natural candidates for sequences of approximate eigenvectors, and Theorem 1.2 (see (2.13) in particular) says that the Nevai condition at x_0 is equivalent to the requirement that truncations of the generalized eigenfunction at x_0 yield a sequence of approximate eigenvectors for J . In fact, from the point of view of spectral theory, for compactly supported measures, the Nevai condition is a simple restatement of this in the “energy representation” of the Jacobi matrix.

This connection is one reason Nevai’s condition is interesting—it has also been used to relate ratios of $\lambda_n(x_0)$ for the measures $d\rho(x)$ and $e^{g(x)} d\rho(x)$ (Nevai’s motivation in [32]). In this context, it was used by Máté–Nevai–Totik [31] to relate CD kernel asymptotics for orthogonal polynomials on the unit circle (OPUC) and OPRL and to study OP asymptotics when a Szegő condition fails (see [28–30]; see also [44, Sect. 3.10]).

Given the form of the Christoffel variational principle, Nevai’s condition seems like something that must always hold for $x_0 \in \text{supp}(d\mu)$. However, (1.11) also provides a basis for some counter-examples to Nevai’s condition. In Sect. 3 (see Theorem 3.2), we will prove that

Theorem 1.3 *If*

$$\liminf_{n \rightarrow \infty} (|p_n(x_0)|^2 + |p_{n+1}(x_0)|^2)^{1/n} > 1, \tag{1.15}$$

then (1.11) fails.

This was the basis for the first counter-example to (1.11) by Szwarz [47]; in Sect. 3 (Example 3.5), we will see that the Anderson model provides an example of a measure on $[-2, 2]$ for which the Nevai condition fails for Lebesgue a.e. $x_0 \in [-2, 2]$!

Of course, (1.15) is associated with a positive Lyapunov exponent. One might guess that a zero Lyapunov exponent implies (1.11). A main impetus for this paper was our realization that this is not true! Recall that a measure on $[-2, 2]$ is called *regular* if and only if $\lim(a_1 \cdots a_n)^{1/n} = 1$ (see Stahl–Totik [45] or Simon [40] for reviews) and that regular measures have a zero Lyapunov exponent, that is, for quasi-every (namely, outside, possibly, a set of zero logarithmic capacity) $x_0 \in [-2, 2]$,

$$\lim_{n \rightarrow \infty} \frac{1}{n} \log(|p_n(x_0)|^2 + |p_{n+1}(x_0)|^2)^{1/2} = 0. \tag{1.16}$$

In Sects. 4 and 5, we provide two examples of regular measures on $[-2, 2]$ for which Nevai’s condition fails for Lebesgue a.e. x_0 in $\pm(1, 2)$. The example in Sect. 4 will be somewhat simpler but will have no a.c. spectrum, while that in Sect. 5 will have pure a.c. spectrum on $(-1, 1)$.

We also want to discuss extensions of Nevai's theorem, Theorem 1.1. In this regard, we should mention some beautiful work of Nevai–Totik–Zhang [34] and Zhang [51] that already expanded this. The first paper proved uniform convergence on $[-2, 2]$ with an elegant approach; this was extended in the second paper to (a_n, b_n) approaching a periodic limit. Somewhat earlier, Lubinsky–Nevai [26] had proven the Nevai condition for this periodic limit case but only uniformly on compact subsets of the interior of the spectrum. That paper also has results on subexponential growth for some cases of measures that do not have compact support, a subject beyond the scope of this paper. Still later, another proof for the periodic limit case was found by Szwarc [48].

In the past four years, it has become clear that the proper analog of the Nevai class for the periodic case is not approach to a fixed periodic element but approach to an isospectral torus. We want to prove that one can do this not only in the periodic case, but also on approach to the isospectral torus of, in general, almost periodic Jacobi matrices that occurs in the general finite gap case. We will also study the Nevai condition on the a.c. spectrum of ergodic Jacobi matrices and for the Fibonacci model.

From what we have said so far, it appears to be a mixed verdict on the Nevai condition, since we have results on when it occurs and also on when it doesn't. But we want to reinterpret the negative results. These all provide examples where it is not true that the Nevai condition holds *everywhere* on the topological support of $d\rho$ or even that it fails for Lebesgue a.e. x_0 on the support. We believe it is likely that the following more refined property could be true:

Conjecture 1.4 For any measure, $d\rho$, with compact support, the Nevai condition holds for $d\rho$ -a.e. x_0 in $\text{supp}(d\rho)$.

Here is a summary of the contents of the rest of the paper. In Sect. 2, we prove Theorem 1.2. In Sect. 3, we discuss cases with a positive Lyapunov exponent, including a proof of Theorem 1.3. Sections 4 and 5 present the details of the examples of regular measures where the Nevai condition fails for Lebesgue a.e. x_0 in a particular open subset. Section 6 presents our version of the NTZ approach, as preparation for our discussion of the finite gap Nevai class in Sect. 7. In Sect. 8, we will relate (1.11) to the absence of ℓ^2 solutions of classical right limits and so recover the results of Sect. 7 and even more (including Fibonacci models). The ideas of Sect. 8 seem to be more generally applicable than Sect. 6, but the constants in Sect. 6 are more explicit. Section 9 has some final remarks, including a discussion of Conjecture 1.4 and a second conjecture (Conjecture 9.5). Section 9 also notes that if $\frac{1}{n}K_n(x_0, x_0)$ has a nonzero limit, then the Nevai condition holds, and so links this to recent work on that question. Thus, for those interested in ergodic Schrödinger operators, by the end of this paper, we will have proven the Nevai condition in the Fibonacci model uniformly on the spectrum and for ergodic models with a.c. spectrum, Lebesgue a.e. on the essential support of the a.c. spectrum.

2 Subexponential Decay

We begin with the equivalence of (1.11), (1.12), (1.13), and more:

Proposition 2.1 *Let c_0, c_1, c_2, \dots be a sequence of non-negative numbers and*

$$S_n = \sum_{j=0}^n c_j. \tag{2.1}$$

Then the following are equivalent:

$$(i) \quad c_n/S_n \rightarrow 0, \tag{2.2}$$

$$(ii) \quad S_n/S_{n+1} \rightarrow 1, \tag{2.3}$$

$$(iii) \quad c_{n+1}/S_n \rightarrow 0, \tag{2.4}$$

$$(iv) \quad (c_n + c_{n+1})/S_n \rightarrow 0, \tag{2.5}$$

$$(v) \quad (c_{n-1} + c_n)/S_n \rightarrow 0. \tag{2.6}$$

Remarks 1. The relevance, of course, is to

$$c_n = p_n(x_0)^2. \tag{2.7}$$

2. If $c_n = e^{n^2}$, $c_n/S_n \rightarrow 1$ and $c_{n-1}/S_n \rightarrow 0$, so (i) is not equivalent to $c_{n-1}/S_n \rightarrow 0$.

Proof (i) \Leftrightarrow (ii). We have

$$S_{n-1}/S_n = 1 - c_n/S_n, \tag{2.8}$$

so

$$(i) \Leftrightarrow S_{n-1}/S_n \rightarrow 1, \tag{2.9}$$

which, by renumbering indices, is equivalent to (ii).

(ii) \Leftrightarrow (iii). We have

$$S_{n+1}/S_n = 1 + c_{n+1}/S_n, \tag{2.10}$$

from which (ii) is equivalent to (iii).

(iv) or (v) \Rightarrow (i). This is immediate, since for $j = \pm 1$,

$$0 \leq c_n/S_n \leq (c_n + c_{n+j})/S_n. \tag{2.11}$$

(i) \Rightarrow (iv). This is immediate from (i) \Rightarrow (iii).

(i) \Rightarrow (v). Since (i) $\Rightarrow S_{n-1}/S_n \rightarrow 1$, (i) $\Rightarrow c_{n-1}/S_n \rightarrow 0$, from which (v) is immediate. □

Theorem 2.2 (\Rightarrow Theorem 1.2) *Suppose that (1.14) holds. Nevai’s condition is equivalent to*

$$\int (x - x_0)^2 d\eta_n^{(x_0)}(x) \rightarrow 0, \tag{2.12}$$

and (2.12) holds if and only if (1.11) holds.

Proof By (1.8), (1.3), and the orthogonality of p_n to p_{n+1} ,

$$\int (x - x_0)^2 d\eta_n^{(x_0)}(x) = \frac{a_{n+1}^2 [p_n(x_0)^2 + p_{n+1}(x_0)^2]}{K_n(x_0, x_0)}. \tag{2.13}$$

By (1.14) and (i) \Leftrightarrow (iv) in Proposition 2.1,

$$(2.12) \Leftrightarrow \frac{p_n(x_0)^2 + p_{n+1}(x_0)^2}{K_n(x_0, x_0)} \rightarrow 0 \Leftrightarrow (1.11).$$

For measures, $\{\nu_n\}$, all supported in a fixed compact, $\nu_n \xrightarrow{w} \delta_{x_0} \Leftrightarrow \int (x - x_0)^2 d\nu_n \rightarrow 0$. □

In the example in Remark 2 after Proposition 2.1, $(c_{n-2} + c_{n-1})/S_n \rightarrow 0$ but (2.2) fails. However, this cannot happen for the case $c_n = p_n(x_0)$. Note that if (1.14) holds,

$$|p_n(x_0)| \leq A_-^{-1} [A_+ + |x_0| + \sup_n |b_n|] [|p_{n-2}(x_0)| + |p_{n-1}(x_0)|], \tag{2.14}$$

since

$$p_n(x_0) = a_n^{-1} ((x_0 - b_n)p_{n-1}(x_0) - a_{n-1}p_{n-2}(x_0)). \tag{2.15}$$

Also note the following obvious fact:

Proposition 2.3 *Under the hypotheses and notation of Proposition 2.1, if there is a constant K so that*

$$c_n \leq K(c_{n-2} + c_{n-1}), \tag{2.16}$$

then (2.2) is equivalent to

$$\frac{c_{n-2} + c_{n-1}}{S_n} \rightarrow 0. \tag{2.17}$$

3 Positive Lyapunov Exponent

We begin by proving a contrapositive of Theorem 1.3:

Proposition 3.1 *If (1.11) holds, then*

$$\limsup_{n \rightarrow \infty} K_n(x_0, x_0)^{1/n} \leq 1. \tag{3.1}$$

So, in particular,

$$\limsup_{n \rightarrow \infty} (|p_n(x_0)|^2 + |p_{n+1}(x_0)|^2)^{1/n} \leq 1. \tag{3.2}$$

Proof Given ε , pick N so that for $n \geq N$,

$$\frac{|p_n(x_0)|^2}{K_n(x_0, x_0)} \leq \varepsilon. \tag{3.3}$$

Then

$$\frac{K_{n-1}(x_0, x_0)}{K_n(x_0, x_0)} \geq 1 - \varepsilon, \tag{3.4}$$

so, for $n \geq N$,

$$K_n(x_0, x_0) \leq (1 - \varepsilon)^{-1} K_{n-1}(x_0, x_0), \tag{3.5}$$

which implies

$$\limsup K_n(x_0, x_0)^{1/n} \leq (1 - \varepsilon)^{-1}. \tag{3.6}$$

Since ε is arbitrary, (3.1) holds. Thus, since

$$\left(|p_n(x_0)|^2 + |p_{n+1}(x_0)|^2\right)^{1/n} \leq K_{n+1}(x_0, x_0)^{1/n}, \tag{3.7}$$

we get (3.2). □

Theorem 3.2 (\equiv Theorem 1.3) (1.15) \Rightarrow not (1.11).

Proof As noted, this is a contrapositive of the last statement in Proposition 3.1. □

Recall that the transfer matrix at x_0 is defined by

$$T_n(x_0) = A_n(x_0) \cdots A_1(x_0), \tag{3.8}$$

$$A_j(x_0) = \frac{1}{a_j} \begin{pmatrix} x_0 - b_j & -1 \\ a_j^2 & 0 \end{pmatrix}, \tag{3.9}$$

so $\det(A_j) = \det(T_n) = 1$ and

$$\begin{pmatrix} p_n(x_0) \\ a_n p_{n-1}(x_0) \end{pmatrix} = T_n(x_0) \begin{pmatrix} 1 \\ 0 \end{pmatrix}. \tag{3.10}$$

Recall also that one says the Lyapunov exponent exists if

$$\gamma(x_0) = \lim_{n \rightarrow \infty} \frac{1}{n} \log \|T_n(x_0)\| \tag{3.11}$$

exists. The Ruelle–Osceledec theorem (see, e.g., [39, Theorem 10.5.29]) says that if $\gamma(x_0) > 0$, then there is a one-dimensional subspace, V , of \mathbb{C}^2 , so $u \in V \setminus \{0\}$ implies

$$\lim_{n \rightarrow \infty} \|T_n(x_0)u\|^{1/n} = e^{-\gamma(x_0)}, \tag{3.12}$$

and if $u \in \mathbb{C}^2 \setminus V$, then

$$\lim_{n \rightarrow \infty} \|T_n(x_0)u\|^{1/n} = e^{\gamma(x_0)}. \tag{3.13}$$

Thus:

Corollary 3.3 *Let (1.14) hold. If $\gamma(x_0) > 0$, then either x_0 is a pure point of $d\rho$, namely, $\rho(\{x_0\}) > 0$, or else (1.11) fails.*

Proof If (3.12) holds for $u = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then by (3.10), $|p_n(x_0)| \leq C e^{-\gamma(x_0)n/2}$, so $p_n \in \ell^2$, and x_0 is a pure point. So if x_0 is not a pure point, then (3.13) holds, and since $A_- > 0$, (3.13) implies

$$\lim_{n \rightarrow \infty} (|p_n(x_0)|^2 + |p_{n+1}(x_0)|^2)^{1/n} = e^{\gamma(x_0)} > 1, \tag{3.14}$$

and therefore (1.11) fails. □

Example 3.4 (Szwarc [47]) This example is mainly of historical interest. This is a modification of [47] but uses the key notion of having an isolated point of $\sigma_{\text{ess}}(J)$, which is not an eigenvalue. We begin by noting that the whole-line Jacobi matrix, J_∞ , with $a_n \equiv 1$, $b_n = 0$ ($n \neq 0$) and $b_0 = \frac{3}{2}$ has $E = \frac{5}{2}$ ($= 2 + \frac{1}{2}$) as an eigenvalue with eigenfunction $u_n = (\frac{1}{2})^{|n|}$, so $\sigma(J_\infty) = [-2, 2] \cup \{\frac{5}{2}\}$.

Now let J be the one-sided Jacobi matrix with $a_n \equiv 1$ and

$$b_n = \begin{cases} \beta & n = 1, \\ \frac{3}{2} & n = k^2, k = 2, 3, \dots, \\ 0 & n \neq k^2 \text{ all } k, \end{cases} \tag{3.15}$$

where β will be adjusted below.

Standard right-limit theorems (see, e.g., [22]) imply

$$\sigma_{\text{ess}}(J) = [-2, 2] \cup \left\{ \frac{5}{2} \right\},$$

so, in particular, $\frac{5}{2} \in \sigma(J)$. Moreover, since the nonzero b_n 's are of zero density, it is easy to see that

$$\gamma\left(\frac{5}{2}\right) = \log\left(\text{spectral radius of } \begin{pmatrix} \frac{5}{2} & -1 \\ 1 & 0 \end{pmatrix}\right) = \log(2), \tag{3.16}$$

since $\begin{pmatrix} \frac{5}{2} & -1 \\ 1 & 0 \end{pmatrix}$ has eigenvalues 2 and $\frac{1}{2}$.

It is easy to see that there is exactly one choice of β for which $\frac{5}{2}$ is an eigenvalue of J . For any other choice, Corollary 3.3 is applicable, and so the Nevai condition fails at $\frac{5}{2} \in \sigma(J)$.

Example 3.5 Let $a_n \equiv \frac{1}{2}$ and $b_n(\omega)$ be i.i.d. random variables uniformly distributed in $[-1, 1]$. This is an Anderson model for which it is well known that for a.e. choice of ω (see, e.g., [7]), the associated measure $d\rho_\omega$ has pure point spectrum filling $[-2, 2]$. Moreover, for a.e. ω and quasi-every $x_0 \in [-2, 2]$, one has a Lyapunov exponent $\gamma(x_0) > 0$. Since the set of eigenvalues is countable, for quasi-every $x_0 \in [-2, 2]$,

Corollary 3.3 is applicable. Thus, $\sigma(J_\omega) = [-2, 2]$, but for quasi-every $x_0 \in [-2, 2]$, the Nevai condition fails.

See Zygmunt [52] for some other examples.

4 A Regular Measure for Which the Nevai Condition Fails on a Set of Positive Lebesgue Measure

Example 4.1 Let J be a Jacobi matrix with $b_n \equiv 0$ and a_n described as follows. Partition $\{1, 2, \dots\}$ into successive blocks $A_1, C_1, A_2, C_2, \dots$, where

$$\#(A_j) = 3^{j^2}, \quad \#(C_j) = 2^{j^2}. \tag{4.1}$$

On A_j , $a_n \equiv 1$ and on C_j , $a_n \equiv \frac{1}{2}$. Since the 3^{j^2} blocks are much larger than the 2^{j^2} blocks,

$$\lim_{n \rightarrow \infty} (a_1 \cdots a_n)^{1/n} = 1, \tag{4.2}$$

and since $|a_n| \leq 1$, $\|J\| \leq 2$, so $\sigma(J) \subset [-2, 2]$. Also, $[-2, 2] \subset \sigma(J)$ can be seen immediately by constructing approximate eigenvectors, supported in blocks where $a_n = 1$, and using Weyl’s criterion [36, Theorem VII.12]. Thus, J is regular for $[-2, 2]$, but, of course, not in Nevai class, since $a_n \not\rightarrow 1$.

We will prove that:

Theorem 4.2 *For the Jacobi matrix J , described above, the Nevai condition fails for Lebesgue a.e. $x_0 \in [-2, 2] \setminus [-1, 1]$.*

The intuition, which we will implement, goes as follows: For x_0 in the specified set, in A_j regions, solutions of the eigenfunction equation are linear combinations of plane waves. So in such regions, $p_n(x_0)$ hardly grows or decays. In C_j regions, they are linear combinations of growing and decaying exponentials, so usually, the growing exponentials will win and the $p_n(x_0)$ will grow exponentially. C_{j+1} is much bigger than C_j (indeed, $\#(C_{j+1}) = 2 \times 4^j \#(C_j)$), so at the center of C_{j+1} , $|p_n(x_0)|^2$ will be much bigger than $\sum_{k \in \cup_{\ell=1}^j (A_\ell \cup C_\ell) \cup A_{j+1}} |p_k(x_0)|^2$ and comparable to $\sum_{\substack{k \in C_{j+1} \\ k \leq n-1}} |p_k(x_0)|^2$, which will prevent (1.11) from holding.

We will say more about the general strategy shortly, but we first implement the initial step:

Proposition 4.3 *Suppose for a given x_0 , there are C, D and $\alpha, \beta > 0$ so that*

- (i) *For all j and $n \in C_j \cup A_{j+1}$,*

$$|p_n(x_0)| \leq C^{j+1} \exp(\alpha 2^{j^2}); \tag{4.3}$$

(ii) For all large j and n_j , the “center” of C_j , we have

$$\left(|p_{n_j-1}(x_0)|^2 + |p_{n_j}(x_0)|^2\right)^{1/2} \geq D^{-(j+1)} \exp(\beta 2^{j^2}). \tag{4.4}$$

Then (1.11) (and so the Nevai condition) fails at x_0 .

Remarks 1. In fact, the proof shows that (4.4) must only hold for infinitely many j 's.

2. Since $\#(C_j)$ is even, it does not have a strict center. By “center,” we mean one half unit prior to the midpoint.

Proof Suppose (1.11) holds. Then, since (1.13) holds, for any $\varepsilon > 0$, there exists $N(\varepsilon)$, so for $n \geq N(\varepsilon)$,

$$|p_{n+1}(x_0)|^2 \leq \varepsilon \sum_{k=1}^n |p_k(x_0)|^2, \tag{4.5}$$

so

$$\sum_{k=1}^{n+1} |p_k(x_0)|^2 \leq (1 + \varepsilon) \sum_{k=1}^n |p_k(x_0)|^2, \tag{4.6}$$

and thus, for $n \geq m \geq N(\varepsilon)$,

$$|p_{n-1}(x_0)|^2 + |p_n(x_0)|^2 \leq (1 + \varepsilon)^{n-m} \sum_{k=1}^m |p_k(x_0)|^2. \tag{4.7}$$

Now, suppose C_j is such that its left-most point, $m_j + 1$, has $m_j \geq N(\varepsilon)$, and let n_j be the center of C_j . Then by (4.3),

$$\sum_{k=1}^{m_j} |p_k(x_0)|^2 \leq C^{2j} \exp(2\alpha 2^{(j-1)^2}) \left[\sum_{k=1}^j 2^{k^2} + 3^{k^2} \right], \tag{4.8}$$

and by (4.4),

$$|p_{n_j-1}(x_0)|^2 + |p_{n_j}(x_0)|^2 \geq D^{-2(j+1)} \exp(2\beta 2^{j^2}). \tag{4.9}$$

Pick ε so that $(1 + \varepsilon) \leq e^{2\beta}$. Since $n_j - m_j = \frac{1}{2}(2^{j^2})$, (4.7) says that as long as $m \geq N(\varepsilon)$,

$$D^{-2(j+1)} \exp(2\beta 2^{j^2}) \leq \exp(\beta 2^{j^2}) C^{2j} \exp(2\alpha 2^{(j-1)^2}) (2j 3^{j^2}). \tag{4.10}$$

Since $2\beta > \beta + 4 \times 4^{-j} \alpha$ for j large, (4.10) cannot hold for large j . This contradiction implies that (4.5) cannot hold for this value of ε , and thus, (1.11) fails. \square

The upper bound, (4.3), will be easy from transfer matrix arguments. The lower bound, (4.4), is much more subtle. Indeed, it implies that for any ℓ , asymptotically

$|p_{n_j}(x_0)| \geq n_j^\ell$. On the other hand, for a.e. x_0 with respect to a spectral measure, $p_n(x_0)$ is polynomially bounded. Thus, (4.3) must fail for a dense set of x_0 's! Fortunately, this kind of problem has been faced before in spectral theory contexts, and we will be able to borrow a technique from Jitomirskaya–Last [17]. We turn first to the upper bound.

For $x_0 \in (-2, 2) \setminus [-1, 1]$, define $\theta(x_0), \eta(x_0)$ by

$$2 \cos \theta(x_0) = x_0, \quad \cosh \eta(x_0) = |x_0|. \tag{4.11}$$

Let $Q(x_0), R(x_0)$ be the one-unit transfer matrices for $b_n \equiv 0, a_n \equiv 1$ and $b_n \equiv 0, a_n \equiv \frac{1}{2}$. Then $Q(x_0)$ has $e^{\pm i\theta_0}$ as eigenvalues, and $R(x_0)$ has eigenvalues $e^{\pm \eta(x_0)}$ if $x_0 > 0$ and $-e^{\pm \eta(x_0)}$ if $x_0 < 0$. It follows that for some constants $c(x_0), d(x_0)$,

$$\|Q(x_0)^k\| \leq c(x_0), \tag{4.12}$$

$$\|R(x_0)^k\| \leq d(x_0)e^{k\eta(x_0)}, \tag{4.13}$$

with c, d bounded uniformly on compacts of $(-2, 2) \setminus [-1, 1]$. This lets us prove that:

Proposition 4.4 *For any compact subset K of $(-2, 2) \setminus [-1, 1]$, there are constants C, α so that for all n, j and $x_0 \in K$ with $n \in C_j \cup A_{j+1}$, we have*

$$\|T_n(x_0)\| \leq C^{j+1} \exp(\alpha 2^{j^2}). \tag{4.14}$$

Remark Since $p_n(x_0)$ is a matrix element of T_n , we immediately have (4.3).

Proof For $n \in A_{j+1}$, $T_n(x_0)$ is a product of $(j + 1)$ factors of products of $Q(x_0)$ and j factors of products of $R(x_0)$. Thus, by (4.12) and (4.13),

$$\|T(x_0)\| \leq c(x_0)^{j+1} d(x_0)^j \exp\left(\eta(x_0) \sum_{\ell=1}^j 2^{\ell^2}\right). \tag{4.15}$$

For $n \in C_j$, the estimate is similar, but $c(x_0)^{j+1}$ is replaced by $c(x_0)^j$ and $\sum_{\ell=1}^j 2^{\ell^2}$ by a smaller sum.

Since c, d , and η are bounded on K and

$$\sum_{\ell=1}^j 2^{\ell^2} \leq 2^{j^2} \left[1 + \frac{1}{2} + \frac{1}{4} + \dots\right] = 2 \times 2^{j^2}, \tag{4.16}$$

we obtain (4.14). □

To get the lower bound, following Jitomirskaya–Last [17], we need to consider Weyl solutions and Green’s functions. For $\text{Im } z > 0$, there is a unique solution $u_n^+(z)$ of

$$a_n u_{n+1} + (b_n - z)u_n + a_{n-1}u_{n-1} = 0 \tag{4.17}$$

defined for $n = 1, 2, \dots$ (with $a_0 \equiv 1$) which is ℓ^2 at infinity and normalized by

$$u_0^+(z) = -1. \tag{4.18}$$

This is the Weyl solution. The spectral theorist’s Green’s function (different from the Green’s function of potential theory!) is defined for $n, m \geq 1$ by

$$G_{nm}(z) = \langle \delta_n, (J - z)^{-1} \delta_m \rangle. \tag{4.19}$$

Then $G_{nm} = G_{mn}$ and, for $1 \leq n \leq m$,

$$G_{nm}(z) = p_{n-1}(z)u_m^+(z). \tag{4.20}$$

As usual, the Wronskian is constant and, by $a_0 \equiv 1$ and (4.18) plus $p_{-1} = 0, p_0 = 1$, this constant is 1. So for all $n = 0, 1, 2, \dots$,

$$a_n(u_{n+1}^+(z)p_{n-1}(z) - u_n^+(z)p_n(z)) = 1. \tag{4.21}$$

By (4.19), G_{nm} is the Borel transform of a signed measure, and so for Lebesgue a.e. x_0 , it has boundary values $G_{nm}(x_0 + i0)$. In particular, since $p_0 = 1, u_n^+ = G_{1n}$ has a.e. boundary values, $u_n^+(x_0 + i0)$. Also (4.20) and (4.21) still hold for $z = x_0 + i0$.

In particular, for our example where a_n is 1 or $\frac{1}{2}$, (4.21) and the Schwarz inequality imply that

$$(p_n(x_0)^2 + p_{n-1}(x_0)^2) \geq (|u_{n+1}^+(x_0 + i0)|^2 + |u_n^+(x_0 + i0)|^2)^{-1}. \tag{4.22}$$

To get an exponentially growing lower bound on $p_n(x_0)^2 + p_{n-1}(x_0)^2$, we only need to get an exponentially decaying upper bound on $|u_n^+(x_0 + i0)|$ and $|u_{n+1}^+(x_0 + i0)|$.

Now fix $1 < k < \ell < \infty$ and define \tilde{J} to be the Jacobi matrix obtained by replacing a_k and a_ℓ by 0. Thus, under

$$\begin{aligned} \ell^2(\{1, 2, \dots\}) &= \ell^2(\{1, \dots, k\}) \oplus \ell^2(\{k + 1, \dots, \ell\}) \\ &\oplus \ell^2(\{\ell + 1, \ell + 2, \dots\}), \end{aligned} \tag{4.23}$$

$$\tilde{J} = J_L \oplus J_M \oplus J_R \tag{4.24}$$

(L, M, R for left, middle, right).

Proposition 4.5 *Let \tilde{G} be the Green’s function for \tilde{J} . Let*

$$k + 1 \leq n \leq \ell. \tag{4.25}$$

Then

$$G_{1n} = -a_k G_{1k} \tilde{G}_{k+1n} - a_\ell G_{1\ell+1} \tilde{G}_{\ell n}. \tag{4.26}$$

Proof Define

$$\Gamma = J - \tilde{J}. \tag{4.27}$$

Then

$$(J - z)^{-1} = (\tilde{J} - z)^{-1} - (J - z)^{-1} \Gamma (\tilde{J} - z)^{-1}, \tag{4.28}$$

so

$$G_{1n} = \tilde{G}_{1n} - \sum_{m,r} G_{1m} \Gamma_{mr} \tilde{G}_{rn}. \tag{4.29}$$

Since 1 and n lie in different blocks in the direct sum (4.24), $\tilde{G}_{1n} = 0$. Γ is a rank four operator; since $\tilde{G}_{kn} = \tilde{G}_{\ell+1n} = 0$, two terms in the sum in (4.29) vanish. The result is (4.26). \square

Proposition 4.6 *Let $J^{(k)}$ be the $k \times k$ matrix with 0's on the diagonal and $\frac{1}{2}$ in each of the two principal off-diagonals. Let $G^{(k)}(z)$ be the matrix $(J^{(k)} - z)^{-1}$. Then for $x_0 \notin [-1, 1]$ and $m \leq n$,*

$$G_{mn}^{(k)}(x_0) = \frac{2(w^{-m} - w^m)(w^{-(k+1-n)} - w^{(k+1-n)})}{(w^{-1} - w)(w^{-(k+1)} - w^{(k+1)})}, \tag{4.30}$$

where $|w| > 1$ and solves

$$w^{-1} + w = 2x_0. \tag{4.31}$$

In particular, for any compact $K \in (-2, 2) \setminus [-1, 1]$, there are $\gamma > 0$ and C so that for all $x_0 \in K$ and all $k = 2r$,

$$|G_{1n}^{(2r)}(x_0)| \leq C e^{-\gamma r} \tag{4.32}$$

for

$$n = r, r - 1. \tag{4.33}$$

Proof Since w^n and w^{-n} solve

$$\frac{1}{2} (u_{n+1} + u_{n-1}) = x_0 u_n, \tag{4.34}$$

$w^{-n} - w^n$ solves (4.34) with $u_0 = 0$ boundary condition, while $w^{-(k+1-n)} - w^{(k+1-n)}$ solves it with $u_{k+1} = 0$ boundary condition. The numerator in (4.30) is twice their product, and the denominator twice their Wronskian, proving (4.30). Then (4.32) follows by noting that the dominant term in the numerator is w^r , while in the denominator, w^{2r} . \square

Thus, in (4.26), where $k + 1$ and ℓ are taken to be the edge of a C_j block and n to be the center or one less, the \tilde{G} terms are exponentially small. We thus need estimates on the set of x_0 for which $G_{1k}(x_0)$ can be large. Here we recall Loomis's theorem:

Theorem 4.7 (Loomis [23]) *Let μ be a complex measure on \mathbb{R} of finite total variation $\|\mu\|$. Let*

$$F_\mu(x) = \lim_{\varepsilon \downarrow 0} \int \frac{d\mu(y)}{y - (x + i\varepsilon)}, \tag{4.35}$$

which exists for Lebesgue a.e. x . Then with $|\cdot| = \text{Lebesgue measure}$,

$$|\{x \mid |F_\mu(x)| > M\}| \leq \frac{C\|\mu\|}{M} \tag{4.36}$$

for a universal constant C .

Remarks 1. The history is complicated and is partly described in Loomis [23]. The result for measures μ that are absolutely continuous is due earlier to Kolmogorov.

2. For $d\mu = f(x) dx$ with $f \in L^1$, the optimal constant was found by Davis [12]. For purely singular positive measures, the result is essentially due to Boole [2] with constant $C = 2$, which is optimal—indeed, one has equality.

3. In [17], they only used Boole’s equality, since their measure is purely singular. That is true here also, but not in the next section. In any event, it is useful to know that pure singularity is not needed a priori, although it follows on $[-2, 2] \setminus [-1, 1]$ from the estimates here.

Proposition 4.8 *Let $n_1 < n_2 < \dots$ be an arbitrary sequence of indices. Let $\delta > 0$. Then for a.e. $x_0 \in \mathbb{R}$, $\exists J(x_0)$, so that*

$$j > J(x_0) \implies |G_{1n_j}(x_0 + i0)| \leq e^{\delta j}. \tag{4.37}$$

Proof This is a standard Borel–Cantelli argument. Let χ_j be the characteristic function of $\{x_0 \mid |G_{1n_j}(x_0)| > e^{\delta j}\}$. Since $G_{1n_j}(x_0 + i0)$ is of the form (4.35) for a measure of variation at most 1,

$$\int \chi_j(x) dx \leq C e^{-\delta j}. \tag{4.38}$$

Thus,

$$\int \sum_{j=1}^\infty \chi_j(x) dx < \infty, \tag{4.39}$$

which implies that for a.e. x_0 , $\sum_{j=1}^\infty \chi_j(x_0) < \infty$. Since each $\chi_j(x_0)$ is 0 or 1, only finitely many are nonzero, that is, for all large j , (4.37) holds. \square

Proof of Theorem 4.2 By Propositions 4.3 and 4.4, we only need to prove that (4.4) holds for Lebesgue a.e. x_0 . By (4.22), it suffices to prove exponentially decaying upper bounds on $u_{n_j}^+, u_{n_j+1}^+$ (for the same n_j as (4.4)). As noted, $u_n^+ = G_{1n}$, so it suffices to prove exponentially decaying upper bounds on G_{1n_j}, G_{1n_j+1} .

We use (4.26) with k, ℓ the lower and upper edge of the C_j block. By Proposition 4.6,

$$|\tilde{G}_{k+1n_j}| + |\tilde{G}_{k+1n_j+1}| + |\tilde{G}_{\ell n_j}| + |\tilde{G}_{\ell n_j+1}| \leq C \exp(-\gamma 2^{j^2}) \tag{4.40}$$

for some $\gamma > 0$.

Thus, the result follows from Proposition 4.8, for eventually each of the complementary Green’s functions in (4.26) is bounded by $\exp(\frac{1}{2}\gamma j) \leq \exp(\frac{1}{2}\gamma 2^{j^2})$, and (4.4) holds with $\beta = \frac{1}{2}\gamma$. □

Finally, we want to note that J has a two-sided right limit which has $a_n = \frac{1}{2}$ for $n \leq -1$ and $a_n = 1$ for $n \geq 0$. There is no set of positive Lebesgue measures on which J is reflectionless, so, by a theorem of Remling [37], J has purely singular spectrum.

5 A Regular Measure with Some A.C. Spectrum

As we noted at the end of the last section, Example 4.1 has no a.c. spectrum. Of course, if Conjecture 1.4 is true, then Lebesgue a.e. on the a.c. spectrum, the Nevai condition holds. So an example like Example 4.1 cannot have a.c. spectrum on $[-2, 2] \setminus [-1, 1]$, but it can on $[-1, 1]$. The example in this section shows that a.c. spectrum is indeed possible on $[-1, 1]$.

Example 5.1 Let J be a Jacobi matrix with $b_n \equiv 0$ and a_n described as follows: Partition $\{1, 2, \dots\}$ into successive blocks $A_1, B_1, C_1, D_1, A_2, B_2, \dots$, where

$$\#(A_j) = 3^{j^2}, \quad \#(C_j) = 2^{j^2}, \quad \#(B_j) = \#(D_j) = j^6 - 1. \tag{5.1}$$

On A_j , $a_n \equiv 1$; on C_j , $a_n \equiv \frac{1}{2}$; and on B_j and D_j , $\log(a_n^2)$ linearly interpolates from $\log(\frac{1}{4})$ to $\log(1)$; that is, for $n \in B_j$,

$$\frac{a_n^2}{a_{n-1}^2} = c_j, \tag{5.2}$$

and for $n \in D_j$,

$$\frac{a_{n-1}^2}{a_n^2} = c_j, \tag{5.3}$$

where

$$c_j^{j^6} = \frac{1}{4}, \tag{5.4}$$

so that

$$1 - c_j = kj^{-6} + o(j^{-6}) \tag{5.5}$$

for a suitable nonzero constant k . In particular,

$$|1 - c_j| \leq E_0 j^{-6} \tag{5.6}$$

for some E_0 .

As in Example 4.1, this J is regular with spectrum $[-2, 2]$. We will prove that:

Theorem 5.2 *On $[-2, 2] \setminus [-1, 1]$, J has purely singular spectrum and for Lebesgue a.e. x_0 in this set, the Nevai condition fails. On $(-1, 1)$, J has purely a.c. spectrum, and for all $x_0 \in (-1, 1)$, the Nevai condition holds uniformly on compact subsets.*

The technical key to the new elements of this example is:

Theorem 5.3 *For $x_0 \in (-1, 1)$, let $u_n(x_0, \theta_0)$ be the solution of (4.17) (with $a_0 \equiv 1$ and $z = x_0$) for $n = 1, 2, \dots$, with*

$$u_0 = \cos \theta_0, \quad u_1 = \sin \theta_0. \tag{5.7}$$

Then for any compact set $K \subset (-1, 1)$, there is a constant, C , so that for all $x_0 \in K$, all θ_0 and all n ,

$$|u_n(x_0, \theta_0)| \leq C. \tag{5.8}$$

Proof of Theorem 5.2 given Theorem 5.3 J has as one of its right limits J_r , the two-sided matrix with $b_n \equiv 0$, $a_n \equiv \frac{1}{2}$ whose a.c. spectrum is $\Sigma_{ac}(J_r) = [-1, 1]$. By a theorem of Last–Simon [21], $\Sigma_{ac}(J) \subset [-1, 1]$, so J has purely singular spectrum on $[-2, 2] \setminus [-1, 1]$. The results on this set for the Nevai condition follow the arguments in Sect. 4 without change.

Theorem 5.3 implies that the transfer matrix $T_n(x_0)$ is uniformly bounded in n and $x_0 \in K \subset (-1, 1)$ compact. Carmona’s formula (see, e.g., [39, Theorem 10.7.5]; also [4, 21, 38, 41]) then implies the spectrum is purely a.c. on $(-1, 1)$.

The fact that the transfer matrix is bounded also implies $p_n(x_0)^2$ bounded above, and, given constancy of the Wronskian (i.e., $\det(T_n) = 1$), uniform lower bounds on $p_n(x_0)^2 + p_{n+1}(x_0)^2$. Thus, on $(-1, 1)$,

$$\frac{p_n(x_0)^2}{\sum_{j=0}^n p_j(x_0)^2} \leq \frac{C}{n} \rightarrow 0, \tag{5.9}$$

proving (1.11). □

The situation we need to control for Theorem 5.3 has much in common with those studied by Kiselev–Last–Simon [19], and their techniques will work here. We note that in our situation, $\sum_n (a_{n+1} - a_n)^2 + (b_{n+1} - b_n)^2 < \infty$, a general condition studied recently by Denisov [13] but under the additional assumptions that $a_n \equiv 1$, $b_n \rightarrow 0$. It would be interesting to see if one can extend his ideas to this context (see Conjecture 9.5 and the discussion following it below).

We depend on the EFGP transform, as does [19], but we need to allow modification for our case where a_n is not identically one, as it is in [19]. Since $b_n \equiv 0$ for us, we state the equations for that case. One defines R_n, θ_n by

$$R_n \sin(\theta_n) = a_n u_n \sin(k_n), \tag{5.10}$$

$$R_n \cos(\theta_n) = a_n (u_{n+1} - u_n \cos(k_n)), \tag{5.11}$$

where k_n is given by

$$2 \cos(k_n) = \frac{x_0}{a_n}. \tag{5.12}$$

We note, since $a_n \geq \frac{1}{2}$ and $\sup_{x_0 \in K} |x_0| < 1$, that uniformly for $x_0 \in K$ and all n ,

$$\varepsilon \leq k_n \leq \pi - \varepsilon \tag{5.13}$$

for some $\varepsilon > 0$ (depending on K).

As in [19], straightforward manipulations of the eigenfunction equation show (4.17) is equivalent to

$$\frac{R_{n+1}^2}{R_n^2} = 1 + (a_{n+1}^2 - a_n^2) \frac{\sin^2(\theta_n + k_n)}{a_n^2 \sin^2(k_n)}, \tag{5.14}$$

$$\cot(\theta_{n+1}) = \frac{a_n}{a_{n+1}} \frac{\sin(k_n)}{\sin(k_{n+1})} \cot(\theta_n + k_n). \tag{5.15}$$

R_1 and θ_1 are functions of θ_0 (given $a_1 = a_2 = 1$), and R_1 is, for $x_0 \in K$, uniformly bounded above and below. Moreover, by (5.10) and (5.13), for C depending only on K ,

$$|u_n| \leq C R_n. \tag{5.16}$$

Define

$$X_n = \frac{(a_{n+1}^2 - a_n^2) \sin^2(\theta_n + k_n)}{a_n^2 \sin^2(k_n)}. \tag{5.17}$$

By Lemma 3.5 of [19] and $\sup_n |X_n| < \infty$, it suffices to prove that

$$\sup_N \left| \sum_{j=1}^N X_j \right| < \infty. \tag{5.18}$$

Define \tilde{B}_j, \tilde{D}_j by adding to B_j, D_j the index one before (i.e., the top index of A_j and C_j). Then X_n is only nonzero on $\bigcup_j (\tilde{B}_j \cup \tilde{D}_j)$. On $\tilde{B}_j \cup \tilde{D}_j$, by (5.2), (5.3), and (5.6),

$$|X_n| \leq E(x_0) j^{-6}, \tag{5.19}$$

where

$$E(x_0) = E_0 \sup_n \frac{1}{\sin^2(k_n)} \tag{5.20}$$

is bounded above on K by (5.13). Theorem 5.3 is reduced to proving

$$\sup_N \left| \sum_{n=1}^N X_n \right| < \infty \tag{5.21}$$

uniformly in θ_1 and $x_0 \in K$.

Next, we note that one can write

$$X_n = X_n^\sharp + \tilde{X}_n \tag{5.22}$$

using

$$\sin^2(\theta_n + k_n) = \frac{1}{2}(1 - \cos(2(\theta_n + k_n))). \tag{5.23}$$

The X_n^\sharp terms are independent of θ_n , and there is a symmetry between points in \tilde{B}_j and \tilde{D}_j which, given the opposite signs of $1 - a_n^2/a_{n+1}^2$, causes a partial cancellation, that is, since $c_j^{-1} - c_j = O(j^{-6})$,

$$\sum_{n \in \tilde{B}_j} X_n^\sharp + \sum_{n \in \tilde{D}_j} X_n^\sharp = O(j^{-6}). \tag{5.24}$$

Moreover,

$$\sum_{n \in \tilde{B}_j} |X_n^\sharp| \leq E(x_0). \tag{5.25}$$

These together (plus the approximate cancellation) imply

$$\sup_N \left| \sum_{n=1}^N X_n^\sharp \right| \leq E(x_0) + O\left(\sum j^{-6}\right), \tag{5.26}$$

so we have reduced the proof of (5.21), and so of Theorem 5.3, to proving

$$\sup_N \left| \sum_{n=1}^N \tilde{X}_n \right| < \infty \tag{5.27}$$

uniformly in θ_1 and $x_0 \in K$.

We want to use cancellations of sums of cosines—more explicitly, that sums of M cosines with suitably varying phase are of order 1, not M . Here is what we need:

Lemma 5.4 *For any $q \in (0, 2\pi)$, any θ , and M ,*

$$\left| \sum_{\ell=1}^M \cos(q\ell + \theta) \right| \leq \left[\sin\left(\frac{q}{2}\right) \right]^{-1}. \tag{5.28}$$

Proof Since $\cos(\psi) = \text{Re}(e^{i\psi})$, it suffices to prove this if $\cos(q\ell + \theta)$ is replaced by $e^{i(q\ell + \theta)}$. By summing a geometric series,

$$\begin{aligned} \left| \sum_{\ell=1}^M e^{i(q\ell + \theta)} \right| &= \left| \frac{e^{i[(M+1)q + \theta]} - e^{i[q + \theta]}}{e^{iq} - 1} \right| \\ &\leq \frac{2}{2|(e^{iq/2} - e^{-iq/2})/2|} = \frac{1}{\sin(q/2)}. \end{aligned} \quad \square$$

In $\cos(\theta_n + k_n)$, both k_n and θ_n are n -dependent. But over sub-blocks small compared to j^6 , k_n is close to constant, and $\theta_{n+1} - \theta_n$ is close to constant. Thus, we break \tilde{B}_j and \tilde{D}_j into j^4 blocks, each with j^2 members, call them $\{\tilde{B}_{j,\ell}\}_{\ell=1}^{j^4}$ and $\{\tilde{D}_{j,\ell}\}_{\ell=1}^{j^4}$. For any n in some \tilde{B}_j or \tilde{D}_j , let β_n be the first element of the sub-block containing n and

$$\kappa_n = k_{\beta_n}.$$

Clearly, with constants uniformly bounded over K (below, C will stand for a generic constant bounded on any compact $K \subset (-1, 1)$),

$$|n - \beta_n| \leq Cj^2, \tag{5.29}$$

$$|k_n - \kappa_n| \leq Cj^{-4}. \tag{5.30}$$

Note that (5.30) comes from the fact that over a sub-block, a_n changes by at most $j^2 O(j^{-6})$.

In (5.15), the ratio of a 's is 1 plus an error of order j^{-6} , so given that arc cot has bounded derivatives,

$$|\theta_{n+1} - (\theta_n + k_n)| \leq Cj^{-6}, \tag{5.31}$$

and so,

$$|\theta_{n+1} - \theta_n - \kappa_n| \leq Cj^{-4}. \tag{5.32}$$

This implies that if

$$\tilde{\theta}_n = \theta_{\beta_n} + (n - \beta_n)\kappa_n, \tag{5.33}$$

then

$$|\theta_n - \tilde{\theta}_n| \leq Cj^{-2}. \tag{5.34}$$

Define Y_n to be \tilde{X}_n with $\cos(2(\theta_n + k_n))$ replaced by $\cos(2(\tilde{\theta}_n + \kappa_n))$ and $a_n^2 \sin^2(k_n)$ by $a_{\beta_n}^2 \sin^2(\kappa_n)$. By (5.30) and (5.34) (since $a_{n+1}^2 - a_n^2 \sim j^{-6}$), on $\tilde{B}_j \cup \tilde{D}_j$,

$$|Y_n - \tilde{X}_n| \leq Cj^{-8} \tag{5.35}$$

and

$$\sum_{n \in \tilde{B}_j \cup \tilde{D}_j} |Y_n - \tilde{X}_n| \leq Cj^{-2},$$

which is summable in j . Thus, to prove (5.27), we need

$$\sup_N \left| \sum_{n=1}^N Y_n \right| < \infty \tag{5.36}$$

uniformly in θ_1 and K .

Proof of Theorem 5.3 As noted, we are reduced to proving (5.36). Now $\{1, \dots, N\}$ can be broken into sums over 0 (i.e., A_j and C_j , except for their final indices), sums

over some number of sub-blocks, and one further partial sub-block. Summing over a single sub-block is, by Lemma 5.4 (given that, by (5.13), $2\kappa_n$ is bounded away from 0 and 2π), bounded by Cj^{-6} (from the fact that $a_{n+1}^2 - a_n^2 \sim j^{-6}$). Since there are $2j^4$ sub-blocks in $\tilde{B}_j \cup \tilde{D}_j$, we see that

$$\left| \sum_{n=1}^N Y_n \right| \leq \sum_j (2j^4)(Cj^{-6}) < \infty. \quad \square$$

6 The NTZ Argument

Here we begin with the key lemma of Nevai–Totik–Zhang [34] and apply it to extend the result of Zhang [51] to allow approach to an isospectral torus.

Proposition 6.1 [34] *For any positive r , any $\theta, \varphi \in [0, 2\pi]$, and L ,*

$$\frac{12}{L} \sum_{j=0}^{L-1} |1 - re^{i(j\theta+\varphi)}|^2 \geq |1 - re^{i\varphi}|^2. \quad (6.1)$$

Remarks 1. The authors of [34] allow general $p > 0$ where we take $p = 2$; but for $p = 2$, their constant is 32, not 12.

2. We include a proof at the end of this section for the reader’s convenience and because we want to emphasize the concepts in the context of what we cannot do in the next section.

3. For θ not near 0 or 2π , the idea behind a bound of this form is the same as the idea behind Lemma 5.4. As $\theta \rightarrow 0$ or 2π , for this argument to work, the constant 12 has to be replaced by larger and larger numbers. The idea for small θ is instead to use the fact that enough terms need to be close to the initial one.

Corollary 6.2 [51] *Let A be a 2×2 matrix with*

$$\det(A) = 1, \quad |\text{Tr}(A)| \leq 2. \quad (6.2)$$

Then for any vector $v \in \mathbb{C}^2$ (with $v = (v_1, v_2)$ the components of v),

$$|(A^{L-1}v)_1|^2 \leq \frac{12}{L} \sum_{j=0}^{L-1} |(A^j v)_1|^2. \quad (6.3)$$

Proof If A obeys (6.2), so does $B = A^{-1}$, and if $w = A^{L-1}v$, (6.3) is equivalent to

$$|w_1|^2 \leq \frac{12}{L} \sum_{j=0}^{L-1} |(B^j w)_1|^2, \quad (6.4)$$

so we need only prove (6.4).

Any B obeying (6.2) is a limit of B 's with $|\text{Tr}(B)| < 2$, so we can suppose

$$\det(B) = 1, \quad |\text{Tr}(B)| < 2. \tag{6.5}$$

In that case, B is diagonalizable and has eigenvalue $e^{\pm i\theta}$ with $2 \cos(\theta) = \text{Tr}(B)$.

In particular, for any v ,

$$(B^\ell v)_1 = \alpha e^{i\ell\theta} + \beta e^{-i\ell\theta} \tag{6.6}$$

for some α, β . By replacing θ by $-\theta$, we can suppose $\alpha \neq 0$ (if $\alpha = \beta = 0$, (6.4) is trivial!). Write $-\beta/\alpha = r e^{-i\varphi}$. Then (6.4) is equivalent to (after multiplying by $|\alpha|^{-2}$)

$$|1 - r e^{-i\varphi}|^2 \leq \frac{12}{L} \sum_{j=0}^{L-1} |1 - r e^{-i(2j\theta + \varphi)}|^2, \tag{6.7}$$

which, after a change of names of θ, φ , is (6.1). □

Recall that if $\{a_n, b_n\}_{n=1}^\infty$ are Jacobi parameters, a two-sided set $\{a_n^{(r)}, b_n^{(r)}\}_{n=-\infty}^\infty$ is called a right limit if for some $m_j \rightarrow \infty$ and all $n = 0, \pm 1, \dots$,

$$a_{m_j+n} \rightarrow a_n^{(r)}, \quad b_{m_j+n} \rightarrow b_n^{(r)}. \tag{6.8}$$

If $\sup_n (|a_n| + |b_n|) < \infty$, there are right limits by compactness and, indeed, any sequence m_k has a subsequence defining a right limit. Right limits are described in [44, Chap. 7] and references quoted there.

Any finite gap set

$$\epsilon = [\alpha_1, \beta_1] \cup \dots \cup [\alpha_{\ell+1}, \beta_{\ell+1}] \tag{6.9}$$

with

$$\alpha_1 < \beta_1 < \alpha_2 < \beta_2 < \dots < \beta_{\ell+1} \tag{6.10}$$

defines an ℓ -dimensional isospectral torus, \mathcal{T}_ϵ , of almost periodic two-sided Jacobi matrices, J , with $\sigma(J) = \epsilon$. The torus \mathcal{T}_ϵ can be defined using minimal Herglotz functions [44, Chap. 5] or reflectionless requirements [44, Chap. 7]. If ρ_ϵ is the potential theoretic equilibrium measure for ϵ (see, e.g., [40, 45]), we say ϵ is ‘‘periodic’’ if and only if each $\rho_\epsilon([\alpha_j, \beta_j])$ is rational; equivalently, all $J \in \mathcal{T}_\epsilon$ have a common period p .

The Nevai class for ϵ is defined to be those one-sided J 's whose right limits are all in \mathcal{T}_ϵ . For $\epsilon = [-2, 2]$, \mathcal{T}_ϵ has a single point (with period 1!) and the Nevai class for ϵ is the usual Nevai class.

Theorem 6.3 *If J lies in the Nevai class for a periodic ϵ , then the Nevai condition holds uniformly for J on ϵ .*

Remark If J has a single p element orbit, $J^{(r)} \in \mathcal{T}_\epsilon$, as right limits (i.e., J is asymptotically periodic); this is a result of [48, 51].

Proof Let p be the period of ϵ . We will prove that for any $x_n \in \epsilon$ and any L , we have

$$\limsup_{n \rightarrow \infty} \frac{|p_n(x_n)|^2}{\sum_{j=n-pL}^n |p_j(x_n)|^2} \leq \frac{12}{L}, \tag{6.11}$$

from which

$$\limsup_{n \rightarrow \infty} \frac{|p_n(x_n)|^2}{\sum_{j=0}^n |p_j(x_n)|^2} = 0, \tag{6.12}$$

proving the claimed uniform Nevai condition.

Without loss, we can pass to a subsequence so that $x_n \rightarrow x_\infty$; the ratio in (6.11) still converges to the \limsup ; $a_{n+k} \rightarrow a_k^{(r)}$, $b_{n+k} \rightarrow b_k^{(r)}$ for some periodic right limit; and $(p_n(x_n), p_n(x_{n-1})) / \|(p_n(x_n), p_{n-1}(x_{n-1}))\|$ has a limit in \mathbb{C}^2 .

The transfer matrix over p units starting at 0 for that x_∞ is a matrix A obeying (6.2). So, by (6.3),

$$\limsup_{n \rightarrow \infty} \frac{|p_n(x_n)|^2}{\sum_{j=0}^{L-1} |p_{n-jp}(x_n)|^2} \leq \frac{12}{L}, \tag{6.13}$$

which implies (6.11). □

We turn to the proof of Proposition 6.1. Without loss, we can (by taking complex conjugates) suppose

$$0 < \theta \leq \pi \tag{6.14}$$

(since $\theta = 0$ is trivial). There are three cases to consider:

Case 1 $L \leq 12$, which is trivial.

Case 2

$$\theta L \geq 2\pi, \quad L \geq 13. \tag{6.15}$$

Case 3

$$\theta L < 2\pi. \tag{6.16}$$

Proof of Proposition 6.1 Consider Case 2 first, expanding

$$\sum_{j=0}^{L-1} |1 - r e^{i(j\theta + \varphi)}|^2 = L(1 + r^2) - 2r \operatorname{Re}[X], \tag{6.17}$$

where

$$X = \frac{e^{i(L\theta + \varphi)} - e^{i\varphi}}{e^{i\theta} - 1}, \tag{6.18}$$

so

$$|X| \leq \frac{1}{\frac{1}{2}|1 - e^{i\theta}|} = \frac{1}{|\sin(\frac{\theta}{2})|} \leq \frac{\pi}{\theta}, \tag{6.19}$$

since

$$\inf_{0 \leq y \leq \pi} \left[\frac{\sin(\frac{y}{2})}{y} \right] = \frac{1}{\pi}. \tag{6.20}$$

By (6.15), $\pi/\theta \leq L/2$, so by (6.17),

$$\text{LHS of (6.1)} \geq L(1+r^2) - Lr \geq \frac{L}{2}(1+r^2) > 6(1+r^2) \tag{6.21}$$

(since $L > 12$). Clearly,

$$\text{RHS of (6.1)} \leq |1+r|^2 \leq 2(1+r^2), \tag{6.22}$$

so (6.1) holds in Case 2.

That leaves Case 3. We will consider $\varphi < 0$ ($\varphi > 0$ is even easier). Consider the L points

$$T = \{\varphi + j\theta\}_{j=0}^{L-1}. \tag{6.23}$$

Since $L\theta < 2\pi$, they do not make it back around the circle. Consider the three sets: $S_1 = \{\eta \mid \varphi \leq \eta < \frac{\varphi}{2}\}$, $S_2 = \{\frac{\varphi}{2} \leq \eta < 0\}$, and $S_3 = \{0 \leq \eta < -\frac{\varphi}{2}\}$. Clearly, $\#(S_1 \cap T) \geq \max(\#(S_2 \cap T), \#(S_3 \cap T))$, so at most two-thirds of the points in T lie in $S_2 \cup S_3$.

By the lemma below, if $\eta \in T \setminus (S_2 \cup S_3)$,

$$|1 - re^{i\eta}|^2 \geq \frac{1}{4}|1 - re^{i\varphi}|^2, \tag{6.24}$$

so

$$\text{LHS of (6.1)} \geq \frac{L}{3} \left(\frac{12}{L} \right) \frac{1}{4} |1 - re^{i\varphi}|^2 = \text{RHS of (6.1)}. \quad \square$$

Lemma 6.4

$$\inf_{\substack{\pi \geq |\eta| \geq \frac{\varphi}{2} \\ 0 < r}} \frac{|1 - re^{i\eta}|}{|1 - re^{i\varphi}|} \geq \frac{1}{2}. \tag{6.25}$$

Proof Note that $|1 - re^{i\eta}|/|1 - re^{i\varphi}|$ is invariant under $r \rightarrow r^{-1}$, so we can suppose $0 < r \leq 1$. Moreover, $|1 - re^{i\eta}|$ is invariant under $\eta \rightarrow -\eta$ and increasing in η for $0 < \eta < \pi$, so the inf occurs at $\eta = \frac{\varphi}{2}$.

A straightforward calculation shows $|1 - re^{i\varphi/2}|/|1 - re^{i\varphi}|$ is decreasing in r in $r \in (0, 1]$, so the inf is $|1 - e^{i\varphi/2}|/|1 - e^{i\varphi}| = |\sin(\frac{\varphi}{4})|/|\sin(\frac{\varphi}{2})| = 1/|2 \cos(\frac{\varphi}{4})| \geq \frac{1}{2}$. \square

7 The Nevai Class of a General Finite Gap Set

In this section, we will discuss the extension of Theorem 6.3 to general finite gap sets. We will only be able to prove the weaker result that the Nevai condition holds

uniformly on compact subsets of ϵ^{int} . In the next section, using different methods, we will prove the result uniformly on all of ϵ .

We begin by noting the following abstraction of the argument we used in the proof of Theorem 6.3:

Proposition 7.1 *Let J be a half-line Jacobi matrix, and let \mathcal{R} be the set of its right limits. Let $K \subset \mathbb{R}$ be a compact set. For $v \in \mathbb{C}^2$ and $J^{(r)} \in \mathcal{R}$, let $u_n(v, J^{(r)}, z)$ solve*

$$a_n^{(r)} u_{n+1} + b_n^{(r)} u_n + a_{n-1}^{(r)} u_{n-1} = z u_n \tag{7.1}$$

with

$$(u_0, u_1) = (v_1, v_2). \tag{7.2}$$

Suppose that for all ϵ , there is N so that for all unit vectors $v \in \mathbb{C}^2$, all $J^{(r)} \in \mathcal{R}$, all $x_0 \in K$, and all $n > N$,

$$\frac{|u_n(v, J^{(r)}, x_0)|^2}{\sum_{j=0}^n |u_j(v, J^{(r)}, x_0)|^2} \leq \epsilon. \tag{7.3}$$

Then J obeys the Nevai condition uniformly on K .

Proposition 7.2 *Let J be a half-line Jacobi matrix obeying (1.14), and let \mathcal{R} be the set of its right limits. Suppose that there is a compact subset $K \subset \mathbb{R}$ such that for each $x_0 \in K$ and $J^{(r)} \in \mathcal{R}$, there is a solution $u_n^+(J^{(r)}, x_0)$ of (7.1) (with $z = x_0$) so that*

$$(i) \quad \sup_{n, x_0, J^{(r)}} |u_n^+(J^{(r)}, x_0)| < \infty, \tag{7.4}$$

$$(ii) \quad \inf_{x_0, J^{(r)}} a_0^{(r)} |u_1^+ \overline{u_0^+} - \overline{u_1^+} u_0^+| > 0. \tag{7.5}$$

Then the Nevai condition holds for J uniformly on K .

Remark These are very strong conditions, but they hold in the finite gap case.

Proof Define

$$U_n(J^{(r)}, x_0) = \frac{1}{d(J^{(r)}, x_0)} \begin{pmatrix} u_{n+1}^+ & \overline{u_{n+1}^+} \\ a_n^{(r)} u_n^+ & a_n^{(r)} \overline{u_n^+} \end{pmatrix}, \tag{7.6}$$

where $d(J^{(r)}, x_0)$ is a square root of

$$a_n^{(r)} (u_{n+1}^+ \overline{u_n^+} - \overline{u_{n+1}^+} u_n^+), \tag{7.7}$$

which is n -independent. Then U_n is uniformly bounded in $x_0 \in K$, $J^{(r)} \in \mathcal{R}$, n by (7.4)/(7.5) and has determinant 1, so the same is true of U_n^{-1} .

Moreover, the transfer matrix for $J^{(r)}$ is

$$T_n = U_n U_0^{-1}, \tag{7.8}$$

so it is bounded in n , $J^{(r)}$, x , and has a bounded inverse. This shows

$$|u_n(v, J^{(r)}, x_0)|^2 + |u_{n+1}(v, J^{(r)}, x_0)|^2 \tag{7.9}$$

is uniformly bounded above and below as v runs through unit vectors.

The ratio in (7.3) is thus uniformly bounded by c/n , so Proposition 7.1 is applicable. □

Theorem 7.3 *If J lies in the Nevai class for a finite gap set ϵ , then the Nevai condition holds uniformly on compact subsets of ϵ^{int} .*

Proof In [6] (see also [44, Chap. 9]), Jost solutions are constructed on the isospectral torus, \mathcal{T}_ϵ , that obey (7.4)/(7.5). □

8 Absence of Pure Points in Right Limits

In this section, we want to note and apply the following:

Theorem 8.1 *Let J be a bounded half-line Jacobi matrix with (1.14), and let \mathcal{R} be the set of its right limits. Let \mathcal{E} be the set of $x_0 \in \mathbb{R}$ so that for every $J^{(r)} \in \mathcal{R}$ and every nonzero solution u_n of (7.1) with $z = x_0$, we have*

$$\sum_{n=-\infty}^0 |u_n|^2 = \infty. \tag{8.1}$$

Then

- (i) *The Nevai condition holds uniformly on any compact subset of \mathcal{E} .*
- (ii) *If \mathcal{E} contains $\sigma_{\text{ess}}(J)$, then the Nevai condition holds uniformly on $\sigma(J)$.*

We will provide a proof below. We first discuss some consequences.

Theorem 8.2 *If J lies in the Nevai class for a finite gap set ϵ , then the Nevai condition holds uniformly on $\sigma(J)$.*

Proof In [6] (see also [44, Chap. 9]), it is proven that for any $J^{(r)}$ in the isospectral torus, \mathcal{T}_ϵ , and any $x_0 \in \epsilon^{\text{int}}$, every solution is almost periodic; and for $x_0 \in \{\alpha_j, \beta_j\}_{j=1}^{\ell+1}$, every solution is the sum of an almost periodic function and n times an almost periodic function. Nonzero almost periodic functions obey (8.1) and $\sigma_{\text{ess}}(J) = \epsilon$, so Theorem 8.1 is applicable. □

There is a class of whole-line stochastic Jacobi matrices called subshifts, with work reviewed in [8]. The most famous is the Fibonacci model which has $(\chi_I =$ characteristic function of the set $I)$

$$a_n \equiv 1, \quad b_n \equiv \chi_{[1-\alpha, 1]}((n\alpha + \theta)_{\text{mod } 1}),$$

where $\alpha = \frac{1}{2}(\sqrt{5} - 1)$ and θ is a parameter (e.g., 0). The name comes from the fact that the transfer matrix, T_n , has special properties when n is a Fibonacci number. Damanik–Lenz [9] showed that there are no solutions ℓ^2 at $-\infty$ for any θ and any x_0 in the spectrum, and it is not hard to see the right limits for any half-line Fibonacci problem are again Fibonacci models or such models modified at a single site. Thus, Theorem 8.1 is applicable, and:

Theorem 8.3 *Any Fibonacci model restricted to a half line obeys the Nevai condition uniformly on the spectrum.*

Remark Results of Damanik–Killip–Lenz [11] allow extension of this to general Sturmian models.

Theorem 8.1 was motivated by our trying to understand Szwarz [48]. He noted that one could use results of Nevai [32] on weak limits of the measure

$$\sum_j \frac{p_n(x_j^{(n+1)})^2}{\sum_{k=0}^n p_k(x_j^{(n+1)})^2} \delta_{x_j^{(n+1)}}, \tag{8.2}$$

where $x_j^{(n+1)}$ are the solutions of

$$p_{n+1}(x_j^{(n+1)}) = 0, \tag{8.3}$$

that is, the ratios in (1.11) are weights in some natural measures. Thus, a failure of (1.11) should imply a suitable half-line limit has a pure point, and that is what is forbidden by (8.1). We begin with:

Lemma 8.4 *Let $J_{n;F}$ be the $n \times n$ truncated Jacobi matrix. Then the spectral measure of $J_{n+1;F}$ and vector δ_{n+1} is (8.2), where the $x_j^{(n+1)}$ solve (8.3).*

Proof It is well known (see [44]) that

$$\det(x - J_{n+1;F}) = P_{n+1}(x), \tag{8.4}$$

so the eigenvalues are the solutions of (8.3). The unnormalized eigenvector for $x_j^{(n+1)}$ is $(v_1, \dots, v_{n+1})^T$, where

$$v_k = p_{k-1}(x_j^{(n+1)}), \tag{8.5}$$

so (8.2) has the squares of the normalized components for δ_{n+1} . □

Lemma 8.5 Let $J_F^{(n)}$ be a family of $m_n \times m_n$ finite Jacobi matrices with coefficients associated to $\{a_j^{(n)}\}_{j=1}^{m_n-1} \cup \{b_j^{(n)}\}_{j=1}^{m_n}$. Suppose

- (i) $m_n \rightarrow \infty$,
- (ii)

$$\sup_{j,n} |a_j^{(n)}| + |b_j^{(n)}| < \infty, \tag{8.6}$$

(iii) for each fixed j ,

$$a_j^{(n)} \rightarrow a_j^{(\infty)}, \quad b_j^{(n)} \rightarrow b_j^{(\infty)}. \tag{8.7}$$

Let $J^{(\infty)}$ be the infinite Jacobi matrix with parameters $\{a_j^{(\infty)}, b_j^{(\infty)}\}_{j=1}^{\infty}$. Let $d\rho^{(n)}$ be the spectral measure for $J_F^{(n)}$ and δ_1 , and $d\rho^{(\infty)}$ for $J^{(\infty)}$ and δ_1 . Then

$$\text{w-lim}_{n \rightarrow \infty} d\rho^{(n)} = d\rho^{(\infty)}. \tag{8.8}$$

Remark This generalizes Theorem 3 in [32, p. 17].

Proof Extend $J_F^{(n)}$ to an infinite matrix by setting all other matrix elements to 0. Then (8.6) implies that

$$\sup_n \|J_F^{(n)}\| < \infty, \tag{8.9}$$

and (8.7) implies that for any finite support vector, v ,

$$\|(J_F^{(n)} - J^{(\infty)})v\| \rightarrow 0. \tag{8.10}$$

It follows that

$$\text{s-lim } J_F^{(n)} = J^{(\infty)}. \tag{8.11}$$

So, by (8.9),

$$\text{s-lim } (J_F^{(n)})^k = (J^{(\infty)})^k \tag{8.12}$$

for all k . Thus,

$$\lim \langle \delta_1, (J_F^{(n)})^k \delta_1 \rangle = \langle \delta_1, (J^{(\infty)})^k \delta_1 \rangle, \tag{8.13}$$

so

$$\lim \int x^k d\rho^{(n)} = \int x^k d\rho^{(\infty)}, \tag{8.14}$$

from which (8.8) follows. □

Proof of Theorem 8.1 Let \tilde{K} be a compact subset of \mathcal{E} . If (1.11) does not hold uniformly, we can find $n(j) \rightarrow \infty$, $x_j \in \tilde{K}$, and $\varepsilon > 0$ so that for all j ,

$$\frac{p_{n(j)}(x_j)^2}{\sum_{k=0}^{n(j)} p_k(x_j)^2} \geq \varepsilon. \tag{8.15}$$

By passing to a subsequence, we can suppose

$$x_j \rightarrow x_\infty \in \tilde{K}. \tag{8.16}$$

Notice that (8.15) implies

$$p_{n(j)-1}(x_j)^2 \leq \sum_{k=0}^{n(j)} p_k(x_j)^2 \leq \varepsilon^{-1} p_{n(j)}(x_j)^2. \tag{8.17}$$

Define $\tilde{b}_{n(j)+1}$ by

$$a_{n(j)} p_{n(j)-1}(x_j) + (\tilde{b}_{n(j)+1} - x_j) p_{n(j)}(x_j) = 0. \tag{8.18}$$

Thus, for the OPs with Jacobi parameters $(a_1, \dots, a_{n(j)}), (b_1, \dots, b_{n(j)}, \tilde{b}_{n(j)+1})$, we have

$$\tilde{p}_{n(j)+1}(x_j) = 0. \tag{8.19}$$

Moreover, by (8.18) and (8.17),

$$\limsup |\tilde{b}_{n(j)+1}| \leq \varepsilon^{-1/2} \sup_k |a_k| + \sup_k |x_k| \tag{8.20}$$

is finite.

Let $J_F^{(j)}$ be the Jacobi matrix which is $(n(j) + 1) \times (n(j) + 1)$ with $a_{n(j)}, a_{n(j)-1}, \dots, a_1$ off diagonal and $\tilde{b}_{n(j)+1} - x_j + x_\infty, b_{n(j)} - x_j + x_\infty, \dots, b_1 - x_j + x_\infty$ on diagonal. By (8.15) and Lemma 8.4 (turning $J_{n(j)+1;F}$ on its head!), the spectral measure for $J_F^{(j)}, \delta_1$ has a pure point at x_∞ of mass at least ε .

By passing to a further subsequence, we can suppose for all q that $a_{n(j)+q} \rightarrow a_q^{(r)}, b_{n(j)+q} \rightarrow b_q^{(r)}$ for some right limit, $J^{(r)}$, and, by (8.20) and a further subsequence, that $\tilde{b}_{n(j)+1} \rightarrow \tilde{b}_1^{(r)}$.

The coefficients of $J_F^{(j)}$ clearly obey (8.6), and there is a $J^{(\infty)}$, so (8.7) holds. This is given by the reversed left side of $J^{(r)}$ (from $-\infty$ to 1), with $b_1^{(r)}$ replaced by $\tilde{b}_1^{(r)}$.

For any positive function f ,

$$\int f(y) d\rho^{(n)}(y) \geq \varepsilon f(x_\infty). \tag{8.21}$$

So, by (8.8),

$$\int f(y) d\rho^{(\infty)}(y) \geq \varepsilon f(x_\infty). \tag{8.22}$$

Thus,

$$\rho^{(\infty)}(\{x_\infty\}) \geq \varepsilon, \tag{8.23}$$

and x_∞ is an eigenvalue of $J^{(\infty)}$. Thus, $J^{(r)}$ has an eigensolution at x_∞ which is ℓ^2 at $-\infty$, violating (8.1). This proves statement (i) of Theorem 8.2.

For (ii), by passing to a subsequence, (8.16) holds for some x_∞ . If x_∞ is an isolated eigenvalue, x_j must be equal to x_∞ for large j , so $\limsup |p_j(x_\infty)| > 0$, violating the condition that x_∞ is an isolated eigenvalue. If $x_\infty \in \sigma_{\text{ess}}(J)$, the argument in the first part produces a contradiction. Thus, (ii) is proven. \square

9 Some Comments

We end this paper with some final results and comments. The following must be well known in the ergodic theory community:

Proposition 9.1 *Let a_n be a sequence of reals and*

$$C_n = \frac{1}{n} \sum_{j=1}^n a_j. \tag{9.1}$$

If

$$\lim_{n \rightarrow \infty} C_n = C_\infty \neq 0 \tag{9.2}$$

exists, then

$$\frac{a_n}{nC_n} \rightarrow 0. \tag{9.3}$$

Proof Since

$$a_n = nC_n - (n - 1)C_{n-1}, \tag{9.4}$$

$$n^{-1}a_n = C_n - C_{n-1} + n^{-1}C_{n-1}, \tag{9.5}$$

so

$$\frac{a_n}{nC_n} = \frac{C_n - C_{n-1} + n^{-1}C_{n-1}}{C_n} \tag{9.6}$$

goes to zero if (9.2) holds. \square

It is an idea associated especially with Freud and Nevai (see [33]) that on the a.c. spectrum,

$$\frac{1}{n} \sum_{j=0}^{n-1} p_j(x_0)^2 \rightarrow \frac{\rho_\infty(x_0)}{w(x_0)}, \tag{9.7}$$

where ρ_∞ is the density of zeros and w is the a.c. part of the underlying measure, with concrete results both classical [31, 49] and recent [1, 24, 25, 43, 50].

Via Proposition 9.1, this gives several results on the Nevai condition. Totik’s result [49] implies:

Theorem 9.2 *Let $d\rho$ have the form*

$$d\rho(x) = w(x) dx + d\rho_s(x), \tag{9.8}$$

with $d\rho_s$ Lebesgue singular. Let ϵ be the essential support of $d\rho$ and suppose ρ is regular for ϵ . Suppose I is an open interval in ϵ on which a Szegő condition holds:

$$\int_I \log(w(x)) dx > -\infty. \tag{9.9}$$

Then for Lebesgue a.e. $x_0 \in I$, the Nevai condition holds.

More recent work on uniform convergence [24, 43] implies:

Theorem 9.3 *Under the hypotheses of Theorem 9.2, if (9.9) is replaced by w continuous on I and $\inf_I w(x) > 0$, then the Nevai condition holds on all of I uniformly on compact subsets of I .*

Recent work on ergodic Jacobi matrices ([1]; see that paper for the definition of ergodic Jacobi matrices) implies:

Theorem 9.4 *Let J_ω be a family of ergodic Jacobi matrices obeying (1.14). Suppose Σ_{ac} , the essential support of the a.c. spectrum, is nonempty. Then for a.e. $\omega, x_0 \in \Sigma_{ac}$, the Nevai condition holds.*

We also want to make a comment regarding the unbounded case (where ρ is not compactly supported). In this case, there exist measures, ρ , of various types (including pure point!) with $\lim_{n \rightarrow \infty} \frac{a_{n+1}^2 [p_n(x_0)^2 + p_{n+1}(x_0)^2]}{K_n(x_0, x_0)} \neq 0$ for x in a set of positive ρ measure. Indeed, the power law behavior of the generalized eigenfunctions in [3] and those associated with the absolutely continuous part of the measure in [10] imply the limit in these cases is actually ∞ for certain values of the relevant parameters. In the introduction, it was noted that this means that truncations of the generalized eigenfunction do not form a sequence of approximate eigenvectors. Although (2.13) is still true in this case, Theorem 1.2 does not hold for measures that are not compactly supported, so this does not constitute a counter-example to Conjecture 1.4. In any case, it seems to be an interesting challenge to study the Nevai condition in the unbounded case.

Next, we turn to some remarks on Conjecture 1.4. It is standard to break $d\rho$ into three parts: a.c., pure point, and singular continuous. For compactly supported measures, the Nevai condition always holds at pure points, since if x_0 is a pure point, $\sum_{j=0}^\infty p_j(x_0)^2 < \infty$, so $p_n(x_0)^2 \rightarrow 0$. For $d\rho_c$ (= the continuous part of $d\rho$), using Egorov’s Theorem together with the fact that $\lambda_n \rightarrow 0$ $d\rho_c$ -a.e., it is not hard to see that $\lim_{n \rightarrow \infty} \int \frac{p_n(x)^2}{\sum_{j=0}^n p_j(x)^2} d\rho_c(x) = 0$, so the issue is going from convergence of integrals to pointwise convergence.

If x_0 is not a pure point but $\sup_n |p_n(x_0)| < \infty$, then since $\sum_{j=0}^\infty p_j(x_0)^2 = \infty$, the Nevai condition holds. Thus, our Conjecture 1.4 is related to a famous conjecture

of Steklov and what is sometimes called the Schrödinger conjecture [27]. There are known counter-examples to both conjectures (see [35] and [16]), but the counter-example in [35] has failure of boundedness at a single point and the counter-example in [16] does not seem to violate the Nevai condition, so Conjecture 1.4 could be true.

We note that the currently open version of the Schrödinger conjecture, that for a.e. x_0 in the essential support of the a.c. spectrum one has bounded eigenfunctions, would imply the Nevai condition a.e. on the essential support of the a.c. spectrum.

Finally, in connection with the example of Sect. 5, we would like to point out that we believe the following is true:

Conjecture 9.5 Let $q \in \mathbb{N}$, let $\{a_n, b_n\}_{n=1}^\infty$ be Jacobi parameters obeying

$$\sum_{n=1}^{\infty} (a_{n+q} - a_n)^2 + (b_{n+q} - b_n)^2 < \infty, \quad (9.10)$$

and let \mathcal{R} be the set of corresponding right limits. Then

$$\Sigma_{ac} = \bigcap_{\mathcal{R}} \sigma(J^{(r)}),$$

where Σ_{ac} is the corresponding essential support of the a.c. spectrum.

We note that (9.10) implies \mathcal{R} is made of q -periodic Jacobi matrices. The inclusion $\Sigma_{ac} \subset \bigcap_{\mathcal{R}} \sigma(J^{(r)})$ follows from a general result of [21], so the point here is the inclusion in the other direction. Conjecture 9.5 generalizes a conjecture of Kaluzhny–Last [18], who make this conjecture for the special case where \mathcal{R} consists of a single element. Denissov’s result [13] establishes it for the special case where $a_n \equiv 1$ and where the single element of \mathcal{R} is the free Jacobi matrix (namely, $b_n \rightarrow 0$), proving an even earlier variant of this conjecture by Last [20]. Theorem 5.2 shows that Conjecture 9.5 holds for the Jacobi matrix of Example 5.1 (a special case where (9.10) holds for $q = 1$), thus providing some level of confirmation for it.

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