CONSTRUCTIVE
APPROXIMATION

Finite Gap Jacobi Matrices, I. The Isospectral Torus

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Abstract Let $\mathfrak{e} \subset \mathbb{R}$ be a finite union of disjoint closed intervals. In the study of orthogonal polynomials on the real line with measures whose essential support is $\mathfrak{e}$, a fundamental role is played by the isospectral torus. In this paper, we use a covering map formalism to define and study this isospectral torus. Our goal is to make a coherent presentation of properties and bounds for this special class as a tool for ourselves and others to study perturbations. One important result is the expression of Jost functions for the torus in terms of theta functions.

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1 Introduction

Let $\mathfrak{e} \subset \mathbb{R}$ be a union of $\ell + 1$ disjoint closed intervals

$$\mathfrak{e} = \mathfrak{e}_1 \cup \mathfrak{e}_2 \cup \cdots \cup \mathfrak{e}_{\ell+1}, \tag{1.1}$$

$$\mathfrak{e}_j = [\alpha_j, \beta_j], \tag{1.2}$$

$$\alpha_1 < \beta_1 < \alpha_2 < \cdots < \alpha_{\ell+1} < \beta_{\ell+1}, \tag{1.3}$$

where ℓ counts the number of gaps.

For later purposes, we will need to exploit potential theoretic objects associated to $\mathfrak{e}$. The notation cap($\mathfrak{e}$) will be its logarithmic capacity, $d\rho_{\mathfrak{e}}$ the equilibrium measure (normalized by $\rho_{\mathfrak{e}}(\mathbb{R}) = 1$)

$$d\rho_{\mathfrak{e}}(x) = \rho_{\mathfrak{e}}(x)\,dx, \tag{1.4}$$

and $\rho_{\mathfrak{e}}(\mathfrak{e}_j)$ the harmonic measures. For reasons that become clear soon, we say $\mathfrak{e}$ is periodic if all harmonic measures, $\rho_{\mathfrak{e}}(\mathfrak{e}_j)$, $j = 1, \ldots, \ell + 1$, are rational. See [35, 41, 44, 55, 62, 69, 74] for discussions of potential theory.

We will be interested in one- and two-sided Jacobi matrices: one-sided with parameters labeled $\{a_n, b_n\}_{n=1}^{\infty}$,

$$J = \begin{pmatrix} b_1 & a_1 & 0 & 0 & \cdots \\ a_1 & b_2 & a_2 & 0 & \cdots \\ 0 & a_2 & b_3 & a_3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}, \tag{1.5}$$

and two-sided with $\{a_n, b_n\}_{n=-\infty}^{\infty}$ extended to the top and left in the obvious way. And, of course, we want to consider the orthogonal polynomials on the real line (OPRL) [28, 66, 71] defined by

$$\begin{aligned} p_{-1}(x) &= 0, \qquad p_0(x) = 1, \\ x p_n(x) &= a_{n+1} p_{n+1}(x) + b_{n+1} p_n(x) + a_n p_{n-1}(x). \end{aligned} \tag{1.6}$$

If $d\mu$ is the spectral measure for J and vector $(1, 0, 0, \ldots)^t$, then the p_n's are orthonormal

$$\int p_n(x) p_m(x)\,d\mu(x) = \delta_{nm}. \tag{1.7}$$

We will also want to consider monic OPs, P_n, the multiple of p_n with leading coefficient 1,

$$p_n(x) = (a_1 \cdots a_n)^{-1} P_n(x), \tag{1.8}$$

$$x P_n(x) = P_{n+1}(x) + b_{n+1} P_n(x) + a_n^2 P_{n-1}(x). \tag{1.9}$$

We want to analyze the case where

$$\sigma_{\mathrm{ess}}(J) \equiv \sigma_{\mathrm{ess}}(d\mu) = \mathfrak{e}. \tag{1.10}$$

Here $\sigma_{\text{ess}}(J)$ is the essential spectrum of J, also known as the derived set of supp($d\mu$). We will use $\sigma(J)$ (or $\sigma(d\mu)$) for the spectrum of J and $\Sigma_{\text{ac}}(d\mu) = \{x \mid \frac{d\mu}{dx} \neq 0\}$ for the essential support of the a.c. part of $d\mu$. In this paper, we will focus on the isospectral torus, in [16] on the Szegő class, and in [17] on results that go beyond the Szegő class. Some of our results were announced in [15].

The goal is to extend what is known about the case $\mathfrak{e} = [-1, 1]$. This can be viewed as a problem in approximation theory where polynomial asymptotics is critical or as a problem in spectral theory where Jacobi parameter asymptotics is critical. As usual, there are three main levels from the point of view of polynomial asymptotics:

(a) *Root asymptotics*. Asymptotics of $|P_n(x)|^{1/n}$. For $[-1, 1]$, the theory is due to Erdös–Turán [23] and Ullman [75–77]. For general sets, including finite gap sets, the theory is due to Stahl–Totik [69] (see Simon [62] for a review). One has for $x \notin \sigma(d\mu)$ and $d\mu$ regular (i.e., $\sigma_{\text{ess}}(d\mu) = \mathfrak{e}$ and $\lim(a_1 \cdots a_n)^{1/n} = \text{cap}(\mathfrak{e})$) that

$$\left|P_n(x)\right|^{1/n} \to \exp\left(\int \log|x - y| \, d\rho_{\mathfrak{e}}(y)\right). \tag{1.11}$$

(b) *Ratio asymptotics*. Traditionally, this involves the ratio $P_{n+1}(x)/P_n(x)$ having a limit. Nevai [48] showed that if $a_n \to a$, $b_n \to b$ ($a \neq 0$) so that $\sigma_{\text{ess}}(d\mu) = [b - 2a, b + 2a]$, then the limit exists for $x \notin \sigma(d\mu)$. Simon [59] proved a converse: if the limit exists at a single point in $\mathbb{C}_+ = \{z \mid \text{Im } z > 0\}$, then for some a, b, we have that $a_n \to a$, $b_n \to b$. Thus, the proper analog for $\sigma_{\text{ess}}(d\mu) = \mathfrak{e}$ will not be existence of a limit but something more subtle. This is an interesting open question which we will not address.

(c) *Szegő asymptotics*. This says that for $z \notin \sigma(d\mu)$, $P_n(z)/D(z)E(z)^n \to 1$ for an explicit function E (($z + \sqrt{z^2 - 1}$) for $\mathfrak{e} = [-1, 1]$) and a function D which is μ-dependent. The proper analog for general finite gap sets was obtained by Widom [82] (see also Aptekarev [3]) and by Peherstorfer–Yuditskii [51] using variational methods. The ratio is only asymptotically (almost) periodic. One of our main goals in this series is to provide a new nonvariational approach to this result. In addition, following Damanik–Simon [19] for $[-1, 1]$, we want to consider cases where the Szegő condition fails.

From the spectral theory point of view, the analogs of $a_n \to \frac{1}{2}$, $b_n \to 0$ (also known as the Nevai class) concern the isospectral torus, an object we will discuss extensively in this paper. For now, we note that if $\mathfrak{e}$ is periodic, the J's in the isospectral torus are all periodic Jacobi matrices with $\sigma_{\text{ess}}(J) = \mathfrak{e}$. In the general case, it is an ℓ-dimensional torus of almost periodic J's with $\sigma_{\text{ess}}(J) = \mathfrak{e}$. It can be singled out via minimal Herglotz functions [66] or reflectionless potentials [56]; see Sect. 6 below.

The key realization is that the Nevai class needs to be replaced by approach to an isospectral torus. This was first noted by Simon [60, 61] as conjectures in the context of the orthogonal polynomials on the unit circle (OPUC) case. In turn, Simon was motivated by work of López and collaborators [6, 9] who studied the case of a single gap for OPUC.

From a spectral point of view, the analogs of the asymptotics results are:

(a) Regularity implies more restrictions on the Jacobi parameters than $(a_1 \cdots a_n)^{1/n} \to \text{cap}(\mathfrak{e})$. For example, for $\mathfrak{e} = [-1, 1]$, it is known that $\frac{1}{n} \sum_{j=1}^{n}(a_j -$

$\frac{1}{2})^2 + b_j^2 \to 0$ and, for $\mathfrak{e}$ periodic, a similar Cesàro convergence result for distances to the isospectral torus is proven in [65]. The analog for general finite gap sets remains an interesting open question.

(b) The key result here in the case $\mathfrak{e} = [-1, 1]$ is the theorem of Denisov–Rakhmanov [21] stating that if $\Sigma_{ac}(d\mu) = \sigma_{ess}(d\mu) = [-1, 1]$, then $a_n \to \frac{1}{2}$, $b_n \to 0$. Simon [61] conjectured that for periodic $\mathfrak{e}$, the proper result is that if $\Sigma_{ac}(d\mu) = \sigma_{ess}(d\mu) = \mathfrak{e}$, then all right limits lie in the isospectral torus. For periodic $\mathfrak{e}$, this was proven by Damanik–Killip–Simon [20] who conjectured the result for general $\mathfrak{e}$. It was then proven for general finite gap sets by Remling [56]. Remling's result plays a key role in our work in paper II [16]. We note that in the opposite direction, Last–Simon [42] have shown that if all right limits lie in the isospectral torus, then $\sigma_{ess}(d\mu) = \mathfrak{e}$.

(c) Here there are two main results. When $\sigma(d\mu) = \mathfrak{e}$ (no bound states), Widom proved that a Szegő condition implies

$$\liminf \frac{a_1 \cdots a_n}{\text{cap}(\mathfrak{e})^n} > 0, \tag{1.12}$$

$$\limsup \frac{a_1 \cdots a_n}{\text{cap}(\mathfrak{e})^n} < \infty. \tag{1.13}$$

The Szegő condition in this situation is

$$\int_{\mathfrak{e}} \text{dist}(x, \mathbb{R} \setminus \mathfrak{e})^{-1/2} \log\left(\frac{d\mu}{dx}\right) dx > -\infty. \tag{1.14}$$

Widom allowed no eigenvalues outside $\mathfrak{e}$. Peherstorfer–Yuditskii [51] had eigenvalues, but only in a later note [52] did they have the natural (from their paper [50]) condition

$$\sum_j \text{dist}(x_j, \mathfrak{e})^{1/2} < \infty, \tag{1.15}$$

where x_j are the point masses of $d\mu$ (or eigenvalues of J) outside $\mathfrak{e}$. Thus, Peherstorfer–Yuditskii [52] showed

$$(1.14) + (1.15) \Rightarrow (1.12) + (1.13). \tag{1.16}$$

One of our main results in paper II [16] is to show

$$(1.12) + (1.15) \Rightarrow (1.14) + (1.13). \tag{1.17}$$

Peherstorfer remarked to us that, while this result is new, it can also be derived from the results of [51].

The key to our analysis is a machinery developed by Sodin–Yuditskii [67] and exploited by Peherstorfer–Yuditskii [51, 52]. To explain it, we note that the key to recent sum rule discussions (summarized in [66]) is to take the m-function given by

$$m(z) = \int \frac{d\mu(x)}{x - z} \tag{1.18}$$

and in the case $\mathfrak{e} = [-2, 2]$, move it to $\mathbb{D} = \{z \mid |z| < 1\}$ via

$$M(z) = -m(z + z^{-1}). \qquad (1.19)$$

The map

$$\mathbf{x}(z) = z + z^{-1} \qquad (1.20)$$

is the unique analytic bijection of $\mathbb{D}$ to $\mathbb{C} \cup \{\infty\} \setminus [-2, 2]$ with

$$\mathbf{x}(0) = \infty, \qquad \lim_{\substack{z \to 0 \\ z \neq 0}} z\mathbf{x}(z) > 0. \qquad (1.21)$$

The minus sign in (1.19) comes from the fact that $\mathbf{x}$ maps $\mathbb{D} \cap \mathbb{C}_+$ to $-\mathbb{C}_+$ (where $\mathbb{C}_+ = \{z \mid \operatorname{Im} z > 0\}$).

In our case, there cannot be an analytic bijection of $\mathbb{D}$ to $\mathbb{C} \cup \{\infty\} \setminus \mathfrak{e}$, since $\mathbb{C} \cup \{\infty\} \setminus \mathfrak{e}$ is not simply connected. However, because the holomorphic universal cover of $\mathbb{C} \cup \{\infty\} \setminus \mathfrak{e}$ is $\mathbb{D}$, there is an analytic map $\mathbf{x}: \mathbb{D} \to \mathbb{C} \cup \{\infty\} \setminus \mathfrak{e}$ which is locally one-to-one and obeys (1.21). Moreover, there is a group Γ of Möbius transformations of $\mathbb{D}$ to $\mathbb{D}$ so that

$$\mathbf{x}(z) = \mathbf{x}(w) \quad \Leftrightarrow \quad \exists \gamma \in \Gamma \text{ so that } z = \gamma(w). \qquad (1.22)$$

This group is isomorphic to $\pi_1(\mathbb{C} \cup \{\infty\} \setminus \mathfrak{e}) = \mathbb{F}_\ell$, the free non-Abelian group on ℓ generators. We mention that $\mathbf{x}$ is uniquely determined if (1.21)–(1.22) hold and $\mathbf{x}$ is locally one-to-one.

Our goal in this paper is to discuss the isospectral torus in terms of this formalism. It turns out that basic objects for the isospectral torus, like Bloch waves and Green's function behavior, are not discussed in detail anywhere. We will remedy that here. While these results will not be surprising to experts, they are exceedingly useful both in our further works [16, 17] and in [11, 27, 36, 64].

We should expand on the point we already remarked upon that there are two distinct ways of describing the isospectral torus: as a set of minimal Herglotz functions or as the family of reflectionless Jacobi matrices with spectrum $\mathfrak{e}$. The view as minimal Herglotz functions goes back to the earliest periodic KdV work [22, 45] (see also [25, 40, 79]), while the reflectionless definition goes back at least to Sodin–Yuditskii [67] (see also [56]).

There is an important distinction: reflectionless objects are natural whole-line (doubly infinite) Jacobi matrices, while minimal Herglotz functions are associated to half-line objects. Of course, the passage from whole-line to half-line objects is by restriction—but the converse is not so simple. From our point of view, the key is that the J's associated to minimal Herglotz functions are quasiperiodic, and such functions are determined by their values on a half-line (because a quasiperiodic function vanishing on a half-line is identically zero). Alternatively, if $m(z)$ is a minimal Herglotz function, the demand that

$$m_0(z) = \mathcal{M}(a_0, b_0, m(z)), \qquad (1.23)$$

where

$$\mathcal{M}\big(a, b, f(z)\big) = \frac{1}{-z + b - a^2 f(z)}, \tag{1.24}$$

be a minimal Herglotz function determines a_0 and b_0, and so inductively, minimality allows a unique continuation from the half-line.

In Sect. 2, we describe the map $\mathbf{x}$ in (1.21)–(1.22) and its natural extension to a covering (albeit not universal covering) map of the two-sheeted Riemann surface, $\mathcal{S}$, that the m-function for elements of the isospectral torus lives on. In Sect. 3, we describe a critical result of Beardon [8] on the Poincaré index of Γ. Section 4 reviews the facts about character automorphic Blaschke products and their connection to potential theory. We will also present estimates on these products needed in later papers [16, 17]. In Sect. 5, we use this machinery to prove Abel's theorem. In Sect. 6, we describe the isospectral torus as the family of minimal Herglotz functions on $\mathcal{S}$. Sections 7 and 8 will describe the Jost functions of elements of the isospectral torus and will prove that the natural map from the isospectral torus to the group of characters of Γ, given by the character of the Jost function, is an isomorphism of tori. We will also relate Jost functions to theta functions, one of the more significant results of the present paper. Section 9 will discuss Jost solutions and the associated Bloch waves. Finally, Sect. 10 will apply these solutions to the study of Green's function. Some of the material in Sects. 2, 4, and 6 is in suitable texts but included here because we wish to make this paper more accessible to approximation theorists who may be unfamiliar with it.

We also mention the enormous debt this paper owes to the seminal work of Sodin–Yuditskii [67] and Peherstorfer–Yuditskii [51]. About the only real advantage of our presentation in this first paper over ideas implicit in [51, 67] is that we are more explicit and our Jost functions, unlike the close relatives in [51, 67], are strictly character automorphic. We emphasize that [51, 67] had as their focus the theory of certain infinite gap sets for which $\mathfrak{e}$ is typically a Cantor set of positive Lebesgue measure. But they include finite gap sets and provide useful tools in that special case. Our work makes use of some results special to this finite gap situation.

We note that while we discuss Jost functions and solutions here for the isospectral torus, in [16, 17] we will present them for any J in the Szegő class. For us, they are the key to understanding Szegő asymptotics in this finite gap situation.

2 The Covering Map and the Fuchsian Group

In this section, we describe the basic objects and setup that we will use. We emphasize that these constructs are not new here, and more than anything else, this section sets up notation and gives a pedagogical introduction. The Riemann surface, $\mathcal{S}$, was introduced for finite gap KdV in [22, 45] and for finite gap Jacobi matrices in [25, 40, 79]. The Fuchsian group formalism is from [67].

Let $\mathcal{S}_+$ be the set $\mathbb{C} \cup \{\infty\} \setminus \mathfrak{e}$ viewed as a Riemann surface. First of all, we want to view this as one sheet of the Riemann surface of the function

$$w = \big(R(z)\big)^{1/2}, \tag{2.1}$$

where

$$R(z) = \prod_{j=1}^{\ell+1} (z - \alpha_j)(z - \beta_j). \tag{2.2}$$

More explicitly, we consider pairs (w, z) in $\mathbb{C}^2$ obeying

$$w^2 - R(z) \equiv G(w, z) = 0. \tag{2.3}$$

Since $\frac{\partial G}{\partial z} \neq 0$ at those $2\ell + 2$ points where $\frac{\partial G}{\partial w} = 0$, this set is a one-dimensional complex manifold, also known as a Riemann surface.

With two points at infinity added, this set becomes a compact surface $\mathcal{S}$. One can formally define $\mathcal{S}$ by looking in $\mathbb{C}^3 \setminus \{0\}$ at triples, (w, z, u), with

$$w^2 u^{2\ell} = \prod_{j=1}^{\ell+1} (z - \alpha_j u)(z - \beta_j u) \tag{2.4}$$

and regarding (w, z, u) as equivalent to (w', z', u') if there is $\lambda \in \mathbb{C} \setminus \{0\}$, so $w = \lambda w'$, $z = \lambda z'$, $u = \lambda u'$. Rather than this formal projective space view, we will think of two points, $\infty_{\pm} \in \mathcal{S}$, obtained by using $\zeta = 1/z$ coordinates on $\mathcal{S}_{\pm}$ and adding the missing point $\zeta = 0$.

There is a natural map $\pi \colon \mathcal{S} \to \mathbb{C} \cup \{\infty\}$ given by $(w, z) \to z$. It sets up $\mathcal{S}$ as a branched cover of $\mathbb{C} \cup \{\infty\}$. Here π is two-to-one on all points in $\mathbb{C} \cup \{\infty\}$ except $\{\alpha_j, \beta_j\}_{j=1}^{\ell+1}$—these latter points are the branch points. There is a second natural map $\tau \colon \mathcal{S} \to \mathcal{S}$ that in (w, z) coordinates takes $w \to -w$. Let $\mathcal{S}_-$ denote the image of $\mathcal{S}_+$ under τ. $\tau(\infty_+) = \infty_-$. $\mathcal{S} \setminus (\mathcal{S}_+ \cup \mathcal{S}_-)$ is thus $\pi^{-1}(\mathfrak{e})$. Each $\pi^{-1}(\mathfrak{e}_j)$ is topologically a circle.

There is a close connection between $\mathcal{S}$ and the potential theory associated to $\mathfrak{e}$. In terms of the equilibrium measure, $d\rho_{\mathfrak{e}}$, consider its Borel transform,

$$M_{\mathfrak{e}}(z) = \int \frac{d\rho_{\mathfrak{e}}(x)}{x - z}. \tag{2.5}$$

It is a basic fact (due to Craig [18]; see also [62, 66]) that for suitable points, $x_j \in (\beta_j, \alpha_{j+1})$, we have

$$M_{\mathfrak{e}}(z) = \frac{-\prod_{j=1}^{\ell} (z - x_j)}{(\prod_{j=1}^{\ell+1} (z - \alpha_j)(z - \beta_j))^{1/2}}, \tag{2.6}$$

so $M_{\mathfrak{e}}$ has a natural analytic continuation from $\mathbb{C} \cup \{\infty\} \setminus \mathfrak{e}$ to $\mathcal{S}$.

Topologically, $\mathcal{S}$ is the sphere with ℓ handles attached—the canonical surface of genus ℓ. Its first homology group (see [4, 33, 80] for basic topological notions we use here) is $\mathbb{Z}^{2\ell}$. One way of looking at the generators is picking curves that loop around each $\pi^{-1}(\mathfrak{e}_j)$ but one (the sum of all $\ell + 1$ is homologous to zero) and also curves that loop around each

$$\pi^{-1}([\beta_j, \alpha_{j+1}]) \equiv G_j \quad j = 1, \dots, \ell. \tag{2.7}$$

For a while, we put $\mathcal{S}$ aside and focus on $\mathcal{S}_+$. Note that $\mathcal{S}_+$ is not simply connected. Its fundamental group is the free non-Abelian group on ℓ generators. We will pick ∞ as the base point. One way of picking generators is to pick $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_\ell$ where $\tilde{\gamma}_j$ is the curve that starts at ∞, traverses in $\mathbb{C}_-$ to $\frac{1}{2}(\beta_j + \alpha_{j+1})$ (in the j-th gap), and returns to ∞ in $\mathbb{C}_+$ (see the lower half of Fig. 2).

The universal cover of $\mathcal{S}_+$ inherits the local complex structure of $\mathcal{S}_+$ and so is a Riemann surface. The deck transformations preserve this complex structure, so there is a discrete group, Γ, of complex automorphisms of the universal cover where each $\gamma \in \Gamma$ has no fixed points.

It is a fundamental result in the theory of Riemann surfaces (the uniformization theorem; see [24, 31, 47]) that the only simply connected Riemann surfaces are the Riemann sphere, the complex plane, and the unit disk, $\mathbb{D}$. The sphere has no fixed point free complex automorphism and the only discrete groups of automorphisms on $\mathbb{C}$ are one- and two-dimensional lattices, so the only Riemann surfaces with cover $\mathbb{C}$ are the tori and the punctured disk. Since $\mathcal{S}_+$ is neither of these, its universal cover is $\mathbb{D}$.

Thus, there exists a map $\mathbf{x} \colon \mathbb{D} \to \mathcal{S}_+$, which is locally one-to-one, and a group, Γ, of Möbius transformations on $\mathbb{D}$ so that (1.22) holds. By requiring (1.21) (which, by the action of maps on $\mathbb{D}$, we can always do), we uniquely fix $\mathbf{x}$. There is a lovely proof of the existence of $\mathbf{x}$ due to Radó [54] that follows the standard proof of the Riemann mapping theorem [1]; see [66, Sect. 9.5].

Γ is a discrete group of Möbius transformations leaving $\mathbb{D}$ setwise fixed, also known as a Fuchsian group. For background on such groups, see [38] or [66, Chap. 9].

$\mathcal{S}_+$ is invariant under complex conjugation, as is $\mathbb{D}$, so $\overline{\mathbf{x}(\bar{z})}$ is also a covering map of $\mathbb{D}$ over $\mathcal{S}_+$. But it obeys (1.21), so by uniqueness,

$$\mathbf{x}(\bar{z}) = \overline{\mathbf{x}(z)}. \tag{2.8}$$

We define the fundamental region, $\mathcal{F}^{\text{int}} \subset \mathbb{D}$, as follows: $\mathbf{x}^{-1}(\mathbb{C} \cup \{\infty\} \setminus [\alpha_1, \beta_{\ell+1}])$ consists of connected components on which $\mathbf{x}$ is a bijection (this is because $\mathbb{C} \cup \{\infty\} \setminus [\alpha_1, \beta_{\ell+1}]$ is simply connected, and so contains no closed curve nonhomotopic to the trivial curve in $\mathcal{S}_+$). We let $\mathcal{F}^{\text{int}}$ be the component containing $0 \in \mathbf{x}^{-1}(\{\infty\})$; we will shortly enlarge $\mathcal{F}^{\text{int}}$ to a fundamental set, $\mathcal{F}$.

In $\mathcal{F}^{\text{int}}$, consider $\mathbf{x}^{-1}(\mathbb{R} \cup \{\infty\} \setminus [\alpha_1, \beta_{\ell+1}])$. By (2.8), the set is a subset of $\mathbb{D} \cap \mathbb{R}$. But as $y \to \alpha_1$ or $\beta_{\ell+1}$, $\mathbf{x}^{-1}(y)$ must approach the boundary of $\mathbb{D}$. It follows that $\mathbf{x}^{-1}(\mathbb{R} \cup \{\infty\} \setminus [\alpha_1, \beta_{\ell+1}]) = (-1, 1) \subset \mathbb{D}$. The other inverse images of this set are, by (1.22), images of $(-1, 1)$ under Möbius transformations, so arcs of orthocircles, that is, pieces of circles orthogonal to $\partial \mathbb{D}$.

In place of (1.21), we could have required $\mathbf{x}(0) = \frac{1}{2}(\beta_j + \alpha_{j+1})$ together with $\mathbf{x}'(0) > 0$ and seen that for this $\mathbf{x}$, one has $(-1, 1)$ in the inverse image of the gap (β_j, α_{j+1}). Since the two $\mathbf{x}$'s are related by a Möbius transformation, we conclude that under our $\mathbf{x}$ (normalized by (1.21)) the inverse images of gaps are also arcs of orthocircles. The boundary of $\mathcal{F}^{\text{int}}$ in $\mathbb{D}$ (not $\overline{\mathbb{D}}$) clearly has 2ℓ pieces corresponding to the tops and bottoms of the ℓ gaps. Thus, $\mathcal{F}^{\text{int}}$ is $\mathbb{D}$ with 2ℓ orthocircles (and their interior) removed—ℓ in each half-plane—these are conjugate to one another. We label the boundary pieces in the upper half-disk $C_1^+, \ldots, C_\ell^+$. Figure 1 shows a typical $\mathcal{F}^{\text{int}}$ for $\ell = 2$ with the inverse image of $\mathbb{C}_- \cap \mathcal{S}_+$ shaded.
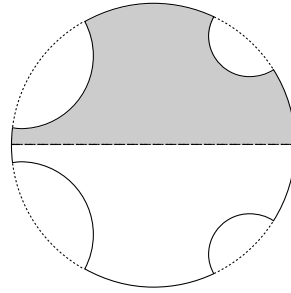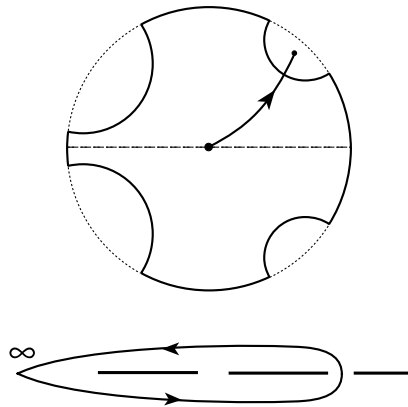
Fig. 1 The fundamental region

Fig. 2 Fuchsian group
generators

We now define a fundamental set, $\mathcal{F}$, by adding the arcs $C_1^+, \ldots, C_\ell^+$ to $\mathcal{F}^{\text{int}}$. With
this definition, every point in $\mathbb{D}$ can be uniquely written as $\gamma(w)$ for some $w \in \mathcal{F}$ and
some $\gamma \in \Gamma$. The fundamental region $\mathcal{F}^{\text{int}}$ is indeed the interior of $\mathcal{F}$. As a subset
of $\mathbb{D}$, $\overline{\mathcal{F}}$ has $C_1^-, \ldots, C_\ell^-$ added. Here C_j^- denotes the complex conjugate of C_j^+.
Sometimes we want to consider the closure of $\mathcal{F}$ in $\overline{\mathbb{D}}$, that is, also add the 2ℓ arcs in
$\partial\mathbb{D}$ at the ends.

To describe the Fuchsian group, Γ, we begin with the ℓ generators: the deck trans-
formations that go into the generators, $\tilde{\gamma}_1, \ldots, \tilde{\gamma}_\ell$, of the homotopy group, $\pi_1(\mathcal{S}_+)$.
Figure 2 shows the lift of the curve associated to $\tilde{\gamma}_2$ in our example. The bottom half
of the curve in $\mathcal{S}_+$ under $\mathbf{x}^{-1}$ goes from 0 to a point on C_2^+. Since that half of $\tilde{\gamma}_2$ lies
in $\mathbb{C}_-$, this piece of curve lies in $\mathbb{C}_+ \cap \mathcal{F}$. The second half must be the inversion in
the curve C_2^+ of the first half, and so it is as shown.

A little thought shows that the Möbius map that corresponds to $\tilde{\gamma}_2$, which we will
call γ_2, is what one gets by composing complex conjugation with inversion in C_2^+.
Inversion in the circle $|z - z_0| = r$ is the map

$$z \to z_0 + \frac{r^2}{\bar{z} - \bar{z}_0}. \tag{2.9}$$

Thus,

$$\gamma_j = r_j^+ c,$$

where $c(z) = \bar{z}$ and r_j^+ is inversion in C_j^+.

Γ is the free non-Abelian group generated by $\{\gamma_j\}_{j=1}^{\ell}$. Every element of Γ can be uniquely written as $\alpha_{w(\gamma)} \cdots \alpha_2 \alpha_1$, where each α_j is either a γ_j or a γ_j^{-1} and no α_j is an α_{j-1}^{-1}. The word length of γ is $w(\gamma)$. It will be convenient to define

$$\Gamma_k = \{\gamma \mid w(\gamma) = k\}. \tag{2.10}$$

We have $\#\Gamma_k = 2\ell(2\ell - 1)^{k-1}$, since α_1 has 2ℓ choices $(\gamma_1, \ldots, \gamma_\ell, \gamma_1^{-1}, \ldots, \gamma_\ell^{-1})$ and each other α_j has $2\ell - 1$ choices. By definition, $\Gamma_0 = \{1\}$.

Alternatively, one can write for $\gamma \in \Gamma_{2m}$,

$$\gamma = s_1 \cdots s_{2m}, \tag{2.11}$$

with each s_k an $r_j^{\pm}$ (r_j^- is inversion in C_j^-), and for $\gamma \in \Gamma_{2m+1}$,

$$\gamma = s_1 \cdots s_{2m+1} c. \tag{2.12}$$

We point out that $\mathcal{F}$ is the Dirichlet fundamental region for Γ, that is,

$$\overline{\mathcal{F}} = \{z \mid |\gamma(z)| \geq |z| \text{ for all } \gamma \in \Gamma\}. \tag{2.13}$$

Moreover, C_j^+ is the perpendicular bisector in the hyperbolic metric of 0 and $\gamma_j(0)$ (see, e.g., [66, Sect. 9.3]).

Since $\gamma \in \Gamma$ has no fixed points in $\mathbb{D}$, it cannot be elliptic, and it is not hard to see [66] that it is, in fact, hyperbolic.

The fact that $\mathcal{F}$ is a fundamental set implies that

$$\mathbb{D} = \bigcup_{\gamma \in \Gamma} \gamma(\mathcal{F}). \tag{2.14}$$

We will let

$$\mathbb{D}_k = \bigcup_{w(\gamma) \leq k} \gamma(\mathcal{F}), \tag{2.15}$$

and

$$\mathcal{R}_k = \mathbb{D} \setminus \mathbb{D}_k, \tag{2.16}$$

and finally define $\partial \mathcal{R}_k \subset \partial \mathbb{D}$ as

$$\partial \mathcal{R}_k = \overline{\mathcal{R}_k} \cap \partial \mathbb{D}, \tag{2.17}$$

where the closure is taken in $\overline{\mathbb{D}}$. Thus, $\mathbb{D}_k$ is connected, while $\mathcal{R}_k$ consists of $2\ell(2\ell - 1)^k$ disks (intersected with $\mathbb{D}$) with only some boundaries included and $\partial \mathcal{R}_k$
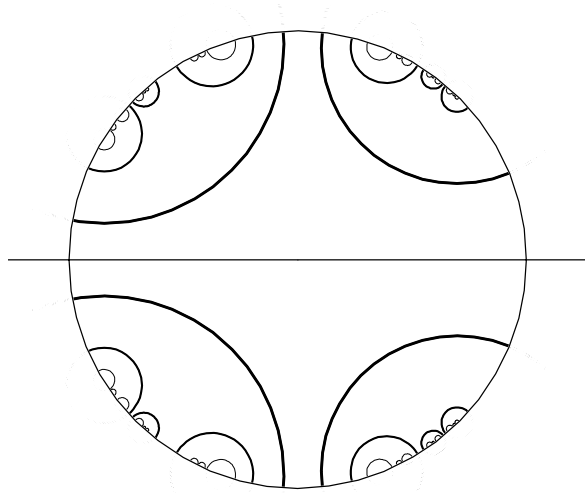
is $2\ell(2\ell - 1)^k$ connected arcs in $\partial\mathbb{D}$. Figure 3 shows the arcs $C_j^{\pm}$ and their images under Γ_1 and Γ_2.

In Fig. 3, the union of the large partial disks is $\mathcal{R}_0$, the union of the medium partial disks is $\mathcal{R}_1$, and the union of the tiny partial disks is $\mathcal{R}_2$.

Notice the geometry is such that

$$z \in \mathcal{R}_k \quad \Rightarrow \quad \frac{z}{|z|} \in \partial\mathcal{R}_k, \tag{2.18}$$

which we will need in Sect. 7 and [16].

We denote by $\mathcal{L}$ the set of limit points of Γ. It is a subset of $\partial\mathbb{D}$ and can be defined via several equivalent definitions:

(i) $\mathcal{L} = \bigcap_k \overline{\mathcal{R}}_k$,
(ii) $\mathcal{L} = \overline{\{z \in \partial\mathbb{D} \mid \gamma(z) = z \text{ for some } \gamma \neq 1; \ \gamma \in \Gamma\}}$,
(iii) $\mathcal{L} = \overline{\{\gamma(0) \mid \gamma \in \Gamma\}} \cap \partial\mathbb{D}$.

In (ii), each $\gamma \in \Gamma$ is hyperbolic, so it has two fixed points on $\partial\mathbb{D}$, each of which is either $\lim_{n\to\infty} \gamma^n(0)$ or $\lim_{n\to\infty} \gamma^{-n}(0)$. This is the key to proving that (ii) and (iii) are equivalent and that (iii) is the same if $\gamma(0)$ is replaced by $\gamma(z_0)$ for any fixed $z_0 \in \mathbb{D}$. The key to understanding (i) is that, by definition, $\mathbb{D}_k$ contains only finitely many $\gamma(0)$, all of which are a finite distance from $\partial\mathbb{D}$. As we will explain in the next section, $\mathcal{L}$ is of one-dimensional Lebesgue measure zero, indeed, of Hausdorff dimension strictly smaller than one.

Now, we return to $\mathcal{S}_+$ and the two-sheeted Riemann surface $\mathcal{S}$. The basic fact here is that the map $\mathbf{x} : \mathbb{D} \to \mathcal{S}_+$ has an analytic continuation both to a map of $\mathbb{C} \cup \{\infty\} \setminus \mathcal{L}$ to $\mathbb{C} \cup \{\infty\}$ and to a map

$$\mathbf{x}^{\sharp} : \mathbb{C} \cup \{\infty\} \setminus \mathcal{L} \to \mathcal{S}. \tag{2.19}$$

By construction, $\mathbf{x}(z)$ approaches $\mathbb{R}$ as $z \to \partial\mathbb{D}$ with $z \in \mathcal{F}$. So, by the strong form of the reflection principle, $\mathbf{x}$ is continuous and real-valued up to $\partial\mathbb{D} \cap \overline{\mathcal{F}}$ and can be

meromorphically continued to $(\partial \mathbb{D} \cap \overline{\mathcal{F}}) \cup \mathcal{F}^{-1}$. Utilizing (1.22), we can thus extend $\mathbf{x}$ to a map of $\mathbb{C} \cup \{\infty\} \setminus \mathcal{L}$ onto $\mathbb{C} \cup \{\infty\}$. Outside $\overline{\mathbb{D}}$, $\mathbf{x}$ is defined by

$$\mathbf{x}(1/z) = \mathbf{x}(z). \tag{2.20}$$

At points, z_0, where $\mathbf{x}(z_0)$ is real and $\mathbf{x}(z) - \mathbf{x}(z_0)$ has a zero of order k, there are $2k$ curves (asymptotically rays) coming out of z_0 on which $\mathbf{x}(z)$ is real. On $\mathbb{C}_+ \cap \mathcal{F}$, $\mathbf{x}$ has negative imaginary part and so, by reflection, at points, z_0, in $\mathbb{C}_+ \cap \partial \mathbb{D} \cap \overline{\mathcal{F}}$— except for the endpoints, there are two rays near z_0 where $\mathbf{x}$ is real. It follows that on the set

$$\left\{ z \in \mathbb{C} \cup \{\infty\} \setminus \mathcal{L} \mid \mathbf{x}(z) \notin \{\alpha_j, \beta_j\}_{j=1}^{\ell+1} \right\}, \tag{2.21}$$

$\mathbf{x}'$ is nonzero. At points, z_0, where $\mathbf{x}(z_0) \in \{\alpha_j, \beta_j\}_{j=1}^{\ell+1}$, images of $\mathbb{R}$ or a C_j^+ under an element $\gamma \in \Gamma$ intersect $\partial \mathbb{D}$ orthogonally, so four rays on which $\mathbf{x}$ is real come out of z_0. Hence, $\mathbf{x}(z) - \mathbf{x}(z_0)$ has a double zero, that is, $\mathbf{x}'(z_0) = 0$ and $\mathbf{x}''(z_0) \neq 0$, and the extended map $\mathbf{x}$ is therefore not a local bijection at such points z_0. The same is true for the canonical projection $\pi : \mathcal{S} \to \mathbb{C} \cup \{\infty\}$, so if we think of $\mathbf{x}: \mathbb{D} \to \mathcal{S}_+ \subset \mathcal{S}$ (rather than into a subset of $\mathbb{C} \cup \{\infty\}$), we can extend it to a map $\mathbf{x}^\sharp : \mathbb{C} \cup \{\infty\} \setminus \mathcal{L} \to \mathcal{S}$ via

$$\mathbf{x}^\sharp(1/z) = \tau\big(\mathbf{x}(z)\big). \tag{2.22}$$

Then the maps $\mathbf{x}$ and $\mathbf{x}^\sharp$ are related via

$$\mathbf{x} = \pi \circ \mathbf{x}^\sharp. \tag{2.23}$$

The elements of Γ are rational functions and so maps of $\mathbb{C} \cup \{\infty\}$ to itself. It is easy to see that each γ maps $\mathcal{L}$ to $\mathcal{L}$ (for if $\gamma_n(0) \to z_0$, then $\gamma \circ \gamma_n(0) \to \gamma(z_0)$) and so $\mathbb{C} \cup \{\infty\} \setminus \mathcal{L}$ to itself. Of course, we have

$$\mathbf{x}\big(\gamma(z)\big) = \mathbf{x}(z), \tag{2.24}$$

since γ is analytic and this holds on $\mathbb{D}$. The map $\mathbf{x}^\sharp$ has a similar relation. Indeed, since (2.22) holds, we have (1.22) for $\mathbf{x}^\sharp$ on all of $\mathcal{S}$. By the unfolding discussed above, $\mathbf{x}^\sharp$ is a local bijection on all of $\mathbb{C} \cup \{\infty\} \setminus \mathcal{L}$, that is, a covering map, albeit not the universal cover.

A major role will be played by automorphic and character automorphic functions. These are functions, f, defined on $\mathbb{D}$ (usually analytic but occasionally meromorphic and occasionally only harmonic) or on $\mathbb{C} \cup \{\infty\} \setminus \mathcal{L}$ (always meromorphic) which obey

$$f\big(\gamma(z)\big) = c(\gamma) f(z) \tag{2.25}$$

for all γ and z. Here $c \neq 0$ and must obey

$$c(\gamma \gamma') = c(\gamma) c(\gamma'). \tag{2.26}$$

If $c \equiv 1$, f is called *automorphic*. If $|c| = 1$ so that c is a unitary group character, we call f *character automorphic*. A character is determined by $\{c(\gamma_j)\}_{j=1}^{\ell}$ which can

be chosen independently so the set of all characters is an ℓ-dimensional torus, Γ^*. This set has a group structure if $c\tilde{c}(\gamma) = c(\gamma)\tilde{c}(\gamma)$ is the product and $c(\gamma) = 1$ the identity. Moreover, $c^{-1}(\gamma) = \overline{c(\gamma)}$.

Notice that $\mathbf{x}$ is automorphic on $\mathbb{D}$ and $\mathbf{x}^\sharp$ is automorphic on $\mathbb{C} \cup \{\infty\} \setminus \mathcal{L}$ if we extend the notion to include $\mathcal{S}$-valued functions. Moreover, f is automorphic and analytic (respectively meromorphic) on $\mathbb{D}$ if and only if there is a function, F, on $\mathcal{S}_+$ which is analytic (respectively meromorphic) with

$$F\big(\mathbf{x}(z)\big) = f(z). \tag{2.27}$$

Similarly, (2.27) with $\mathbf{x}$ replaced by $\mathbf{x}^\sharp$ sets up a one-to-one correspondence between meromorphic functions on $\mathcal{S}$ and meromorphic automorphic functions on $\mathbb{C} \cup \{\infty\} \setminus \mathcal{L}$.

In particular, we have that the analog of (1.19),

$$M(z) = -m\big(\mathbf{x}(z)\big), \tag{2.28}$$

is an automorphic function with $\operatorname{Im} M(z) > 0$ for $z \in \mathcal{F}^{\text{int}} \cap \mathbb{C}_+$.

We will require the following result:

Theorem 2.1 *Fix ℓ and let $Q_\ell \subset \mathbb{R}^{2\ell+2}$ be the set of $(\alpha_1, \beta_1, \ldots, \alpha_{\ell+1}, \beta_{\ell+1})$ for which* (1.3) *holds. Given $q \in Q_\ell$, let $\mathbf{x}_q$ be the covering map and $\gamma_{j;q}$ the Fuchsian group generators. Then $\mathbf{x}_q$ and $\gamma_{j;q}$ are continuous in q on Q_ℓ.*

This is a special case of a theorem of Hejhal [34] who noted that one can also base a proof on ideas from Ahlfors–Bers [2]. We have found a proof for the case at hand and given it in [66, Sect. 9.8]. We note that the Blaschke product, $B(z)$, that we discuss in Sect. 4 below is also continuous in q.

We will also need the following well-known fact about functions on $\mathcal{S}$ (see [24, 31] or [66, Theorem 5.12.5]):

Theorem 2.2 *Let F be a nonconstant meromorphic function on $\mathcal{S}$. Then F has a degree, d, so that for all a, $\{w \mid F(w) = a\}$ has d points, counting multiplicity (i.e., d is the sum of the orders of zeros of $F(w) - a$ in local coordinates). If $F \circ \tau \not\equiv F$, then the degree of F is at least $\ell + 1$.*

Remark In particular, if F is analytic on $\mathcal{S}$, it must be constant.

3 Beardon's Theorem

From our point of view, a theorem of Beardon [8] plays a critical role. To state the theorem, we need some notions. Fix $s > 0$ and a Fuchsian group, Γ. The *Poincaré series* is given by

$$\sum_{\gamma \in \Gamma} \big|\gamma'(0)\big|^s. \tag{3.1}$$

We are interested in when this series is convergent. It is a basic fact (see [38, 66]) that if the series in (3.1) converges, then uniformly for z in compacts of $\mathbb{D}$, the series

$$\sum_{\gamma \in \Gamma} \left| \gamma'(z) \right|^s \tag{3.2}$$

converges, as does uniformly on compacts of $\mathbb{D}$ the series

$$\sum_{\gamma \in \Gamma} \left(1 - \left| \gamma(z) \right| \right)^s. \tag{3.3}$$

Indeed, convergence of (3.2) for one z implies convergence uniformly on compacts. What is also true (see, e.g., [66, Sect. 9.4]) is that if $K \subset \overline{\mathbb{D}} \setminus \mathcal{L}$ is compact, then there is $C > 0$ so that for all $z \in K$ and all $\gamma \in \Gamma$,

$$\left| \gamma'(z) \right| \leq C \left| \gamma'(0) \right|, \tag{3.4}$$

so convergence of (3.1) implies convergence of (3.2) uniformly for $z \in K$. In particular, since $\overline{\mathbb{D}} \setminus \mathcal{L}$ is connected, we see that the series

$$\sum_{\gamma \in \Gamma} \left| \gamma(z) - \gamma(w) \right|^s \tag{3.5}$$

converges uniformly for z, w in compacts $K \subset \overline{\mathbb{D}} \setminus \mathcal{L}$.

Poincaré [53] proved that for any Fuchsian group, (3.1) converges if $s = 2$, and Burnside [13, 14] proved that if the set of limit points is not all of $\partial\mathbb{D}$, then (3.1) converges if $s = 1$. Beardon proved:

Theorem 3.1 (Beardon [8]) *If Γ is a finitely generated Fuchsian group whose limit points are not all of $\partial\mathbb{D}$, then there is some $s < 1$ so that* (3.1) *converges.*

As Beardon noted, this is equivalent to the set of limit points having Hausdorff dimension less than 1. Indeed, it is known (in work later than Beardon, see [49, 70] and also [10, Chap. 14]) that the infimum over s for which (3.1) converges is the Hausdorff dimension of $\mathcal{L}$. We note that Beardon's proof is very involved, in part because of the need to consider issues such as elliptic and parabolic elements that are irrelevant to our setup. The result for our case is proven using some simple geometry in Simon [66].

There is an important consequence of Beardon's theorem that we need. Let

$$\tilde{\mathcal{R}} = \partial\mathbb{D} \setminus \partial\mathcal{R}_0, \tag{3.6}$$

that is, $\overline{\mathcal{F}} \cap \partial\mathbb{D}$. This set consists of 2ℓ arcs. For each $\gamma \in \Gamma$, $\gamma(\tilde{\mathcal{R}})$ is also 2ℓ arcs, so

$$\partial\mathcal{R}_k = \mathcal{L} \cup \left[\bigcup_{w(\gamma) > k} \gamma(\tilde{\mathcal{R}}) \right]. \tag{3.7}$$

It is not hard to see that on $\tilde{\mathcal{R}}$ and its images, each $\gamma_j^{\pm}$, but one, decreases sizes by a fixed amount so that ($|\cdot|$ is total arc length)

$$\left|\gamma(\tilde{\mathcal{R}})\right| \leq Ce^{-Dw(\gamma)} \tag{3.8}$$

for some fixed constants $C, D > 0$ (proven in [66, Sect. 9.6]).

By (3.4), for some constant Q,

$$\left|\gamma(\tilde{\mathcal{R}})\right| \leq Q\left|\gamma'(0)\right|. \tag{3.9}$$

Hence

$$\left|\gamma(\tilde{\mathcal{R}})\right| \leq \left|Ce^{-Dw(\gamma)}\right|^{1-s}\left|Q\left|\gamma'(0)\right|\right|^{s}. \tag{3.10}$$

So, by Beardon's theorem for some $s < 1$,

$$|\partial\mathcal{R}_k| \leq C^{1-s}Q^{s}e^{-D(1-s)k}\sum_{\gamma\in\Gamma}\left|\gamma'(0)\right|^{s} \tag{3.11}$$

and thus:

Theorem 3.2 *For some constants $C_0, D_0 > 0$, we have*

$$|\partial\mathcal{R}_k| \leq C_0 e^{-D_0 k}. \tag{3.12}$$

4 Blaschke Products and (Potential Theorist's) Green's Function

The initial elements of this section are classical; see, for example, Tsuji [74]. Given $w \in \mathbb{D}$, we define $b(z, w)$ by

$$b(z, w) = \begin{cases} \frac{|w|}{w}\frac{w-z}{1-\bar{w}z} & w \neq 0, \\ z & w = 0, \end{cases} \tag{4.1}$$

which is meromorphic in z on $\mathbb{C} \cup \{\infty\}$, analytic in z on $\mathbb{D}$, and is the unique bijective map of $\mathbb{D} \to \mathbb{D}$ with

$$b(w, w) = 0 \tag{4.2}$$

and

$$b(0, w) > 0 \quad (w \neq 0); \qquad b'(0, w) > 0 \quad (w = 0). \tag{4.3}$$

Note that b is continuous in z on $\overline{\mathbb{D}}$ and

$$\left|b\left(e^{i\theta}, w\right)\right| = 1. \tag{4.4}$$

The following is standard (see Rudin [57]):

Lemma 4.1 *Let $(w_j)_{j=1}^{\infty}$ be a sequence of points in $\mathbb{D}$.*

(a) *If*

$$\sum_{j=1}^{\infty}(1 - |w_j|) = \infty, \tag{4.5}$$

then as $N \to \infty$,

$$\prod_{j=1}^{N} b(z, w_j) \to 0 \tag{4.6}$$

uniformly on compact subsets of $\mathbb{D}$.

(b) *If*

$$\sum_{j=1}^{\infty}(1 - |w_j|) < \infty, \tag{4.7}$$

then as $N \to \infty$,

$$\prod_{j=1}^{N} b(z, w_j) \to B(z, (w_j)) \tag{4.8}$$

uniformly on compact subsets of $\mathbb{D}$, *where* B *is analytic in* $\mathbb{D}$ *and obeys*

$$B(z, (w_j)) = 0 \quad \Leftrightarrow \quad z \in (w_j). \tag{4.9}$$

Moreover, for Lebesgue a.e. θ,

$$\lim_{r \uparrow 1} B(re^{i\theta}, (w_j)) \equiv B(e^{i\theta}, (w_j)) \tag{4.10}$$

exists obeying

$$|B(e^{i\theta}, (w_j))| = 1. \tag{4.11}$$

Remarks 1. The refined form of (4.9) says that the order of the zero at some w_j is the number of times it occurs in (w_j).

2. The proof shows that when (4.7) holds, uniformly for $|z| \le \rho < 1$, we have

$$\sum_{j=1}^{\infty}|1 - b(z, w_j)| < \infty. \tag{4.12}$$

3. The proof of (4.12) follows from the simple inequality

$$|1 - b(z, w)| \le \frac{1 + |z|}{|1 - \bar{w}z|}(1 - |w|), \tag{4.13}$$

which also proves that the product converges on $\mathbb{C} \setminus [\overline{\mathbb{D}} \cup (1/\bar{w}_j)]$ and on any set $K \subset \partial\mathbb{D}$ with

$$\inf_{e^{i\theta} \in K, w_j} |e^{i\theta} - w_j| > 0. \tag{4.14}$$

In particular, if there is any set $K \subset \partial \mathbb{D}$ for which (4.14) holds, we can find such an open set and so get a product analytic across K. This product is meromorphic, with poles at the points $1/\bar{w}_j$.

We also need the following:

Lemma 4.2 *Suppose γ is an analytic bijection of $\mathbb{D}$ to $\mathbb{D}$. For any $z, w \in \mathbb{D}$, we have*

$$\text{(i)} \quad \bigl|b(z, w)\bigr| = \bigl|b(w, z)\bigr|, \tag{4.15}$$

$$\text{(ii)} \quad \bigl|b\bigl(\gamma(z), \gamma(w)\bigr)\bigr| = \bigl|b(z, w)\bigr|. \tag{4.16}$$

Proof (i) is immediate. For (4.16), fix w and let

$$h(z) = \frac{b(\gamma(z), \gamma(w))}{b(z, w)}. \tag{4.17}$$

It is easy to see that h has a removable singularity at $z = w$, and so it is analytic in $\mathbb{D}$ and continuous on $\overline{\mathbb{D}}$. By (4.4), $|h(e^{i\theta})| = 1$, so, by the maximum principle, $|h(z)| \leq 1$ on $\mathbb{D}$. But $1/h$ has the same properties as h, so $|1/h(z)| \leq 1$, which implies that $|h(z)| = 1$, that is, (4.16) holds. $\qquad \square$

The following is true for any Fuchsian group whose limit points are not dense in $\partial \mathbb{D}$—but we only care here about the Γ's associated to finite gap sets:

Theorem 4.3 *Let Γ be the Fuchsian group of a finite gap set. For any $w \in \mathbb{D}$, the product*

$$\prod_{\gamma \in \Gamma} b\bigl(z, \gamma(w)\bigr) \equiv B(z, w) \tag{4.18}$$

converges for all $z \in \mathbb{C} \cup \{\infty\} \setminus [\mathcal{L} \cup \{\overline{\gamma(w)}^{-1}\}_{\gamma \in \Gamma}]$ and defines a function analytic there and meromorphic in $\mathbb{C} \cup \{\infty\} \setminus \mathcal{L}$. B has simple poles at the points $\{\overline{\gamma(w)}^{-1}\}_{\gamma \in \Gamma}$, simple zeros at $\{\gamma(w)\}_{\gamma \in \Gamma}$, and no other zeros or poles. Moreover,

(i) *For $z, w \in \mathbb{D}$,*

$$\bigl|B(z, w)\bigr| = \bigl|B(w, z)\bigr|. \tag{4.19}$$

(ii) *Each $B(\,\cdot\,, w)$ is character automorphic, that is, for every $w \in \mathbb{D}$ there is a character, c_w, on Γ so that*

$$B\bigl(\gamma(z), w\bigr) = c_w(\gamma) B(z, w). \tag{4.20}$$

(iii) *If*

$$B(z) \equiv B(z, 0), \tag{4.21}$$

then for $z \in \mathbb{C} \cup \{\infty\} \setminus [\mathcal{L} \cup \{\overline{\gamma(0)}^{-1}\}_{\gamma \in \Gamma}]$,

$$\bigl|B(z)\bigr| = \prod_{\gamma \in \Gamma} |\gamma(z)|. \tag{4.22}$$

(iv) *For $e^{i\theta} \in \partial\mathbb{D} \setminus \mathcal{L}$ (with $' = \partial/\partial\theta$),*

$$\left| B'(e^{i\theta}) \right| = \sum_{\gamma \in \Gamma} \left| \gamma'(e^{i\theta}) \right|. \tag{4.23}$$

Remarks 1. We will see below (Theorem 4.4) that $c_0(\gamma)$ is not the identity.

2. When $z \in \{\overline{\gamma(0)}^{-1}\}_{\gamma \in \Gamma}$, both sides of (4.22) are infinite.

Proof By Theorem 3.1 (or Burnside's theorem), for any $w \in \mathbb{D}$,

$$\sum_{\gamma \in \Gamma} \left| 1 - \gamma(w) \right| < \infty. \tag{4.24}$$

So, by Lemma 4.1 and (4.13), the product converges where claimed and has the claimed analytic/zero/pole properties.

(i) is immediate from (4.15)–(4.16). By the proof of (4.16), there exists $\eta(w, \gamma) \in \partial\mathbb{D}$ so that

$$b(\gamma(z), \gamma(w)) = \eta(w, \gamma) b(z, w). \tag{4.25}$$

Thus, for any finite subset, $G \subset \Gamma$,

$$\prod_{\gamma' \in G} b(\gamma(z), \gamma'(w)) = \prod_{\gamma' \in G} b(\gamma(z), \gamma(\gamma^{-1}\gamma'(w)))$$

$$= \prod_{\gamma' \in G} \eta(\gamma^{-1}\gamma'(w), \gamma) \prod_{\gamma'' \in \gamma^{-1}(G)} b(z, \gamma''(w)). \tag{4.26}$$

If $G_n \subset G_{n+1}$ with $\bigcup_n G_n = \Gamma$ and γ is fixed, $\gamma^{-1}(G_n) \subset \gamma^{-1}(G_{n+1})$ and $\bigcup_n \gamma^{-1}(G_n) = \Gamma$, so the left side of (4.26) and the last factor on the right converge to $B(\gamma(z), w)$ and $B(z, w)$, respectively. Thus, the product of η's converges to some $c_w(\gamma) \in \partial\mathbb{D}$, that is, (4.20) holds. From (4.20), it is easy to see that $c_w(\gamma\gamma') = c_w(\gamma)c_w(\gamma')$. That proves (ii).

To get (4.22), suppose first $z \in \mathbb{D}$. Then by (4.19),

$$\left| B(z) \right| = \left| B(z, 0) \right| = \left| B(0, z) \right| = \left| \prod_{\gamma \in \Gamma} \gamma(z) \right|,$$

proving (4.22) for $z \in \mathbb{D}$.

Since for $z \in \mathbb{D}$,

$$\overline{B(1/\bar{z})} = B(z)^{-1}, \qquad \overline{\gamma(1/\bar{z})} = \gamma(z)^{-1} \tag{4.27}$$

(on account of $|B(e^{i\theta})| = |\gamma(e^{i\theta})| = 1$ and the reflection principle), we get (4.22) on $\mathbb{C} \cup \{\infty\} \setminus [\overline{\mathbb{D}} \cup \{\overline{\gamma(0)}^{-1}\}_{\gamma \in \Gamma}]$. Finally, on $\partial\mathbb{D} \setminus \mathcal{L}$, both sides of (4.22) are 1. This completes (iii).

To prove (iv), we note that if g is an analytic function on $\mathbb{D}$ with $|g(z)| < 1$ on $\mathbb{D}$ and $I \subset \partial\mathbb{D}$ is an open interval so that g has an analytic continuation across I with $|g(e^{i\theta})| = 1$ on I, then

$$\frac{\partial}{\partial\theta}|g(e^{i\theta})| = 0, \tag{4.28}$$

so, by Cauchy–Riemann equations,

$$\frac{\partial}{\partial r}\arg(g(re^{i\theta}))\bigg|_{r=1} = 0 \tag{4.29}$$

and

$$\frac{\partial}{\partial\theta}\arg(g(e^{i\theta})) = \frac{\partial}{\partial r}|g(re^{i\theta})|\bigg|_{r=1} > 0. \tag{4.30}$$

Hence, with $' = \frac{\partial}{\partial\theta}$,

$$|g'(e^{i\theta})| = \frac{\partial}{\partial r}|g(re^{i\theta})|\bigg|_{r=1} = \left|\frac{dg}{dz}(e^{i\theta})\right| \tag{4.31}$$

and

$$g'(e^{i\theta}) = i|g'(e^{i\theta})|g(e^{i\theta}). \tag{4.32}$$

In (4.30), we have strict positivity by the same argument that shows boundary values of Herglotz functions are strictly monotone.

Let G be a finite subset of Γ and

$$B_G(z) = \prod_{\gamma \in G} b(z, \gamma(0)). \tag{4.33}$$

Then

$$\frac{\partial}{\partial r}|B_G(z)|\bigg|_{z=e^{i\theta}} = \sum_{\gamma \in G} \frac{\partial}{\partial r}|b(z, \gamma(0))|\bigg|_{z=e^{i\theta}} \tag{4.34}$$

by Leibnitz's rule and $|b(e^{i\theta}, \gamma(0))| = 1$. If $z_0 \in \partial\mathbb{D} \setminus \mathcal{L}$, $B_G(z) \to B(z)$ for z in a neighborhood of z_0, so derivatives converge. Since $\frac{\partial}{\partial r}|B_G(z)| = |\frac{d}{dz}B_G(z)|$, the $\frac{\partial}{\partial r}$ derivatives converge. The terms in the sum are positive, so the sum over all of Γ is absolutely convergent, and (4.34) extends to the limit. By (4.30)–(4.31) and the fact that $|b(z, \gamma(0))| = |\gamma(z)|$, we get (4.23). $\qquad\square$

We emphasize for later use that (4.30)–(4.31) imply

$$\frac{\partial}{\partial\theta}\arg(\gamma(e^{i\theta})) = |\gamma'(e^{i\theta})| > 0. \tag{4.35}$$

Recall that the (real-valued) *potential theoretic Green's function*, $G_\mathfrak{e}$, is uniquely determined by requiring $G_\mathfrak{e}(z) - \log|z|$ to be harmonic on $\mathbb{C} \cup \{\infty\} \setminus \mathfrak{e}$, and for quasi-

every $x \in \mathfrak{e}$,

$$\lim_{\substack{z \to x \\ z \notin \mathfrak{e}}} G_{\mathfrak{e}}(z) = 0. \tag{4.36}$$

In fact, when $\mathfrak{e}$ has the form (1.1)–(1.3), $G_{\mathfrak{e}}$ can be chosen globally continuous on $\mathbb{C}$ with

$$G_{\mathfrak{e}} \restriction \mathfrak{e} = 0. \tag{4.37}$$

Moreover, near infinity,

$$G_{\mathfrak{e}}(z) = \log|z| - \log\big(\mathrm{cap}(\mathfrak{e})\big) + O\big(z^{-1}\big) \tag{4.38}$$

and, by the reflection principle, $G_{\mathfrak{e}}(z)$ is real analytic in x and $|y|$ near any $z_0 \in \mathfrak{e}^{\mathrm{int}}$. For $x \in \mathfrak{e}^{\mathrm{int}}$, we have

$$\rho_{\mathfrak{e}}(x) = \lim_{\varepsilon \downarrow 0} \frac{1}{\pi} \frac{\partial G_{\mathfrak{e}}}{\partial y}(x + i\varepsilon), \tag{4.39}$$

which we will write as

$$\frac{1}{\pi} \frac{\partial}{\partial n} G_{\mathfrak{e}}(x + i0), \tag{4.40}$$

the normal derivative in the positive direction.

Theorem 4.4 *Let Γ be the Fuchsian group of a finite gap set, $\mathfrak{e}$. Then for all $z \in \mathbb{D} \setminus \{0\}$,*

$$\big|B(z)\big| = e^{-G_{\mathfrak{e}}(\mathbf{x}(z))}. \tag{4.41}$$

Moreover,

(i) *If x_∞ is given by*

$$\mathbf{x}(z) = \frac{x_\infty}{z} + O(1) \tag{4.42}$$

    near $z = 0$, then, also near $z = 0$,

$$B(z) = \frac{\mathrm{cap}(\mathfrak{e})}{x_\infty} z + O\big(z^2\big). \tag{4.43}$$

(ii) *The character, c_0, of $B(z)$ is given by*

$$c_0(\gamma_j) = \exp\big(2\pi i \rho_{\mathfrak{e}}\big([\alpha_1, \beta_j]\big)\big). \tag{4.44}$$

(iii) *At any $x_0 \in \{\alpha_j, \beta_j\}_{j=1}^{\ell+1}$, $G_{\mathfrak{e}}$ has a square root zero in the sense that*

$$\lim_{x \uparrow \alpha_j} G_{\mathfrak{e}}(x)(\alpha_j - x)^{-1/2} = a_j, \tag{4.45}$$

$$\lim_{x \downarrow \beta_j} G_{\mathfrak{e}}(x)(x - \beta_j)^{-1/2} = b_j \tag{4.46}$$

    for nonzero a_j, b_j.

Proof By (4.20), $|B(z)|$ is automorphic, so there exists a function h on $\mathbb{C} \cup \{\infty\} \setminus \mathfrak{e}$ such that

$$\big|B(z)\big| = h\big(\mathbf{x}(z)\big). \tag{4.47}$$

Since $\mathbf{x}$ is analytic and $\mathbf{x}B$ is nonvanishing and analytic in a neighborhood of $\overline{\mathcal{F}}$, $\log(h(x)) + \log|x|$ is harmonic on $\mathbb{C} \cup \{\infty\} \setminus \mathfrak{e}$. By $|B(z)| = 1$ on $\overline{\mathcal{F}} \cap \partial \mathbb{D}$, $\log(h(x)) \to 0$ as $x \to \mathfrak{e}$. Thus, $-\log(h(x))$ is $G_{\mathfrak{e}}(x)$, proving (4.41).

Equation (4.43) is immediate from (4.42) and (4.38). The limit (4.45) and (4.46) follow from (4.41), $B'(z) \neq 0$ on $\overline{\mathcal{F}} \cap \partial \mathbb{D}$, and the fact that on $\mathbf{x}^{-1}(\{\alpha_j, \beta_j\}_{j=1}^{\ell+1})$, we have $\mathbf{x}'(z) = 0$, $\mathbf{x}''(z) \neq 0$. That leaves (4.44).

Consider the generator, γ_ℓ, whose action takes 0 into the endpoint of the curve in the top of Fig. 2. Since

$$B\big(\gamma_\ell(0)\big) = c_0(\gamma_\ell) B(0), \tag{4.48}$$

we see that

$$\arg\big(c_0(\gamma_\ell)\big) = \oint_0^{\gamma_\ell(0)} \frac{d}{dz} \arg\big(B(z)\big)\, dz. \tag{4.49}$$

$G_{\mathfrak{e}}(z)$ is harmonic on $\mathbb{C} \setminus \mathfrak{e}$ so that, locally, it is the real part of an analytic function, $\tilde{G}_{\mathfrak{e}}(z)$, but that function has a multivalued imaginary part. Thus, $e^{-\tilde{G}_{\mathfrak{e}}(z)} \equiv E(z)$ has a multivalued argument. Clearly,

$$B(z) = E\big(\mathbf{x}(z)\big) \tag{4.50}$$

and the change of argument in (4.49) is given by the change of argument of $E(z)$ over the curve in the bottom of Fig. 2. This is given by a sum of change of argument around curves surrounding each band, since in the gaps and in $(-\infty, \alpha_1)$, there is cancellation between top and bottom.

By a Cauchy–Riemann equation,

$$\frac{\partial}{\partial x} \arg\big(E(z)\big) = -\frac{\partial}{\partial y} \log\big|E(z)\big| = \frac{\partial G_{\mathfrak{e}}}{\partial n}.$$

The normal derivatives on top and bottom of a band have opposite sign, so given the opposite directions,

$$\arg\big(c_0(\gamma_\ell)\big) = \sum_{j=1}^{\ell} \int_{\alpha_j}^{\beta_j} 2 \frac{\partial G_{\mathfrak{e}}(x + i0)}{\partial n}\, dx \tag{4.51}$$

$$= 2\pi \sum_{j=1}^{\ell} \int_{\alpha_j}^{\beta_j} \rho_{\mathfrak{e}}(x)\, dx = 2\pi \rho_{\mathfrak{e}}\big([\alpha_1, \beta_\ell]\big), \tag{4.52}$$

which is (4.44) for $j = \ell$. Equality (4.52) follows from (4.39). The argument for general $j = 1, \ldots, \ell - 1$ is similar. $\square$

Corollary 4.5 $B(z)^p$ *is automorphic if and only if*

$$\rho_{\mathfrak{e}}\big([\alpha_j, \beta_j]\big) = \frac{q_j}{p} \qquad (4.53)$$

for integers q_j.

Remark We will eventually see (Corollary 6.4) that this relates periodic $\mathfrak{e}$ to periodic Jacobi matrices.

Proof $B(z)^p$ is automorphic if and only if $c_0(\gamma_j)^p = 1$ for $j = 1, 2, \ldots, \ell$ and this, given (4.44), is equivalent to (4.53). $\qquad \square$

Corollary 4.6 *Let $\mathfrak{e}$ be a finite gap set and $d\rho_{\mathfrak{e}}$ its equilibrium measure. Then*

$$\int_{\partial\mathbb{D}} f\big(\mathbf{x}(e^{i\theta})\big)\frac{d\theta}{2\pi} = \int_{\mathfrak{e}} f(x)\, d\rho_{\mathfrak{e}}(x) \qquad (4.54)$$

for all continuous f on $\mathfrak{e}$ and, if integrals are allowed to be infinite, for any positive measurable f on $\mathfrak{e}$. In particular, $f \in L^p(\mathfrak{e}, d\rho_{\mathfrak{e}})$ if and only if $f \circ \mathbf{x} \in L^p(\partial\mathbb{D}, \frac{d\theta}{2\pi})$, so the Szegő conditions

$$\int_{\partial\mathbb{D}} \log\big(f\big(\mathbf{x}(e^{i\theta})\big)\big)\frac{d\theta}{2\pi} > -\infty \qquad (4.55)$$

and

$$\int_{\mathfrak{e}} \log\big(f(x)\big)\, d\rho_{\mathfrak{e}}(x) > -\infty \qquad (4.56)$$

are equivalent for $f \in L^1(\mathfrak{e}, d\rho_{\mathfrak{e}})$.

Proof It suffices to prove (4.54) for continuous functions, f, and then use standard approximation arguments. Recall that $\tilde{\mathcal{R}}$ is given by (3.6) and consists of 2ℓ arcs. Except for endpoints, it is a fundamental domain for the action of Γ on $\partial\mathbb{D}$, so

$$\int_{\partial\mathbb{D}} f\big(\mathbf{x}(e^{i\theta})\big)\frac{d\theta}{2\pi} = \sum_{\gamma\in\Gamma} \int_{\gamma(\tilde{\mathcal{R}})} f\big(\mathbf{x}(e^{i\theta})\big)\frac{d\theta}{2\pi} \qquad (4.57)$$

$$= \sum_{\gamma\in\Gamma} \int_{\tilde{\mathcal{R}}} f\big(\mathbf{x}(e^{i\theta})\big)\big|\gamma'\big(e^{i\theta}\big)\big|\frac{d\theta}{2\pi} \qquad (4.58)$$

by a change of variables and the invariance of $\mathbf{x}$ under Γ, that is, $\mathbf{x} \circ \gamma = \mathbf{x}$. Thus, by (4.23),

$$\int_{\partial\mathbb{D}} f\big(\mathbf{x}(e^{i\theta})\big)\frac{d\theta}{2\pi} = \int_{\tilde{\mathcal{R}}} f\big(\mathbf{x}(e^{i\theta})\big)\big|B'\big(e^{i\theta}\big)\big|\frac{d\theta}{2\pi} \qquad (4.59)$$

$$= \int_{\tilde{\mathcal{R}}} f\big(\mathbf{x}(e^{i\theta})\big)\bigg|\frac{\partial}{\partial n}e^{-G_{\mathfrak{e}}(\mathbf{x}(e^{i\theta}))}\bigg|\bigg|\frac{d\mathbf{x}(e^{i\theta})}{d\theta}\bigg|\frac{d\theta}{2\pi}, \qquad (4.60)$$

where we use (4.41), $|B'(e^{i\theta})| = \frac{d}{dr}|B(re^{i\theta})||_{r=1}$, and the chain rule to go from $d\theta$ to $d\mathbf{x}$ derivatives. $\mathbf{x}$ on $\tilde{\mathcal{R}}$ is a two-fold cover of $\mathfrak{e}$, so using $\frac{\partial}{\partial n}e^{-G_{\mathfrak{e}}(\mathbf{x})} = -\frac{\partial}{\partial n}G_{\mathfrak{e}}$ (since $G_{\mathfrak{e}}$ is 0 on $\mathfrak{e}$) and (4.39)–(4.40), we get

$$\int_{\partial\mathbb{D}} f\big(\mathbf{x}(e^{i\theta})\big)\frac{d\theta}{2\pi} = 2\int_{\mathfrak{e}}\pi\rho_{\mathfrak{e}}(x)f(x)\frac{dx}{2\pi} = \int_{\mathfrak{e}} f(x)\,d\rho_{\mathfrak{e}}(x). \qquad (4.61)$$

$\square$

That concludes what we need about Blaschke products in this paper, but we put in some results on products of Blaschke products which will be critical in later papers in this series.

Theorem 4.7 *Let $(w_k)_{k=1}^{\infty}$ be a sequence in $\mathcal{F}$. Then*

$$\sum_{k=1}^{\infty}\big(1 - |w_k|\big) < \infty \quad \Leftrightarrow \quad \sum_{k=1}^{\infty}\big(1 - \big|B(w_k)\big|\big) < \infty. \qquad (4.62)$$

Moreover,

(i) *If $\sum_{k=1}^{\infty}(1 - |w_k|) = \infty$, then $\prod_{k=1}^{N} B(z, w_k)$ converges to 0 uniformly on compact subsets of $\mathbb{D}$.*

(ii) *If*

$$\sum_{k=1}^{\infty}\big(1 - |w_k|\big) < \infty, \qquad (4.63)$$

then for all $z \in \mathbb{D}$,

$$\sum_{k=1}^{\infty}\big|1 - B(z, w_k)\big| < \infty \qquad (4.64)$$

uniformly on compact subsets of $\mathbb{D}$, so $\prod_{k=1}^{N} B(z, w_k)$ converges to an analytic limit vanishing if and only if $z \in \{\gamma(w_k)\}_{\gamma\in\Gamma, k=1,\dots}$.

Remarks 1. If $\{\gamma_k\}$ is a countable set of distinct elements in Γ and $w_k = \gamma_k(0)$, then $\sum_{k=1}^{\infty}(1 - |w_k|) < \infty$ (by Burnside or Beardon), but $|B(z, \gamma_k(0))| = |B(z)|$ and $\prod_{k=1}^{N} B(z, w_k)$ converges to 0 uniformly on compact subsets of $\mathbb{D}$. Thus, the condition $w_k \in \mathcal{F}$ cannot be replaced by $w_k \in \mathbb{D}$ in (ii).

2. As in the case of Theorem 4.3, we can prove convergence on open subsets of $\mathbb{C}\cup\{\infty\}\setminus[\mathcal{L}\cup\{\overline{\gamma(w_k)}^{-1}\}_{\gamma\in\Gamma, k=1,\dots}]$ with poles at $\{\overline{\gamma(w_k)}^{-1}\}_{\gamma\in\Gamma, k=1,\dots}$.

Proof B is analytic in a neighborhood of $\overline{\mathcal{F}}\cap\mathbb{D}$, as noted in (4.30), $|B'(e^{i\theta})| \neq 0$, and, of course, $|B(e^{i\theta})| = 1$. Thus, for nonzero constants, c, d, and for all $w \in \mathcal{F}$,

$$c\big(1 - \big|B(w)\big|\big) \leq 1 - |w| \leq d\big(1 - \big|B(w)\big|\big), \qquad (4.65)$$

from which (4.62) is immediate. (Notice that (4.65) only holds on $\mathcal{F}$, not on $\mathbb{D}$, and is where the condition $w_k \in \mathcal{F}$ is used.)

To prove (i), we need only note that

$$\left| B(z, w_k) \right| \le \left| b(z, w_k) \right| \tag{4.66}$$

and use Lemma 4.1(a).

To prove (ii), we note that $\prod_{k=1}^{\infty} B(z, w_k)$ is a product of Blaschke products, so to prove (4.64), it suffices to prove that

$$\sum_{\gamma \in \Gamma, k} \left(1 - \left| \gamma(w_k) \right| \right) < \infty. \tag{4.67}$$

Since $\sum_{\gamma \in \Gamma} (1 - |\gamma(0)|) < \infty$, and zero occurs at most finitely often in the sequence $(w_k)_{k=1}^{\infty}$ if (4.63) holds, we can suppose that no w_k is zero, in which case, since $w_k \in \mathcal{F}$ implies (by (2.13))

$$|w_k| = \inf_{\gamma \in \Gamma} \left| \gamma(w_k) \right|, \tag{4.68}$$

we have

$$\inf_{\gamma \in \Gamma, k} \left| \gamma(w_k) \right| > 0. \tag{4.69}$$

This implies that (4.67) is equivalent to

$$\prod_{\gamma \in \Gamma, k} \left| \gamma(w_k) \right| > 0, \tag{4.70}$$

which, by (4.22), is equivalent to

$$\prod_k \left| B(w_k) \right| > 0. \tag{4.71}$$

Now, (4.71) is implied by

$$\sum_k \left(1 - \left| B(w_k) \right| \right) < \infty. \tag{4.72}$$

As we have seen, (4.63) implies (4.72), and thus (4.67).   □

We are especially interested in the case where w_k is determined by $w_k \in \mathcal{F}$ and $\mathbf{x}(w_k)$ real (so $w_k \in [\bigcup_{j=1}^{\ell} C_{\ell}^{+}] \cup (-1, 1)$). In that case:

Proposition 4.8 *Let $(x_k)_{k=1}^{\infty}$ be a sequence in $\mathbb{R} \setminus \mathfrak{e}$, and let $w_k \in \mathcal{F}$ be uniquely determined by*

$$\mathbf{x}(w_k) = x_k. \tag{4.73}$$

Then the following are equivalent:

$$\text{(i)} \quad \sum_{k=1}^{\infty} \left(1 - |w_k| \right) < \infty, \tag{4.74}$$

$$\text{(ii)} \quad \sum_{k=1}^{\infty} G_{\mathfrak{e}}(x_k) < \infty, \tag{4.75}$$

$$\text{(iii)} \quad \sum_{k=1}^{\infty} \text{dist}(x_k, \mathfrak{e})^{1/2} < \infty. \tag{4.76}$$

Proof By (4.45)–(4.46), we have (ii) ⇔ (iii). By (4.62), we have that (i) is equivalent to

$$\sum_{k=1}^{\infty} \big(1 - |B(w_k)|\big) < \infty, \tag{4.77}$$

which, by (4.41) and (4.73), is equivalent to

$$\sum_{k=1}^{\infty} \big|1 - e^{-G_{\mathfrak{e}}(x_k)}\big| < \infty. \tag{4.78}$$

In turn, (4.78) is easily seen to be equivalent to (4.75). $\qquad\square$

Finally, we need to discuss alternating Blaschke products. We will discuss a case with points approaching the top of a gap (or α_1). A similar result holds for approach to a β_j.

Theorem 4.9 *Suppose* $(\zeta_k)_{k=1}^{\infty}$, $(\rho_k)_{k=1}^{\infty}$ *obey, for some* j,

$$\beta_{j-1} < \zeta_1 < \rho_1 < \zeta_2 < \rho_2 < \cdots < \alpha_j \tag{4.79}$$

(where $\beta_0 \equiv -\infty$*) and*

$$\lim_{k \to \infty} \zeta_k = \alpha_j. \tag{4.80}$$

Let $\{z_k\} \cup \{p_k\} \subset \mathcal{F}$ *and* $a_j \in \overline{\mathcal{F}}$ *be given by*

$$\mathbf{x}(z_k) = \zeta_k, \qquad \mathbf{x}(p_k) = \rho_k, \qquad \mathbf{x}(a_j) = \alpha_j.$$

Then as $N \to \infty$,

$$\prod_{k=1}^{N} \frac{B(z, z_k)}{B(z, p_k)} \to B_{\infty}(z) \tag{4.81}$$

uniformly in z *on compact subsets of*

$$\mathbb{C} \cup \{\infty\} \setminus \big[\mathcal{L} \cup \big\{\gamma(p_k), \gamma\big(z_k^{-1}\big)\big\}_{\gamma \in \Gamma, k=1,\dots} \cup \big\{\gamma(a_j)\big\}_{\gamma \in \Gamma}\big] \tag{4.82}$$

to a function which is analytic on the set in (4.82) *with simple poles at points in* $\{\gamma(p_k), \gamma(z_k^{-1})\}_{\gamma \in \Gamma, k=1,\dots}$ *and with simple zeros only at the points* $\{\gamma(p_k^{-1}), \gamma(z_k)\}_{\gamma \in \Gamma, k=1,\dots}$.

Moreover,

$$z \in \partial\mathbb{D} \setminus \left[\mathcal{L} \cup \{\gamma(a_j)\}_{\gamma \in \Gamma}\right] \quad \Rightarrow \quad |B_\infty(z)| = 1 \tag{4.83}$$

and for some constant C (ε-dependent),

$$z \in \mathcal{F} \quad \Rightarrow \quad |\arg(B_\infty(z))| \le C \tag{4.84}$$

if $\arg(B_\infty)$ *is determined by requiring one value of* $\arg(B_\infty(0))$ *to be zero.*

In addition, if we remove the arc of C_{j-1}^+ (or segment of $[-1, 0)$ if $j = 1$) that runs from z_1 to a_j and all its images under $\gamma \in \Gamma$, we get a region, $\mathcal{B}$, free of zeros and poles of B_∞, on which

$$z \in \mathcal{B} \setminus \mathcal{R}_n \quad \Rightarrow \quad |\arg(B_\infty(z))| \le (2n + 1)C. \tag{4.85}$$

Remarks 1. If $z_k \in C_j^+$ for some j, then $\gamma_j^{-1}(z_k) = \bar{z}_k$, so $\{\gamma(z_k^{-1})\}_{\gamma \in \Gamma} = \{\gamma(\bar{z}_k^{-1})\}_{\gamma \in \Gamma}$, which is why we do not need to put complex conjugates in (4.82).

2. The analog of this result for $\mathfrak{e} = [-2, 2]$ is from Simon [58].

Sketch (See [66] for details.) One first shows that if ζ, ω run through a compact set, Q, in a single C_j^+ (including the endpoints) or $[-1, 0)$ or $(0, 1]$ and z through a compact subset, K, of $\mathbb{C} \setminus \mathcal{L} \cup [\{\gamma(Q)\}_{\gamma \in \Gamma} \cup \{\gamma(Q^{-1})\}_{\gamma \in \Gamma}]$, then there is $C < \infty$ so that for all $\gamma \in \Gamma$, $\zeta, \omega \in Q$ and $z \in K$,

$$|b(z, \gamma(\zeta)) - b(z, \gamma(\omega))| \le C|\gamma(\zeta) - \gamma(\omega)|. \tag{4.86}$$

This comes from looking at the three parts of

$$b(z, w) = \frac{|w|}{w} \frac{w - z}{1 - \bar{w}z}. \tag{4.87}$$

From this and telescoping, one gets

$$|B(z, \zeta) - B(z, \omega)| \le C \sum_{\gamma \in \Gamma} |\gamma(\zeta) - \gamma(\omega)| \le C_1|\zeta - \omega|. \tag{4.88}$$

Since

$$\inf_{\substack{z \in K \\ w \in Q}} |B(z, w)| > 0, \tag{4.89}$$

(4.88) implies that

$$\left|1 - \frac{B(z, \zeta)}{B(z, \omega)}\right| \le C_2|\zeta - \omega|, \tag{4.90}$$

which leads to the convergence of (4.81) if we note that

$$\sum_{k=1}^{\infty} |z_k - p_k| < \infty, \tag{4.91}$$

since the sum is bounded by the arclength of C_j^+ (or by 1 if $\alpha_j = \alpha_1$). This easily leads to all the statements except those about $\arg(B_\infty)$.

For any smooth function, f, on a circle $C = \{z \mid z = z_0 + re^{i\theta}\}$, define

$$\mathrm{Var}_C(f) = \int_0^{2\pi} \left| \frac{d}{d\theta} f(z) \right| d\theta. \tag{4.92}$$

For w outside C, let

$$f_w(z) = \arg(w - z). \tag{4.93}$$

Then this arg is increasing on one arc between the tangents to C from w and decreasing on the other, so

$$\mathrm{Var}_C(f_w) = 2 \times \text{angle between tangents.} \tag{4.94}$$

This shows that

$$\mathrm{Var}_C(f_w) \leq 2\pi \tag{4.95}$$

and

$$\mathrm{Var}_C(f_w) \leq \frac{4\,\mathrm{radius}(C)}{\mathrm{dist}(w, C)}. \tag{4.96}$$

Since $\arg(B(z, w))$ is built out of such $\arg(z - \gamma(w))$ and $\arg(z - \overline{\gamma(w)}^{-1})$, and the radii of the circles containing $\gamma(C_j^+)$ decrease so fast that, by Beardon's theorem,

$$\sum_{\gamma \in \Gamma} \mathrm{radius}\big(\gamma\big(C_j^+\big)\big) < \infty, \tag{4.97}$$

we find, uniformly for $z \in \mathcal{F}$, that

$$\mathrm{Var}_{C_j^+}\big(\arg\big(B(z, \cdot)\big)\big) \leq C_0 \tag{4.98}$$

for some finite constant C_0.

Consider $\arg(B_\infty(z; \{z_k\}, \{p_k\}))$ as p_k is changed from z_k to its final value. At $p_k \equiv z_k$, this arg is 0 and the total change is bounded by the variation of $\arg(B(z, w))$ as w varies over C_{j-1}^+. We conclude that on $\mathcal{F}$,

$$\big|\arg\big(B_\infty(z)\big)\big| \leq C_0, \tag{4.99}$$

proving (4.84). Since B_∞ is character automorphic, the variation of $\arg(B_\infty(z))$ over any $\gamma(\mathcal{F})$ is at most $2C_0$, from which (4.85) is immediate.

5 Theta Functions and Abel's Theorem

Given a general compact Riemann surface, $\mathcal{S}$, of genus ℓ, one can construct a natural map, $\mathfrak{A}$, called the *Abel map* from $\mathcal{S}$ to a 2ℓ-dimensional real torus, called the Jacobi

variety, realized as $\mathbb{C}^\ell/\mathbb{L}$ where $\mathbb{L}$ is a 2ℓ-dimensional lattice. Once a base point in $\mathcal{S}$ is fixed, the group structure comes into play. The theory of meromorphic functions—essentially, which finite subsets of $\mathcal{S}$ can occur as zeros and poles—is described using $\mathfrak{A}$ via a result called *Abel's theorem*.

As we will see in the next section, certain m-functions of Jacobi matrices with $\sigma_{\mathrm{ess}}(J) = \mathfrak{e}$ define meromorphic functions on the Riemann surface, $\mathcal{S}$, constructed in Sect. 2. Their zeros and poles lie only at ∞_+, ∞_-, or in the sets G_j of (2.7). $\mathfrak{A}$ takes $\bigcup_j G_j$ into an ℓ-dimensional torus inside the 2ℓ-dimensional Jacobi variety (the real part of the Jacobi variety), which is also a subgroup with a suitable choice of base point. The traditional construction of the isospectral torus [22, 25, 40, 45, 79] uses this general theory of the Abel map and Abel's theorem.

Here, following Sodin–Yuditskii [67], we restrict ourselves to meromorphic functions with poles and zeros only at ∞_+, ∞_-, and in $\bigcup_j G_j$. In that case everything can be made explicit in a way that the real part of the Jacobi variety becomes just the ℓ-dimensional torus, Γ^*, of characters for the Fuchsian group, Γ. The key is the definition of some natural functions on $\mathbb{C} \cup \{\infty\} \setminus \mathcal{L}$ parametrized by points in $\bigcup_{j=1}^\ell \tilde{C}_j^+$ (defined below). Our construction is motivated by the one in [67] but is more explicit.

We will need to use a fundamental set of the action of Γ on $\mathbb{C} \cup \{\infty\} \setminus \mathcal{L}$,

$$\tilde{\mathcal{F}} = \left(\overline{\mathcal{F}} \cup \overline{\mathcal{F}}^{-1}\right) \setminus \bigcup_{j=1}^\ell \tilde{C}_j^-, \tag{5.1}$$

where closure is taken in $\mathbb{C}$, and $\tilde{C}_j^\pm$ are the *complete orthocircles* (obtained by adding the two missing points on $\partial\mathbb{D}$ to $C_j^\pm \cup (C_j^\mp)^{-1}$, respectively); see Fig. 4. $\tilde{\mathcal{F}}^{\mathrm{int}}$ will denote its interior; this is a fundamental region. $\tilde{\mathcal{F}}$ is then $\tilde{\mathcal{F}}^{\mathrm{int}}$ with $\bigcup_{j=1}^\ell \tilde{C}_j^+$ added.

Fig. 4 Complete orthocircles
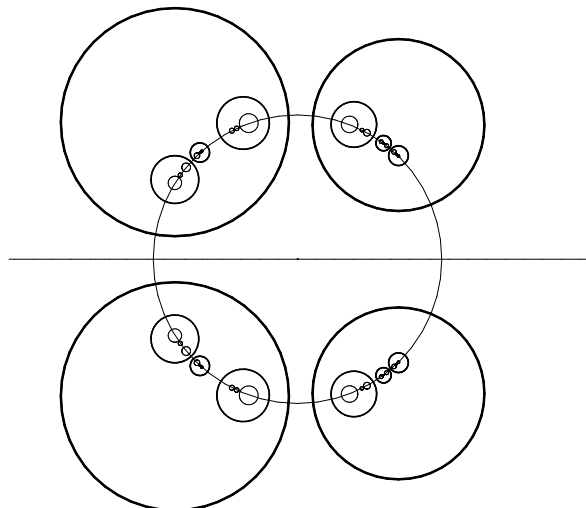
$\mathbf{x}^\sharp$ maps $\tilde{C}_j^+$ onto G_j bijectively. We will use $\zeta_1, \ldots, \zeta_\ell$ for the unique points on $\tilde{C}_1^+, \ldots, \tilde{C}_\ell^+$ that map to $\beta_1, \ldots, \beta_\ell$, that is,

$$\zeta_j \in \tilde{C}_j^+ \cap \partial \mathbb{D}, \qquad \mathbf{x}^\sharp(\zeta_j) = \beta_j. \tag{5.2}$$

We need the following lemma:

Lemma 5.1 *Let f be a character automorphic meromorphic function on $\mathbb{C} \cup \{\infty\} \setminus \mathcal{L}$. Suppose*

(i) *f has no zeros or poles in $\tilde{\mathcal{F}}^{\mathrm{int}}$ (so in $\tilde{\mathcal{F}}$, the only zeros and poles are on $\bigcup_{j=1}^\ell \tilde{C}_j^+$).*
(ii) *Every zero or pole of f has even order.*
(iii) *If D_j is a counterclockwise contour that is just outside $\tilde{C}_j^+$ (say, a circle with the same center but a slightly larger radius), then*

$$\frac{1}{2\pi i} \oint_{D_j} \frac{f'(z)}{f(z)} \, dz = 0. \tag{5.3}$$

(iv)

$$f(0) > 0. \tag{5.4}$$

Then there is a (unique) character automorphic function, g, which we will denote as $\sqrt{f}$, with

$$g(0) > 0 \tag{5.5}$$

and so that for all $z \in \mathbb{C} \cup \{\infty\} \setminus \mathcal{L}$,

$$g(z)^2 = f(z). \tag{5.6}$$

Proof By (5.3), we can define a single-valued function $h(z)$ on $\tilde{\mathcal{F}}^{\mathrm{int}}$ by

$$h(z) = \log\big(f(0)\big) + \oint_0^z \frac{f'(w)}{f(w)} \, dw \tag{5.7}$$

with any contour in $\tilde{\mathcal{F}}^{\mathrm{int}}$ used. Then

$$g(z) = \exp\left(\frac{1}{2} h(z)\right) \tag{5.8}$$

obeys (5.5)–(5.6) and is defined and analytic on $\tilde{\mathcal{F}}^{\mathrm{int}}$.

Since all poles and zeros on $\tilde{C}_j^+$ are of even order, $g(z)$ can be meromorphically continued to a neighborhood, N, of the closure of $\tilde{\mathcal{F}}$. Then for each j, $S_j \equiv \{z \in \tilde{\mathcal{F}}^{\mathrm{int}} \mid \gamma_j(z) \in N\}$ is open and nonempty, and by decreasing N, one can suppose each S_j is connected.

If $z \in S_j$, we have $g(\gamma_j(z))^2 = c_f(\gamma_j) g(z)^2$. Hence, by continuity and connectedness, there is a single square root, $c_g(\gamma_j)$, so that

$$z \in S_j \quad \Rightarrow \quad g\big(\gamma_j(z)\big) = c_g(\gamma_j) g(z). \tag{5.9}$$

We can use this to extend g to $\bigcup_j \gamma_j(\tilde{\mathcal{F}})$ and also to $\bigcup_j \gamma_j^{-1}(\tilde{\mathcal{F}})$, and so (5.9) holds for all z with z and $\gamma_j(z)$ in the domain of current definition. In this way, one gets a character automorphic continuation of g to $\mathbb{C} \cup \{\infty\} \setminus \mathcal{L}$. $\square$

Lemma 5.2 *Let* $\zeta \in \tilde{C}_j^+$ *for some* j. *Then*

$$f(z) = \frac{\mathbf{x}(z) - \mathbf{x}(\zeta)}{\mathbf{x}(z) - \mathbf{x}(\zeta_j)} \eta(z) \eta(0)^{-1}, \tag{5.10}$$

where

$$\eta(z) = \begin{cases} B(z, \zeta) & \text{if } \zeta \in \mathbb{D}, \\ 1 & \text{if } \zeta \in \partial\mathbb{D}, \\ B(z, \bar{\zeta}^{-1})^{-1} & \text{if } \zeta \in \mathbb{C} \setminus \overline{\mathbb{D}} \end{cases} \tag{5.11}$$

obeys properties (i)–(iv) *of Lemma* 5.1. *If* $\zeta \neq \zeta_j$, f *has double zeros at* $\{\gamma(\zeta)\}_{\gamma \in \Gamma}$, *double poles at* $\{\gamma(\zeta_j)\}_{\gamma \in \Gamma}$, *and is otherwise finite and nonvanishing.*

Remarks 1. If $\zeta = \zeta_j$, $f \equiv 1$.
   2. When $z \in \{\gamma(0)\}_{\gamma \in \Gamma}$, $\mathbf{x}(z) = \infty$, and the first factor in (5.10) is interpreted as 1. Thus, because of $\eta(0)^{-1}$, we have $f(0) = 1$.

Proof (i) and (iv) are obvious. Moreover, if $\zeta \neq \zeta_j$ then $f(z)/\eta(z)$ has double poles at $\{\gamma(\zeta_j)\}_{\gamma \in \Gamma}$ (since $\mathbf{x}'(z) = 0$, $\mathbf{x}''(z) \neq 0$ at such points), double zeros at $\{\gamma(\zeta)\}_{\gamma \in \Gamma}$ if ζ is the other point on $\tilde{C}_j^+$ in $\partial\mathbb{D}$ (i.e., if $\mathbf{x}^\sharp(\zeta) = \alpha_{j+1}$), and simple zeros at $\{\gamma(\zeta)\}_{\gamma \in \Gamma} \cup \{\gamma(\bar{\zeta}^{-1})\}_{\gamma \in \Gamma}$ if $\zeta \notin \partial\mathbb{D}$ (since $\mathbf{x}'(z) \neq 0$ at such points and $\mathbf{x}(\zeta)$ is real). Thus, there are precisely the claimed zeros/poles for f since η cancels the zeros at $\{\gamma(\bar{\zeta}^{-1})\}_{\gamma \in \Gamma}$ and doubles the zeros at $\{\gamma(\zeta)\}_{\gamma \in \Gamma}$. This proves (ii).

To prove (iii), we need only check (5.3) if f is replaced by η or by f/η. The function f/η is real on $\partial\mathbb{D}$, so if q is the composition of this function and a conformal map of $\mathbb{C}$ taking $\mathbb{R}$ to $\partial\mathbb{D}$, q is real on the set of points in its domain which lie on $\mathbb{R}$. So

$$\frac{1}{2\pi i} \oint_{\tilde{D}} q(z)\, dz = 0 \tag{5.12}$$

for any conjugate symmetric curve, and so by contour deformation, for D_j and $(f/\eta)'/(f/\eta)$.

For η, we note that if f in (5.3) is replaced by a finite product $\prod_{\gamma \in G} b(z, \gamma(\zeta))$, the integral is zero, since the finite product is meromorphic inside D_j with an equal number of (simple) zeros and (simple) poles. By taking limits, (5.3) holds for $B(z, \zeta)$, and by $(1/g)'/(1/g) = -g'/g$ for $B(z, \bar{\zeta}^{-1})^{-1}$. Thus, (5.3) holds for f. $\square$

Definition Let $y \in G_j$ for some j and let ζ be the unique point in $\tilde{C}_j^+$ with

$$\mathbf{x}^\sharp(\zeta) = y. \tag{5.13}$$

We define $\Theta(\,\cdot\,;y)$ to be the character automorphic function

$$\Theta(z;y) = \left[\frac{\mathbf{x}(z)-\mathbf{x}(\zeta)}{\mathbf{x}(z)-\mathbf{x}(\zeta_j)}\,\eta(z)\eta(0)^{-1}\right]^{1/2} \tag{5.14}$$

and denote by $\mathfrak{A}(y) \in \Gamma^*$ its character. Moreover, we define $\mathfrak{A}(\infty)$ to be the character of $B(z)$.

By the lemma, $\Theta(\,\cdot\,;y)$ is indeed a character automorphic function on $\mathbb{C} \cup \{\infty\} \setminus \mathcal{L}$ with simple zeros at $\{\gamma(\zeta)\}_{\gamma \in \Gamma}$ and simple poles at $\{\gamma(\zeta_j)\}_{\gamma \in \Gamma}$ (and otherwise nonvanishing and finite). By definition,

$$\Theta(0;y) = 1. \tag{5.15}$$

We also define

$$\tilde{\mathfrak{A}} : \mathbb{G} \to \Gamma^* \tag{5.16}$$

by

$$\tilde{\mathfrak{A}}(y_1, \ldots, y_\ell) = \mathfrak{A}(y_1)\cdots\mathfrak{A}(y_\ell) \tag{5.17}$$

using the product in Γ^*. Here

$$\mathbb{G} = G_1 \times \cdots \times G_\ell. \tag{5.18}$$

Theorem 5.3 *The map $\tilde{\mathfrak{A}}$ of (5.16)–(5.17) is a real analytic homeomorphism of ℓ-dimensional tori.*

Remark By real analytic functions, we do not mean real-valued but functions of real parameters which are given locally by convergent series of those parameters—they are, of course, C^∞.

Proof By construction, $\Theta(z;y)$, as a map of $\mathbb{D} \times \bigcup_{j=1}^\ell G_j$ to $\mathbb{C}$, is jointly real analytic. Since

$$\mathfrak{A}(y)(\gamma) = \frac{\Theta(\gamma(0);y)}{\Theta(0;y)} = \Theta\big(\gamma(0);y\big), \tag{5.19}$$

$\mathfrak{A}$ and so $\tilde{\mathfrak{A}}$ are real analytic maps.

Suppose $\vec{y} = (y_1, \ldots, y_\ell)$ and $\vec{w} = (w_1, \ldots, w_\ell)$ lie in $\mathbb{G}$ and $\tilde{\mathfrak{A}}(\vec{y}) = \tilde{\mathfrak{A}}(\vec{w})$. Then

$$f(z) = \prod_{j=1}^\ell \frac{\Theta(z;y_j)}{\Theta(z;w_j)} \tag{5.20}$$

is automorphic since the characters cancel. Hence, there is a unique meromorphic function F on $\mathcal{S}$ so that

$$f(z) = F\big(\mathbf{x}^\sharp(z)\big). \tag{5.21}$$

Let m be the number of j's with $y_j \neq w_j$. The function F has exactly m poles and m zeros, all simple (the poles/zeros where $y_j = w_j$ cancel) and so has degree $m \leq \ell$. If $m \neq 0$, $F \circ \tau \not\equiv F$ since there is a gap with a single simple zero (and if $F \circ \tau = F$, F has either two zeros or a double zero at a branch point). Thus, if $m \neq 0$, we get a contradiction to Theorem 2.2. It follows that $m = 0$, that is, $\vec{y} = \vec{w}$ and $\tilde{\mathfrak{A}}$ is one-to-one.

Any smooth one-to-one map between two smooth, orientable compact manifolds of the same dimension has degree ± 1, and so is also surjective (see [26, 32, 39, 43, 46, 68] for expositions of degree theory). $\qquad\square$

We saw above that Theorem 2.2 is the key to the proof of Theorem 5.3. It is also very powerful in connection with Theorem 5.3 as the following theorems show:

Theorem 5.4 *Let f be a character automorphic function on $\mathbb{C} \cup \{\infty\} \setminus \mathcal{L}$ with no zeros or poles. Then f is constant.*

Proof Let $c_f \in \Gamma^*$ be the character of f. By Theorem 5.3, find $\vec{y} \in \mathbb{G}$ with $\tilde{\mathfrak{A}}(\vec{y}) = c_f$. Let

$$h(z) = \frac{f(z)}{\prod_{j=1}^{\ell} \Theta(z; y_j)}. \tag{5.22}$$

Then h is automorphic, so there is H meromorphic on $\mathcal{S}$ with

$$h(z) = H\big(\mathbf{x}^\sharp(z)\big). \tag{5.23}$$

H has degree m where $m = \#\{j \mid y_j \neq \beta_j\}$. By Theorem 2.2, $m = 0$, that is, $\vec{y} = (\beta_1, \ldots, \beta_\ell)$ so $\tilde{\mathfrak{A}}(\vec{y}) = 1$ and f is automorphic. But then

$$f(z) = F\big(\mathbf{x}^\sharp(z)\big) \tag{5.24}$$

with F analytic on $\mathcal{S}$, and therefore f is constant. $\qquad\square$

Corollary 5.5 *Let $\zeta \in \tilde{C}_j^+$, and suppose h is a character automorphic meromorphic function with zeros only at $\{\gamma(\zeta)\}_{\gamma \in \Gamma}$ and poles only at $\{\gamma(\zeta_j)\}_{\gamma \in \Gamma}$, all simple. If $h(0) = 1$, then*

$$h(z) = \Theta\big(z; \mathbf{x}^\sharp(\zeta)\big). \tag{5.25}$$

Moreover,

$$\Theta(\bar{z}; y) = \overline{\Theta(z; y)} \tag{5.26}$$

and, in particular, $\Theta(\,\cdot\,; y)$ is real and strictly positive on $\mathbb{R}$.

Remark Thus, Θ and so $\mathfrak{A}$ are unique.

Proof Apply Theorem 5.4 to $h(z)/\Theta(z; \mathbf{x}^\sharp(\zeta))$. This gives (5.25). To get (5.26), use (2.8) and the fact that for $\zeta \in \tilde{C}_j^+$

$$B(z, \zeta) = \prod_{\gamma \in \Gamma} b(z, \gamma(\zeta)) = \prod_{\gamma \in \Gamma} b(z, \overline{\gamma(\zeta)}) = \prod_{\gamma \in \Gamma} \overline{b(\overline{z}, \gamma(\zeta))} = \overline{B(\overline{z}, \zeta)}. \qquad \square$$

Definition By a *divisor*, we mean a finite subset $\Delta \subset \bigcup_{j=1}^\ell G_j$ and the assignment of a nonzero integer n_x to each $x \in \Delta$ plus an assignment of an integer n_+ to ∞_+ and

$$n_- = -n_+ \tag{5.27}$$

to ∞_-. We write the divisor formally as

$$n_+ \delta_{\infty_+} + n_- \delta_{\infty_-} + \sum_{x \in \Delta} n_x \delta_x. \tag{5.28}$$

Definition By a *special meromorphic function*, we mean a meromorphic function on $\mathcal{S}$ so that

(i) All zeros and poles lie in $[\bigcup_{j=1}^\ell G_j] \cup \{\infty_\pm\}$.
(ii) If $n_\pm$ are the orders of the zeros and poles at $\infty_\pm$ ($n_\pm > 0$ means a zero of order $n_\pm$, $n_\pm < 0$ means a pole of order $-n_\pm$), then (5.27) holds.

Definition The divisor, $\delta(F)$, of a special meromorphic function F is given by (5.28), where Δ is the set of zeros and poles of F, and n_x is the order of the zero/pole at x.

Notice that we have chosen a base point ζ_j that depends on which G_j the point y lies in. For our later applications, where for each j,

$$\sum_{x \in G_j} n_x = 0, \tag{5.29}$$

that is very convenient. But when (5.29) does not hold, we will need a factor to move the base point to a fixed point, say ζ_1. So we define

$$\Psi_j(z) = \sqrt{\frac{\mathbf{x}(z) - \mathbf{x}(\zeta_j)}{\mathbf{x}(z) - \mathbf{x}(\zeta_1)}}, \tag{5.30}$$

where we can take special roots by Lemma 5.1, verifying (5.3) as we did for $(\mathbf{x}(z) - \mathbf{x}(\zeta))/(\mathbf{x}(z) - \mathbf{x}(\zeta_j))$.

We also need to define $\mathfrak{A}_j$ to be the character of the character automorphic function Ψ_j. For $x \in G_j$, we let

$$\mathfrak{A}^\sharp(x) = \mathfrak{A}_j \mathfrak{A}(x). \tag{5.31}$$

Theorem 5.6 (Abel's theorem for special meromorphic functions) *A divisor is the divisor of a special meromorphic function F if and only if*

$$\text{(a)} \quad \sum_{x \in \Delta} n_x = 0, \tag{5.32}$$

$$\text{(b)} \quad \mathfrak{A}(\infty)^{n_+} \prod_{x \in \Delta} \mathfrak{A}^{\sharp}(x)^{n_x} = 1, \tag{5.33}$$

the identity element of Γ^*.

Remarks 1. The proof provides an explicit formula for F, namely,

$$F\big(\mathbf{x}^{\sharp}(z)\big) = B(z)^{n_+} \prod_{j=1}^{\ell} \prod_{x \in \Delta \cap G_j} \Theta(z; x)^{n_x} \Psi_j(z)^{n_x}. \tag{5.34}$$

2. Notice that if (5.29) holds, we can drop the Ψ_j factors from (5.34) and change $\mathfrak{A}^{\sharp}$ to $\mathfrak{A}$ in (5.33).

Proof (5.32) is an expression of the constancy of degree, that is, that the number of zeros of F is equal, counting multiplicities, to the number of poles. So really we need to prove, assuming (5.32), that a divisor is a $\delta(F)$ if and only if (5.33) holds.

If the divisor obeys (5.33), then the right side of (5.34) is automorphic, so (5.34) holds for some meromorphic F with the proper zeros and poles. Hence, the divisor is a $\delta(F)$. Because of the Ψ_j factors, the poles of $\Theta(z; x)$ at ζ_j are all moved to ζ_1, and the poles at ζ_1 cancel each other by (5.32).

On the other hand, if F is a special meromorphic function and $\delta(F)$ is its divisor, then letting $f(z)$ be the right side of (5.34),

$$\frac{F(\mathbf{x}^{\sharp}(z))}{f(z)} \equiv h(z) \tag{5.35}$$

is character automorphic with no zeros and poles (again, the poles and zeros at ζ_j cancel because of the Ψ_j's and at ζ_1 by (5.32)). By Theorem 5.4, h is constant, so automorphic. Thus, since $F(\mathbf{x}^{\sharp}(z))$ is automorphic, so is f, which implies (5.33). $\square$

6 The Isospectral Torus

Once one has the Abel map and Abel's theorem, the construction of the isospectral torus along the lines pioneered for KdV [22, 45] is straightforward (see [66, Chap. 5] for an exposition of the original papers [25, 40, 79]) but in the covering map guise has a more explicit feel. For additional discussions of the isospectral torus, see [7, 12, 29, 30, 72].

Definition A *minimal Herglotz function* for $\mathfrak{e}$ is a meromorphic function m on $\mathcal{S}$ with degree precisely $\ell + 1$ and which obeys

(i)

$$z \in \mathcal{S}_+ \cap \mathbb{C}_+ \quad \Rightarrow \quad \operatorname{Im} m(z) > 0, \tag{6.1}$$

(ii) near ∞_+,

$$m(z) = -\frac{1}{z} + O(z^{-2}), \tag{6.2}$$

(iii) $m(z)$ has a pole at ∞_-.

In the usual way (see [66]), $m \restriction \mathcal{S}_+ \cap \mathbb{C}_+$ determines a probability measure, $d\mu$, with

$$m(z) = \int \frac{d\mu(x)}{x - z} \tag{6.3}$$

for $z \in \mathcal{S}_+ \setminus \mathbb{R}$. Moreover, the continuity properties of m as one approaches $\mathfrak{e}$ (and the fact that we will see that all poles of m are simple) imply that

$$d\mu(x) = w(x)\,dx + d\mu_{\mathrm{s}}(x), \tag{6.4}$$

where w is real analytic on $\mathfrak{e}^{\mathrm{int}}$ and nonvanishing there, and $d\mu_{\mathrm{s}}$ is a pure point measure with pure points only in the open gaps $\bigcup_{j=1}^{\ell}(\beta_j, \alpha_{j+1})$ and at most one pure point per gap.

Condition (iii) may seem ad hoc. We mention now that one can show ([66, Example 5.13.4]) that if (iii) is dropped, the once-stripped m-function, m_1, (i.e., $m(z) = \mathcal{M}(a_1, b_1, m_1(z))$ in terms of (1.24)) obeys condition (iii). Thus, the extra possibilities allowed if (iii) is dropped result from taking the Jacobi matrix of a minimal Herglotz function as we have defined it and extending by one row and column, with the "wrong" values of a_0 or b_0.

The main elements of the theory are:

(i) The minimal Herglotz functions are in one-to-one correspondence with $\mathbb{G}$, and so form an ℓ-dimensional torus, $\mathcal{T}_{\mathfrak{e}}$.

(ii) The correspondence is that the coordinates of $(y_1, \dots, y_\ell) \in \mathbb{G}$ are the positions of ℓ of the poles of m—the last pole is at ∞_-. The zeros are then determined via the Abel map.

(iii) The Abel map "linearizes" coefficient stripping (i.e., the map (1.24)) since the zeros of m are the poles of the once-stripped m-function, m_1. Explicitly,

$$\tilde{\mathfrak{A}} \colon \mathbb{G} \to \Gamma^*, \tag{6.5}$$

and coefficient stripping corresponds to multiplying by the inverse of the character of B.

(iv) The linearization shows that the corresponding Jacobi parameters, $\{a_n, b_n\}_{n=1}^{\infty}$, are almost periodic sequences with almost periods given by the harmonic measures $\{\rho_{\mathfrak{e}}([\alpha_1, \beta_j])\}_{j=1}^{\ell}$. In particular, one has periodicity if and only if these numbers are all rational.

(v) The construction provides explicit formulae for m and, thus, a_1, b_1 (and so, via the Abel map, a_n, b_n) in terms of theta functions and the logarithmic capacity of $\mathfrak{e}$.

(vi) Uniformly on $\mathcal{T}_{\mathfrak{e}}$, there are bounds on the weight w in (6.4) of the form

$$C\sqrt{|R(x)|} \le w(x) \le D\sqrt{|R(x)|^{-1}}, \qquad (6.6)$$

where $0 < C, D < \infty$.

We begin by recalling what one can get without using the covering or Abel maps.

Theorem 6.1 (i) *Every minimal Herglotz function has exactly one simple pole in each gap G_j, one at ∞_-, and no others.*

(ii) *For every choice $(y_1, \ldots, y_\ell) \in \mathbb{G}$, there is exactly one minimal Herglotz function with poles exactly at $y_1, \ldots, y_\ell$ (and ∞_-).*

(iii) *For every minimal Herglotz function, the once-stripped Herglotz function is also a minimal Herglotz function.*

(iv) *Every minimal Herglotz function has one zero in each gap G_j, one at ∞_+, and no others.*

Remark Of course, (i) and (ii) set up a one-to-one correspondence between $\mathcal{T}_{\mathfrak{e}}$, the set of minimal Herglotz functions, and $\mathbb{G}$. We will often refer to $\mathbb{G}$ as the *isospectral torus*.

Sketch (See [66, Chap. 5] for details.) (i) Every minimal degree meromorphic function, m, on $\mathcal{S}$ with $m \circ \tau \not\equiv m$ has the form

$$m(z) = \frac{p(z) + \sqrt{R(z)}}{a(z)}, \qquad (6.7)$$

where $R(z)$ is given by (2.2), and p, a have degree at most $\ell + 1$. Since m has a zero at ∞_+, $p(z)$ must cancel the leading $O(z^{\ell+1})$ term in $\sqrt{R(z)}$ at ∞_+, so

$$\deg(p) = \ell + 1. \qquad (6.8)$$

Because this cancellation takes place at ∞_+, it does not at ∞_- (since $\sqrt{R}$ flips sign but p does not). For m to have a simple pole at ∞_-, we must have

$$\deg(a) = \ell. \qquad (6.9)$$

m is real on each $[\beta_j, \alpha_{j+1}]$, so, by analyticity, on the entire gap G_j. Thus, $m(z)$ is real on $\mathbb{R} \setminus \mathfrak{e}$ on both sheets. Since $\sqrt{R(z)}$ is real on $\mathbb{R} \setminus \mathfrak{e}$, we conclude first that a is real and then that p is real. In particular, on $\overline{\mathcal{S}}_+ \cap \mathfrak{e}$,

$$\operatorname{Im} m(x + i0) = \frac{\operatorname{Im} \sqrt{R(x)}}{a(x)}. \qquad (6.10)$$

Since R has two zeros between bands, $\operatorname{Im} \sqrt{R(x)}$ changes sign between successive bands. As $\operatorname{Im} m(x + i0) \ge 0$, a must change signs between bands, that is, have an odd number of zeros in each gap. Since, by (6.9), a has only ℓ zeros and there are ℓ gaps, a has one zero per gap.

If a has a zero at $x_0 \in (\beta_j, \alpha_{j+1})$, then

$$\left(p + \sqrt{R}\right) - \left(p - \sqrt{R}\right) = 2\sqrt{R} \neq 0 \qquad (6.11)$$

at x_0, so on one sheet or the other, m must have a pole. If a has a zero at $x_0 \in \{\beta_j, \alpha_{j+1}\}$, the numerator is at best $O((x - x_0)^{1/2})$, and the denominator is $O((x - x_0))$. So again, m has a pole at x_0.

Thus, m has at least one pole per gap. So, since $\deg(m) = \ell + 1$ and m has a pole at ∞_-, m has exactly one simple pole in each gap.

(ii) Write $y_j = (\pi(y_j), \sigma_j)$ if $\pi(y_j) \in (\beta_j, \alpha_{j+1})$ with $\sigma_j = 1$ (respectively -1) if $y_j \in \mathcal{S}_+$ (respectively $\mathcal{S}_-$). Since a has to vanish at $\pi(y_j)$ to get a pole at y_j, we see that m has a pole at y_j and not at $\tau(y_j)$ if and only if $a(\pi(y_j)) = 0$ and

$$p\left(\pi(y_j)\right) = \sigma_j \sqrt{R(y_j)}. \qquad (6.12)$$

If $\pi(y_j) \in \{\beta_j, \alpha_{j+1}\}$, then to avoid a double pole, $p(\pi(y_j)) = 0$, that is, (6.12) still holds (since $\sqrt{R(y_j)} = 0$, it does not matter that σ_j is undefined).

At ∞_+, $a(z)$ is $O(z^\ell)$. Thus, for $m(z)$ to vanish at ∞_+, we must have

$$p(z) + \sqrt{R(z)} = O\left(z^{\ell-1}\right) \qquad (6.13)$$

near ∞_+. Since $\sqrt{R(z)} = O(z^{\ell+1})$, (6.13) determines the top two coefficients (with the top one nonzero) and then, by standard polynomial interpolation, the ℓ conditions (6.12) determine the remaining ℓ coefficients of p.

We have thus proven that given $\vec{y} = (y_1, \ldots, y_\ell) \in \mathbb{G}$, there is a meromorphic function of degree precisely $\ell + 1$ with poles at $y_1, \ldots, y_\ell$ and ∞_-, and a zero at ∞_+. Moreover, it is unique up to a single overall constant—for the above determines p and

$$a(z) = -c \prod_{j=1}^{\ell} \left(z - \pi(y_j)\right) \qquad (6.14)$$

for some constant $c > 0$.

The fact that a changes sign in each gap shows that the sign of c in (6.14) can be picked so that

$$\operatorname{Im} m(x + i0) > 0 \qquad (6.15)$$

on all bands in $\mathcal{S}_+$. Keeping track of the argument of $\sqrt{R(z)}$ as one crosses a branch point shows that with this choice at each $y_j \in \mathcal{S}_+$ such that $\pi(y_j) \in (\beta_j, \alpha_{j+1})$ and $\sigma_j > 0$,

$$m(z) = \frac{c_j}{\pi(y_j) - z} + O(1) \qquad (6.16)$$

with $c_j > 0$. Thus, any limit point in the values of $\operatorname{Im} m(z)$ as z approaches $\mathbb{R}$ is nonnegative. Since $\operatorname{Im} m(z) \to 0$ at ∞_+, the maximum principle applied to the harmonic function $\operatorname{Im} m(z)$ on $\mathcal{S}_+ \cap \mathbb{C}_+$ shows that (6.1) holds.

Any function obeying (6.1) with real boundary values on $\mathbb{R} \setminus \mathfrak{e}$ has (6.2) holding up to a positive constant. We can thus adjust c in (6.14) so that (6.2) holds. We have

thus proven that there exists precisely one meromorphic Herglotz function with poles at $y_1, \ldots, y_\ell$.

(iii) m and the once-stripped function, m_1, are related by

$$m(z) = \frac{1}{-z + b_1 - a_1^2 m_1(z)}, \tag{6.17}$$

where (a_1, b_1) are (and can be) chosen so that (6.2) holds for m_1. It is always true, of course, that m_1 obeys (6.1) and (6.2).

Equation (6.17) sets up a one-to-one correspondence between poles of m_1 in $\mathcal{S} \setminus \{\infty_\pm\}$ and zeros of m there. Since m has degree $\ell + 1$, it has $\ell + 1$ zeros and only a simple zero at ∞_+ by (6.2). Thus, m has precisely ℓ zeros in $\mathcal{S} \setminus \{\infty_\pm\}$. Therefore, m_1 has exactly ℓ poles on $\mathcal{S} \setminus \{\infty_\pm\}$ and, obviously, no pole at ∞_+. Since $m(z)$ has a pole at ∞_-, (6.17) shows that near ∞_-,

$$a_1^2 m_1(z) = -z + b_1 + O(z^{-1}), \tag{6.18}$$

that is, m_1 has a simple pole at ∞_-. Thus, m_1 has degree exactly $\ell + 1$ and we have proven condition (iii) in the definition of a minimal Herglotz function.

(iv) Since m_1 has a pole on each G_j, by (6.17), m has a zero on each G_j. There is a zero at ∞_+, and this accounts for all $\ell + 1$. $\square$

The above construction also lets us prove (6.6):

Theorem 6.2 (i) *There are constants, A, B, so that uniformly in $\vec{y} \in \mathbb{G}$, we have* (*with c the constant in* (6.14))

$$Ac^{-1}\sqrt{|R(x)|} \leq w_{\vec{y}}(x) \leq Bc^{-1}\sqrt{|R(x)|^{-1}}, \tag{6.19}$$

and the residues c_j of (6.16) *obey*

$$0 \leq c_j \leq Bc^{-1}. \tag{6.20}$$

(ii) *Uniformly in $\vec{y} \in \mathbb{G}$, the constant c in* (6.14) *is bounded and bounded away from zero. Moreover, uniformly in $x \in \mathfrak{e}$ and $\vec{y} \in \mathbb{G}$,* (6.6) *holds.*

Proof (i) Since a has the form (6.14), we have

$$\sup_{x \in \mathfrak{e}} |a(x)| \leq c(\beta_{\ell+1} - \alpha_1)^\ell, \tag{6.21}$$

which, given (6.10) and the relation

$$w_{\vec{y}}(x) = \frac{1}{\pi} \operatorname{Im} m(x + i0), \tag{6.22}$$

implies the first inequality in (6.19).

Next, note that if we fix $x \in \mathfrak{e}$, the distance to the nearest zero of a is at least $\mathrm{dist}(x, \mathbb{R} \setminus \mathfrak{e})$ and the distance to all the other zeros of a at least $\frac{1}{2} \min_j |\beta_j - \alpha_j|$. Thus, for $x \in \mathfrak{e}$,

$$\left| a(x) \right| \geq c \left(\frac{1}{2} \min_j |\beta_j - \alpha_j| \right)^{\ell-1} \mathrm{dist}(x, \mathbb{R} \setminus \mathfrak{e}). \tag{6.23}$$

On the other hand, for $x \in \mathfrak{e}$,

$$\left| R(x) \right| \leq (\beta_{\ell+1} - \alpha_1)^{2\ell+1} \mathrm{dist}(x, \mathbb{R} \setminus \mathfrak{e}) \tag{6.24}$$

and

$$\left| R(x) \right| \geq \left(\frac{1}{2} \min_j |\beta_j - \alpha_j| \right)^{2\ell+1} \mathrm{dist}(x, \mathbb{R} \setminus \mathfrak{e}). \tag{6.25}$$

We get the second inequality in (6.19) from (6.23)–(6.25), (6.10), and (6.22).

By (6.10), the residue c_j in (6.16) is given by

$$c_j = 2c^{-1} \sqrt{|R(y_j)|} \prod_{k \neq j} |y_k - y_j|^{-1} \tag{6.26}$$

$$\leq 2c^{-1}(\beta_{\ell+1} - \alpha_1)^{\ell+1} \left(\min_j |\beta_j - \alpha_j| \right)^{-\ell}. \tag{6.27}$$

(ii) By (6.19) and (6.20),

$$Ac^{-1} \int_{\mathfrak{e}} \sqrt{|R(x)|} \, dx \leq \int_{\mathfrak{e}} w_{\vec{y}}(x) \, dx + \sum_j c_j$$

$$\leq Bc^{-1} \left[\ell + \int_{\mathfrak{e}} \sqrt{|R(x)|^{-1}} \, dx \right]. \tag{6.28}$$

The total weight of the measure is 1, so we get the claimed upper and lower bounds on c. Given those, (6.19) yields (6.6). $\qquad\square$

Given $\vec{y} \in \mathbb{G}$, we denote by $m_{\vec{y}}$ the associated minimal Herglotz function. The once-stripped m-function is also a minimal Herglotz function and thus corresponds to some $\vec{w} \in \mathbb{G}$. We define a map $U : \mathbb{G} \to \mathbb{G}$ by

$$U(\vec{y}) = \vec{w}, \tag{6.29}$$

so that $m_{U(\vec{y})}$ is the once-stripped m-function.

Now we can combine Theorem 6.1 with the Abel map:

Theorem 6.3 *Suppose* $\vec{y} = (y_1, \dots, y_\ell) \in \mathbb{G}$ *and let U be defined by* (6.29).

(i) *With $\tilde{\mathfrak{A}}$ defined in* (5.16)–(5.17), *we have*

$$\tilde{\mathfrak{A}}\big(U(\vec{y})\big) = \tilde{\mathfrak{A}}(\vec{y})\mathfrak{A}(\infty)^{-1}. \tag{6.30}$$

(ii) *Let*

$$M_{\vec{y}}(z) = -m_{\vec{y}}\big(\mathbf{x}(z)\big). \tag{6.31}$$

Then we have that

$$M_{\vec{y}}(z) = \frac{B(z)}{\mathrm{cap}(\mathfrak{e})} \prod_{j=1}^{\ell} \frac{\Theta(z; U(\vec{y})_j)}{\Theta(z; y_j)}. \tag{6.32}$$

Remark Since

$$m(z) = -\frac{1}{z} - \frac{b_1}{z^2} - \frac{a_1^2 + b_1^2}{z^3} + O\big(z^{-4}\big),$$

(6.32) implies an explicit formula for b_1 and a_1 in terms of theta functions.

Proof $M_{\vec{y}}(z)$ is a meromorphic function with divisor

$$\delta_{\infty_+} - \delta_{\infty_-} + \sum_{j=1}^{\ell} (\delta_{U(\vec{y})_j} - \delta_{y_j}), \tag{6.33}$$

so (6.30) is just (5.33).

By (5.34) (note that (5.29) holds so there are no Ψ_j factors), we have (6.32) with $\mathrm{cap}(\mathfrak{e})$ replaced by a constant.

Since

$$m_{\vec{y}}\big(\mathbf{x}(z)\big) = -\frac{1}{\mathbf{x}(z)} + O\big(z^2\big) = -\frac{z}{x_{\infty}} + O\big(z^2\big) \tag{6.34}$$

and (4.43) holds, the constant is $\mathrm{cap}(\mathfrak{e})$. $\qquad\square$

Corollary 6.4 *Under the map $\tilde{\mathfrak{A}}$ from $\mathbb{G}$ to Γ^*, $\{U^n(\vec{y})\}_{n=0}^{\infty}$ is mapped to the "equally spaced" orbit $\{\tilde{\mathfrak{A}}(\vec{y})\mathfrak{A}(\infty)^{-n}\}_{n=0}^{\infty}$ in Γ^*. In particular,*

$$n \to m_{U^n(\vec{y})}, \qquad n \to \{a_n, b_n\} \tag{6.35}$$

are almost periodic sequences (indeed, real analytic quasiperiodic sequences) with almost periods $\{\rho_{\mathfrak{e}}([\alpha_1, \beta_j])\}_{j=1}^{\ell}$. These sequences are periodic with period p for one point in $\mathcal{T}_{\mathfrak{e}}$ if and only if they are for all points, and that holds if and only if (4.53) holds.

Remark By a *quasiperiodic sequence*, X_n, we mean a sequence given by

$$X_n = x\big(e^{in\omega_1}, \ldots, e^{in\omega_k}\big), \tag{6.36}$$

where x is a continuous function on the k-torus $(\partial \mathbb{D})^k$. It is called *real analytic* if x is real analytic. $(\omega_1, \ldots, \omega_k)$ are called the *almost periods*.

Proof (6.30) immediately implies that

$$\tilde{\mathfrak{A}}\big(U^n(\vec{y})\big) = \tilde{\mathfrak{A}}(\vec{y})\mathfrak{A}(\infty)^{-n}, \tag{6.37}$$

so the orbit is as claimed.

Realize Γ^* as $(\partial\mathbb{D})^\ell$ by

$$c \sim \big(c(\gamma_1), \ldots, c(\gamma_\ell)\big). \tag{6.38}$$

Then, by (4.44),

$$\tilde{\mathfrak{A}}\big(U^n(\vec{y})\big)_j = \tilde{\mathfrak{A}}(\vec{y})_j e^{-2\pi i \rho_\mathfrak{e}([\alpha_1,\beta_j])n}, \tag{6.39}$$

which, given that $\tilde{\mathfrak{A}}$ is real analytic and $\Theta(z; y)$ (and so, m, a_1, b_1) are real analytic in the y's, proves that the sequences (6.35) are almost periodic, indeed, real analytic quasiperiodic.

For the final statement, note that, by (6.37), periodicity for one or for all $\tilde{\mathfrak{A}}(\vec{y})$ is equivalent to $\mathfrak{A}(\infty)^p = 1$. Now use Corollary 4.5. □

7 Raw Jost Functions and the Jost Isomorphism

Recall that if $d\mu$ is a measure on $\mathbb{R}$ of the form (6.4) so that off $[-2, 2]$, $d\mu$ only has pure points, $\{x_j\}_{j=1}^N$, (N finite or infinite) with

$$\sum_{j=1}^N \big(|x_j| - 2\big)^{1/2} < \infty \tag{7.1}$$

and so that w obeys a Szegő condition,

$$\int_{-2}^2 \big(4 - x^2\big)^{-1/2} \log\big(w(x)\big) > -\infty, \tag{7.2}$$

one defines (see [19, 37, 51, 61]) the Jost function, $u(z)$, on $\mathbb{D}$ by

$$u(z) = z B_\infty(z) \exp\left(\int \frac{z + e^{i\theta}}{z - e^{i\theta}} \log\left[\frac{\sin\theta}{\operatorname{Im} M(e^{i\theta})}\right] \frac{d\theta}{4\pi}\right). \tag{7.3}$$

Here $M(e^{i\theta})$ is the boundary value of

$$M(z) = -m\big(z + z^{-1}\big) \tag{7.4}$$

(expressible in terms of w via $\operatorname{Im} M(e^{i\theta}) = \pi w(2\cos\theta)$), and B_∞ is the Blaschke product of $b(z, \zeta_j)$ with $\zeta_j \in \mathbb{D}$, $\zeta_j + \zeta_j^{-1} = x_j$ (by (7.1) and Lemma 4.1, this is a convergent Blaschke product). In this section and the next, we will begin to discuss the analog for elements of the isospectral torus, but in a way that connects to definitions that work in much greater generality and will be the key to later papers in this series.

Surprisingly, our initial definition will be equal, up to a constant, to an object that is based on the theta function formulae of the last section, and we will find there is a representation of the form (7.3). We regard this as one of the more interesting discoveries in the present paper.

Because there is only one natural choice for the reference measure on $[-2, 2]$, it is somewhat obscured that (7.3) involves not only $d\mu$ but a reference measure which in (7.3) is

$$d\mu_0(x) = \frac{1}{2\pi}\sqrt{4 - x^2}\chi_{[-2,2]}(x)\,dx, \tag{7.5}$$

the measure of the free Jacobi matrix with $a_n \equiv 1$, $b_n \equiv 0$. After a change of variables via $x = 2\cos\theta$ and scaling, $\sqrt{4 - x^2}$ turns into $\sin\theta$, which is where that factor in (7.3) comes from.

When one shifts to multiple gap situations, there is also a reference measure needed. Our eventual choice will be to use a particular point on the isospectral torus—and the next section will explain change of reference measure. In this section, the reference measure will be a different measure on the isospectral torus, so we will call the resulting object the "raw" Jost function.

As a bonus, we will also see that for any point on the isospectral torus,

$$n \to \frac{a_1 \cdots a_n}{\mathrm{cap}(\mathfrak{e})^n} \tag{7.6}$$

is an almost periodic sequence.

Let $\vec{y}_0 = (\beta_1, \beta_2, \ldots, \beta_\ell)$ be the point in $\mathbb{G}$ which serves as the base point for our Θ's. Given $\vec{y} = (y_1, \ldots, y_\ell) \in \mathbb{G}$, we let ξ_j be the unique point on $\tilde{C}_j^+$ such that

$$\mathbf{x}^\sharp(\xi_j) = y_j, \quad j = 1, \ldots, \ell. \tag{7.7}$$

Moreover, we denote by $w_{\vec{y}}(x)$ the weight of the measure $d\mu_{\vec{y}}$ associated to the m-function, $m_{\vec{y}}$, in the isospectral torus.

For each $\vec{y} \in \mathbb{G}$, we define a function on $\mathbb{D}$, the *raw Jost function*, by

$$\mathcal{R}(z; \vec{y}) = \prod_{\{j\,|\,|\xi_j|<1\}} B(z, \xi_j) \exp\left(\frac{1}{4\pi}\int \frac{e^{i\theta} + z}{e^{i\theta} - z}\log\left[\frac{w_{\vec{y}_0}(\mathbf{x}(e^{i\theta}))}{w_{\vec{y}}(\mathbf{x}(e^{i\theta}))}\right]d\theta\right). \tag{7.8}$$

Here we use the estimate (6.6) to be sure that

$$\int \log\big(w_{\vec{y}}(\mathbf{x}(e^{i\theta}))\big)\frac{d\theta}{2\pi} > -\infty \tag{7.9}$$

for all $\vec{y} \in \mathbb{G}$.

We will call the $\exp(\cdots)$ factor in (7.8), the *Szegő part*, and the first factor, the *Blaschke part*. It is easy to see that each is continuous in $\vec{y}$, so $\mathcal{R}(z; \vec{y})$ is also continuous in $\vec{y}$. The Blaschke factor is only piecewise C^1 and not C^1 because whenever ξ_j moves from inside $\mathbb{D}$ to outside, a factor disappears. We have $B(0, \xi_j) = |\xi_j|$ and

$$\xi_j \to \begin{cases} |\xi_j| & \text{if } |\xi_j| \leq 1, \\ 1 & \text{if } |\xi_j| \geq 1 \end{cases} \tag{7.10}$$

has a discontinuous derivative as $|\xi_j|$ passes from below 1 to above. Nevertheless, we will see below that for many cases (where the harmonic measures $\{\rho_{\mathfrak{e}}([\alpha_1, \beta_j])\}_{j=1}^{\ell}$ obey a Diophantine condition), $\vec{y} \mapsto \mathcal{R}(z; \vec{y})$ is real analytic. It is an interesting open question whether this is always true!

We need one more piece of notation. Each $\vec{y} \in \mathbb{G}$ determines a unique $m_{\vec{y}}$ and, thereby, a unique two-sided Jacobi matrix, $\tilde{J}_{\vec{y}}$. Its Jacobi parameters will be denoted $\{a_n(\vec{y}), b_n(\vec{y})\}_{n=-\infty}^{\infty}$. Here is the main theorem of this section:

Theorem 7.1 *There exists a continuous, everywhere strictly positive function, φ, on $\mathbb{G}$ so that*

$$\mathcal{R}(z; \vec{y}) = \varphi(\vec{y}) \prod_{j=1}^{\ell} \Theta(z; y_j). \tag{7.11}$$

Moreover, φ obeys

$$\varphi(\vec{y}_0) = 1, \tag{7.12}$$

$$\frac{a_1(\vec{y})}{\mathrm{cap}(\mathfrak{e})} = \frac{\varphi(U(\vec{y}))}{\varphi(\vec{y})}. \tag{7.13}$$

Remarks 1. In (7.13), U is the map from (6.29).

2. If the harmonic measures $\{\rho_{\mathfrak{e}}([\alpha_1, \beta_j])\}_{j=1}^{\ell}$ are rationally independent, the orbit $\{U^n(\vec{y}_0)\}_{n=0}^{\infty}$ is dense in $\mathbb{G}$ and (7.12)–(7.13) determine φ uniquely. In general, φ is continuous in $\{\alpha_j, \beta_j\}_{j=1}^{\ell}$, so this, in principle, determines it uniquely.

3. It is useful to define, for $\vec{y} = (y_1, \ldots, y_\ell) \in \mathbb{G}$,

$$\tilde{\Theta}(z; \vec{y}) = \prod_{j=1}^{\ell} \Theta(z; y_j). \tag{7.14}$$

We want to note some interesting corollaries before we turn to the proof:

Corollary 7.2 *For each $\vec{y} \in \mathbb{G}$, $\mathcal{R}(z; \vec{y})$ has a meromorphic continuation to $\mathbb{C} \cup \{\infty\} \setminus \mathcal{L}$ with poles, all simple, only at $\{\gamma(\zeta_j)\}_{\gamma \in \Gamma, \, j=1,\ldots,\ell}$, and zeros, all simple, at $\{\gamma(\xi_j)\}_{\gamma \in \Gamma, \, j=1,\ldots,\ell}$, where ζ_j and ξ_j are given by (5.2) and (7.7), respectively.*

Remark Thus, the Szegő part cancels the poles of $B(z, \xi_j)$ at $\{\gamma(\bar{\xi}_j^{-1})\}_{\gamma \in \Gamma}$ for j with $|\xi_j| < 1$.

Proof Obvious from (7.11). $\square$

Define the *raw Jost isomorphism* from $\mathbb{G}$ to Γ^* by

$$\mathcal{J}_r(\vec{y}) = \tilde{\mathfrak{A}}(\vec{y}). \tag{7.15}$$

By Theorem 5.3, it is a real analytic homeomorphism. The following corollary of Theorem 7.1 is so important, we call it a theorem.

Theorem 7.3 *For each* $\vec{y} \in \mathbb{G}$, $\mathcal{R}(\,\cdot\,; \vec{y})$ *is a character automorphic function and its character is* $\mathcal{J}_r(\vec{y})$. *The map* $\mathcal{J}_r$ *is a real analytic bijection between the isospectral torus* $\mathbb{G}$ *and* Γ^*.

Remark The Blaschke part of $\mathcal{R}$ is character automorphic, and we will show directly in Lemma 8.1 that the Szegő part is, too. What is difficult without (7.11) is that the map from $\vec{y}$ to the character of $\mathcal{R}(\,\cdot\,; \vec{y})$ is a bijection. That this map is a bijection will be critical in our proof of Szegő asymptotics in [16].

Proof Immediate from (7.11) and the definition of $\tilde{\mathfrak{A}}$.   □

Corollary 7.4 *For each* $\vec{y} \in \mathbb{G}$, *the sequence*

$$n \to \frac{a_1(\vec{y}) \cdots a_n(\vec{y})}{\mathrm{cap}(\mathfrak{e})^n} \qquad (7.16)$$

is bounded and almost periodic, and also bounded away from 0. *The upper and lower bounds are bounded uniformly in* $\vec{y}$.

Remark That $(a_1 \cdots a_n)^{1/n} \to \mathrm{cap}(\mathfrak{e})$ is a general fact about regular measures, and each $d\mu_{\vec{y}}$ is regular by a theorem of Widom [81] and Van Assche [78] (see [62, 69]). This more subtle result is a special case of a theorem of Widom [82] (see also [3]).

Proof By (7.13),

$$\frac{a_1(\vec{y}) \cdots a_n(\vec{y})}{\mathrm{cap}(\mathfrak{e})^n} = \frac{\varphi(U^n(\vec{y}))}{\varphi(\vec{y})} \qquad (7.17)$$

is given by the values of a continuous function (namely, $\varphi \circ \tilde{\mathfrak{A}}^{-1}$) *on* Γ^* *along the orbit* $\tilde{\mathfrak{A}}(\vec{y}) \mathfrak{A}(\infty)^{-n}$. *This function is bounded and bounded away from* 0.   □

One key to the proof of Theorem 7.1 is a nonlocal step-by-step sum rule. Such sum rules began with Killip–Simon [37] for $[-2, 2]$, formalized by Simon [58], and one version was found for periodic Jacobi matrices by Damanik–Killip–Simon [20]. The extension of the sum rules to the covering map context, which relies on Beardon's theorem, is a major theme in this paper and especially in the second paper [16] in this series. Here is the version for the isospectral torus:

Theorem 7.5 *Let* $m_{\vec{y}}$ *be the m-function for a point* $\vec{y} \in \mathbb{G}$. *Define for* $z \in \mathbb{D}$,

$$M_{\vec{y}}(z) = -m_{\vec{y}}\big(\mathbf{x}(z)\big). \qquad (7.18)$$

Then for all $z \in \mathbb{D}$,

$$a_1(\vec{y}) M_{\vec{y}}(z) = B(z) \frac{\mathcal{R}(z; U(\vec{y}))}{\mathcal{R}(z; \vec{y})}. \qquad (7.19)$$

Proof Let $B^{(\vec{y})}(z)$ be the Blaschke part of $\mathcal{R}$, that is, the Blaschke product in (7.8). Define

$$B_\infty^{(\vec{y})}(z) = B(z)\frac{B^{(U(\vec{y}))}(z)}{B^{(\vec{y})}(z)}. \tag{7.20}$$

Then $B_\infty^{(\vec{y})}(z)$ has zeros and poles precisely at all the zeros and poles of $M_{\vec{y}}(z)$ inside $\mathbb{D}$, so

$$h(z) = \frac{a_1(\vec{y})M_{\vec{y}}(z)}{B_\infty^{(\vec{y})}(z)} \tag{7.21}$$

is analytic and nonvanishing in $\mathbb{D}$.

The same argument that led to (4.84) shows that

$$z \in \mathcal{F} \quad \Rightarrow \quad \left|\arg\bigl(B_\infty^{(\vec{y})}(z)\bigr)\right| \leq C \tag{7.22}$$

for some constant C. Moreover,

$$z \in \mathcal{F} \quad \Rightarrow \quad \left|\arg\bigl(M_{\vec{y}}(z)\bigr)\right| \leq \pi, \tag{7.23}$$

so

$$z \in \mathcal{F} \quad \Rightarrow \quad \left|\arg\bigl(h(z)\bigr)\right| \leq C_1 \tag{7.24}$$

and, in particular, $\arg(h)$ varies by at most $2C_1$ over $\mathcal{F}$.

$B_\infty^{(\vec{y})}$ is character automorphic and $M_{\vec{y}}$ is automorphic, so $h(z)$ is character automorphic. Thus, $\arg(h(z))$ varies by at most $2C_1$ over any $\gamma(\mathcal{F})$, which means that

$$z \in \mathbb{D}_k \quad \Rightarrow \quad \left|\arg\bigl(h(z)\bigr)\right| \leq (2k+1)C_1. \tag{7.25}$$

Thus, by (2.18), for any r,

$$\left\{\theta \mid \left|\operatorname{Im}\log\bigl(h\bigl(re^{i\theta}\bigr)\bigr)\right| \geq (2k+1)C_1\right\} \subset \partial\mathcal{R}_k. \tag{7.26}$$

So, by Beardon's theorem in the form (3.12), for any $p < \infty$,

$$\sup_r \int \left|\operatorname{Im}\log\bigl(h\bigl(re^{i\theta}\bigr)\bigr)\right|^p \frac{d\theta}{2\pi} < \infty. \tag{7.27}$$

By M. Riesz's theorem (see Rudin [57]),

$$\log(h) \in \bigcap_{p<\infty} H^p(\mathbb{D}), \tag{7.28}$$

so by the standard representation for H^p functions, $p \geq 1$ (see, e.g., [57]), we get

$$a_1(\vec{y})M_{\vec{y}}(z) = B_\infty^{(\vec{y})}(z)\exp\left(\frac{1}{2\pi}\int \frac{e^{i\theta}+z}{e^{i\theta}-z}\log\bigl(a_1(\vec{y})\bigl|M_{\vec{y}}\bigl(e^{i\theta}\bigr)\bigr|\bigr)\,d\theta\right), \tag{7.29}$$

where we used that for a.e. θ, $|B_\infty^{(\vec{y})}(e^{i\theta})| = 1$, so

$$\log|h(e^{i\theta})| = \log(a_1(\vec{y})|M_{\vec{y}}(e^{i\theta})|). \tag{7.30}$$

Taking boundary values in

$$M_{\vec{y}}(z)^{-1} = \mathbf{x}(z) - b_1(\vec{y}) - a_1(\vec{y})^2 M_{U(\vec{y})}(z), \tag{7.31}$$

we see that

$$\frac{\operatorname{Im} M_{\vec{y}}(e^{i\theta})}{|M_{\vec{y}}(e^{i\theta})|^2} = a_1(\vec{y})^2 \operatorname{Im} M_{U(\vec{y})}(e^{i\theta}) \tag{7.32}$$

or

$$\begin{aligned}
\log(a_1(\vec{y})|M_{\vec{y}}(e^{i\theta})|) &= \frac{1}{2}\log\left[\frac{\operatorname{Im} M_{\vec{y}}(e^{i\theta})}{\operatorname{Im} M_{U(\vec{y})}(e^{i\theta})}\right] \\
&= \frac{1}{2}\log\left[\frac{\operatorname{Im} M_{\vec{y}}(e^{i\theta})}{\operatorname{Im} M_{\vec{y}_0}(e^{i\theta})}\right] - \frac{1}{2}\log\left[\frac{\operatorname{Im} M_{U(\vec{y})}(e^{i\theta})}{\operatorname{Im} M_{\vec{y}_0}(e^{i\theta})}\right]. \tag{7.33}
\end{aligned}$$

Since $\operatorname{Im} M_{\vec{y}}(e^{i\theta}) = \pi w_{\vec{y}}(\mathbf{x}(e^{i\theta}))$, (7.29) plus (7.33) and the definition (7.8) imply (7.19). $\qquad\square$

Corollary 7.6 *For any $\vec{y} \in \mathbb{G}$, we have*

$$\frac{\mathcal{R}(z; U(\vec{y}))}{\tilde{\Theta}(z; U(\vec{y}))} = \frac{a_1(\vec{y})}{\operatorname{cap}(\mathfrak{e})} \frac{\mathcal{R}(z; \vec{y})}{\tilde{\Theta}(z; \vec{y})}. \tag{7.34}$$

Proof Immediate from (6.32) and (7.19). $\qquad\square$

Proof of Theorem 7.1 For $\vec{y}_0$, we have

$$\tilde{\Theta}(z; \vec{y}_0) = \mathcal{R}(z; \vec{y}_0) = 1. \tag{7.35}$$

Thus, by (7.34), we have (7.11) for $\vec{y} = U^n(\vec{y}_0)$ with

$$\varphi(U^n(\vec{y}_0)) = \frac{a_1(\vec{y}_0) \cdots a_n(\vec{y}_0)}{\operatorname{cap}(\mathfrak{e})^n},$$

since

$$a_1(U^n(\vec{y}_0)) = a_{n+1}(\vec{y}_0). \tag{7.36}$$

Thus, (7.13) also holds for $\vec{y} = U^n(\vec{y}_0)$.

Suppose now $\{\alpha_j, \beta_j\}_{j=1}^{\ell+1}$ are such that $\{\rho_{\mathfrak{e}}([\alpha_1, \beta_j])\}_{j=1}^{\ell}$ are rationally independent. Then $S_0 \equiv \{U^n(\vec{y}_0)\}$ is dense in the isospectral torus. Define

$$\varphi(\vec{y}) \equiv \frac{\mathcal{R}(z=0; \vec{y})}{\tilde{\Theta}(z=0; \vec{y})}. \tag{7.37}$$

Since $\mathcal{R}$ is continuous in $\vec{y}$, so is φ. As (7.11) holds on the dense set S_0 and both sides are continuous, (7.11) holds for all $\vec{y} \in \mathbb{G}$. Similarly, both sides of (7.13) are continuous and as (7.13) holds on S_0, it holds in general.

To handle general $\{\alpha_j, \beta_j\}_{j=1}^{\ell+1}$, we just repeat the continuity argument "at a higher level." Fix $\alpha_1, \ldots, \alpha_{\ell+1}$ and $\beta_{\ell+1}$ and vary $(\beta_1, \ldots, \beta_\ell)$. By a theorem of Totik [73], the map $(\beta_1, \ldots, \beta_\ell) \to \{\rho_\mathfrak{e}([\alpha_j, \beta_j])\}_{j=1}^\ell$ is a C^∞ local bijection, so the set of β's with rationally independent $\{\rho_\mathfrak{e}([\alpha_j, \beta_j])\}_{j=1}^\ell$ is dense. By conveniently labeling $\mathbb{G}$ (say, measure angles in $\tilde{C}_j^+$ about its center starting at the point closest to 0) in a way that is independent of $\{\alpha_j, \beta_j\}_{j=1}^{\ell+1}$, all objects, that is, $\mathcal{R}(z; \vec{y})$ and $\tilde{\Theta}(z; \vec{y})$, are continuous in $(\beta_1, \ldots, \beta_\ell)$ according to Theorem 2.1. Therefore, by repeating the above argument, we get (7.11) and (7.13) by continuity. □

Finally, we want to note that sometimes φ and so $\mathcal{R}$ are real analytic in $\vec{y}$. We say that $\mathfrak{e}$ obeys a *Diophantine condition* if there are a constant, C, and an integer, k, so that for $(n_1, \ldots, n_\ell) \neq (0, \ldots, 0)$ in $\mathbb{Z}^\ell$,

$$\left| \sum_{j=1}^{\ell} n_j \rho_\mathfrak{e}([\alpha_1, \beta_j]) \right| \geq C(1 + |n|)^{-k}. \tag{7.38}$$

As is well known, given Totik's theorem quoted above, Lebesgue a.e. $\{\alpha_j, \beta_j\}_{j=1}^{\ell+1}$ lead to a Diophantine $\mathfrak{e}$.

Theorem 7.7 *If $\mathfrak{e}$ is Diophantine, the function φ of (7.11) is real analytic in $\vec{y}$ and thus, $\mathcal{R}(z; \vec{y})$ is real analytic in $\vec{y}$.*

Proof Θ is real analytic in y, so by (7.11), the statement about φ implies that for $\mathcal{R}$. Let

$$L(\vec{y}) = \log\left(\frac{a_1(\vec{y})}{\mathrm{cap}(\mathfrak{e})}\right). \tag{7.39}$$

Then

$$S(\vec{y}) \equiv \log(\varphi(\vec{y})) \tag{7.40}$$

obeys

$$S(U(\vec{y})) - S(\vec{y}) = L(\vec{y}). \tag{7.41}$$

Since L is real analytic on the torus, its Fourier coefficients, $l_{(n_1, \ldots, n_\ell)}$, obey

$$|l_{(n_1, \ldots, n_\ell)}| \leq C e^{-D|n|} \tag{7.42}$$

for some $C, D > 0$. By (7.41), the Fourier coefficients $s_{(n_1, \ldots, n_\ell)}$ for S obey

$$(e^{in \cdot \omega} - 1) s_{(n_1, \ldots, n_\ell)} = l_{(n_1, \ldots, n_\ell)}, \tag{7.43}$$

where

$$n \cdot \omega = \sum_{j=1}^{\ell} n_j \rho_{\mathfrak{e}}([\alpha_1, \beta_j]). \tag{7.44}$$

Equation (7.43) implies $\ell_0 = 0$, and (7.42)–(7.43) together with (7.38) imply that for any $\varepsilon > 0$,

$$|s_{(n_1,\ldots,n_\ell)}| \le C_\varepsilon e^{-(D-\varepsilon)|n|}. \tag{7.45}$$

This implies that $S(\vec{y})$ and so $\varphi(\vec{y}) = e^{S(\vec{y})}$ are real analytic.   □

8 Change of Reference Measure in Jost Functions

As is well known, OPRL obey a difference equation, (1.6). We will also be interested in other solutions. Since we have labeled the Jacobi parameters starting at $n = 1$, it will be useful to label solutions that way too, that is, to look for solutions of

$$a_n u_{n+1} + (b_n - z)u_n + a_{n-1}u_{n-1} = 0 \tag{8.1}$$

for $n = 1, 2, \ldots$ where a_0 is often picked to be 1, but in the isospectral torus will be the natural two-sided almost periodic a_0. Note that

$$u_n = p_{n-1}(z), \tag{8.2}$$

$n = 1, 2, \ldots$ is a solution of (8.1) with $p_{-1} = 0$ (so a_0 is unimportant).

For any bounded Jacobi matrix, J, and $z \in \mathbb{C}_+$, there is a unique solution of (8.1) which is ℓ^2 at infinity, unique up to an overall constant (see, e.g., [66]). One natural choice is the Weyl solution,

$$w_n(z) = \langle \delta_n, (z - J)^{-1}\delta_1 \rangle. \tag{8.3}$$

It obeys (8.1) for $n = 2, 3, \ldots$. At $n = 1$, one has

$$a_1 w_2 + (b_1 - z)w_1 = -1 \tag{8.4}$$

(since $(J - z)(J - z)^{-1}\delta_1 = \delta_1$), and so (8.1) holds at $n = 1$ if we define

$$w_0 = a_0^{-1}. \tag{8.5}$$

One defect of this solution is that, of course, $w_n(z)$ has a pole at each discrete eigenvalue of J. To get around this, Jost had the idea (for continuum Schrödinger operators) of multiplying $w_n(z)$ by a z-dependent constant $u_0(z)$ with a zero at the eigenvalues. Then $u_0(z)w_n(z)$ can have a removable singularity at such z's. Our raw Jost functions have zeros at the ξ_j's, so when we look at solutions in the next two sections, the poles in the Weyl solutions will be canceled by these zeros. But we are going to want to continue these solutions up to the bands also, and our raw Jost functions have poles at the ζ_j's. These poles are not intrinsic to the y's, but come from the $w_{\vec{y}_0}$ term in (7.8). We thus want to consider modifying that.

Definition By the *Szegő class* for $\mathfrak{e}$, we mean the set of measures $d\mu$ of the form (6.4) where w is supported on $\mathfrak{e}$ and obeys the Szegő condition (4.56) (with $f = w$), and so that outside $\mathfrak{e}$, $d\mu_s$ only has countably many pure points $\{x_j\}_{j=1}^N$ obeying a Blaschke condition of the form (4.76).

We are now prepared to define the Jost functions with reference measure $d\mu_1$.

Definition Suppose $d\mu, d\mu_1$ are two measures in the Szegő class for $\mathfrak{e}$. The *Jost function*, $\mathcal{J}(z; \mu, \mu_1)$, with reference measure $d\mu_1$, is the meromorphic function on $\mathbb{D}$ defined by

$$\mathcal{J}(z; \mu, \mu_1) = \frac{\prod_{j=1}^N B(z, z_j)}{\prod_{j=1}^{N_1} B(z, z_j^{(1)})} \exp\left(\frac{1}{4\pi} \int \frac{e^{i\theta} + z}{e^{i\theta} - z} \log\left[\frac{w_1(\mathbf{x}(e^{i\theta}))}{w(\mathbf{x}(e^{i\theta}))} \right] d\theta \right), \quad (8.6)$$

where $\{z_j\}_{j=1}^N$ are the points in $\mathcal{F}$ with $\mathbf{x}(z_j) = x_j$ (and similarly for $z_j^{(1)}$ and $x_j^{(1)}$).

Notice that, by Proposition 4.8 and condition (4.76), the Blaschke products converge. By Corollary 4.6 and (4.56), the integral in (8.6) converges. Notice that if $d\mu_1$ has pure points in the gaps, then $\mathcal{J}$ has poles in $\mathbb{D}$. For this reason, we will normally consider only $d\mu_1$'s with no such pure points, but since we want to consider the entire isospectral torus later in this section, we allow for the possibility.

We are heading towards proving that $\mathcal{J}$ is character automorphic. The key is:

Lemma 8.1 *Suppose f is a real-valued function on $\partial \mathbb{D}$ so that*

$$\int \left| f\left(e^{i\theta}\right) \right| \frac{d\theta}{2\pi} < \infty \tag{8.7}$$

and for all $\gamma \in \Gamma$,

$$f\left(\gamma\left(e^{i\theta}\right)\right) = f\left(e^{i\theta}\right). \tag{8.8}$$

Define for $z \in \mathbb{D}$,

$$S_f(z) = \exp\left(\int \frac{e^{i\theta} + z}{e^{i\theta} - z} f\left(e^{i\theta}\right) \frac{d\theta}{2\pi} \right). \tag{8.9}$$

Then S_f is character automorphic.

Remark If

$$f\left(e^{i\theta}\right) = \log\left[\frac{w_{\vec{y}_0}(\mathbf{x}(e^{i\theta}))}{w_{\vec{y}}(\mathbf{x}(e^{i\theta}))} \right], \tag{8.10}$$

where $\vec{y}$ has no points in $\mathbb{D}$ (e.g., none of the ξ_j's belongs to $\mathbb{D}$), then S_f is a raw Jost function which, if $\vec{y} \neq \vec{y}_0$, is not automorphic but only character automorphic (by Theorem 7.3). Thus, S_f may have a nontrivial character.

Proof Suppose first there is $a > 0$ so that

$$-a \le f\left(e^{i\theta}\right) \le a \tag{8.11}$$

for all θ. Then, since

$$\int \mathrm{Re}\left(\frac{e^{i\theta} + z}{e^{i\theta} - z}\right)\frac{d\theta}{2\pi} = 1 \tag{8.12}$$

with positive integrand, we have

$$e^{-a} \le \left|S_f(z)\right| \le e^a. \tag{8.13}$$

In particular, if $\gamma \in \Gamma$, then

$$h(z) \equiv \frac{S_f(\gamma(z))}{S_f(z)} \tag{8.14}$$

is analytic and

$$e^{-2a} \le \left|h(z)\right| \le e^{2a}, \tag{8.15}$$

so one can define $\log(h)$ on $\mathbb{D}$ and it belongs to $H^\infty(\mathbb{D})$.

By (8.9), for Lebesgue a.e. θ,

$$\lim_{r\uparrow 1}\left|S_f\left(re^{i\theta}\right)\right| = e^{f(e^{i\theta})}. \tag{8.16}$$

Since γ maps $\mathbb{D}$ to $\mathbb{D}$ and boundary values are nontangential limits, for Lebesgue a.e. θ,

$$\lim_{r\uparrow 1}\left|S_f\left(\gamma\left(re^{i\theta}\right)\right)\right| = e^{f(\gamma(e^{i\theta}))} = e^{f(e^{i\theta})} \tag{8.17}$$

by the hypothesis (8.8).

It follows that for a.e. θ,

$$\lim_{r\uparrow 1}\mathrm{Re}\left(\log\left(h\left(re^{i\theta}\right)\right)\right) = \lim_{r\uparrow 1}\log\left|h\left(re^{i\theta}\right)\right| = 0. \tag{8.18}$$

Since $\log|h|$ is a bounded harmonic function, $\mathrm{Re}(\log(h(z))) = 0$, so $\log(h(z)) = i\psi_\gamma$ for some real ψ_γ, that is,

$$S_f\left(\gamma\left(e^{i\theta}\right)\right) = e^{i\psi_\gamma}S_f\left(e^{i\theta}\right). \tag{8.19}$$

As usual, this implies that $\gamma \to e^{i\psi_\gamma}$ is a character, and so S_f is character automorphic.

If f does not obey (8.11), it is easy to write it as an L^1 limit of functions that do. Thus, S_f is a uniform (on compact subsets of $\mathbb{D}$) limit of character automorphic functions. By the compactness of Γ^*, it is easy to see that any such limit is character automorphic. $\qquad\square$

Theorem 8.2 *For any $d\mu, d\mu_1$ in the Szegő class, $\mathcal{J}(z; \mu, \mu_1)$ is a character automorphic function.*

Remark For $d\mu, d\mu_1$ both in the isospectral torus, this follows from Theorem 7.3 and Theorem 8.3 below.

Proof Immediate from (8.6), (4.20), and Lemma 8.1. □

Theorem 8.3 (Change of reference measure in $\mathcal{J}$) *Let $d\mu, d\mu_1, d\mu_2$ be three measures in the Szegő class. Then for all $z \in \mathbb{D}$,*

$$\mathcal{J}(z; \mu, \mu_1) = \frac{\mathcal{J}(z; \mu, \mu_2)}{\mathcal{J}(z; \mu_1, \mu_2)}. \tag{8.20}$$

In particular, for $\vec{y}, \vec{y}_1 \in \mathbb{G}$, we have

$$\mathcal{J}(z; \mu_{\vec{y}}, \mu_{\vec{y}_1}) = \frac{\mathcal{R}(z; \vec{y})}{\mathcal{R}(z; \vec{y}_1)}. \tag{8.21}$$

Remark By "all $z \in \mathbb{D}$," we either mean except for the discrete set of poles and zeros or else in the sense of meromorphic functions.

Proof In (8.6), the Blaschke products and the $\log[w_1]$, $\log[w]$ factors can be separated out and canceled and recombined. □

Corollary 8.4 *For any Szegő class measure, $d\mu_1$, the character $\mathcal{J}_{\mu_1}(\vec{y})$ of $\mathcal{J}(z; \mu_{\vec{y}}, \mu_1)$ defines a real analytic bijection of $\mathbb{G}$ onto Γ^*.*

Proof By (8.20),

$$\mathcal{J}_{\mu_1}(\vec{y}) = \mathcal{J}_r(\vec{y})\mathcal{J}_{\mu_1}(\vec{y}_0). \tag{8.22}$$

Since $\vec{y} \mapsto \mathcal{J}_r(\vec{y})$ is a bijection between $\mathbb{G}$ and Γ^* (by Theorem 7.3) and $\chi \mapsto \chi \mathcal{J}_{\mu_1}(\vec{y}_0)$ is a bijection of Γ^*, $\vec{y} \mapsto \mathcal{J}_{\mu_1}(\vec{y})$ is a bijection. □

Corollary 8.5 *For any $\vec{y}, \vec{y}_1 \in \mathbb{G}$,*

$$\mathcal{J}(z; \mu_{\vec{y}}, \mu_{\vec{y}_1}) = \varphi(\vec{y})\varphi(\vec{y}_1)^{-1} \frac{\tilde{\Theta}(z; \vec{y})}{\tilde{\Theta}(z; \vec{y}_1)}, \tag{8.23}$$

where φ is the function of Theorem 7.1, and $\tilde{\Theta}$ is given by (7.14).

Proof Immediate from (7.11) and (8.21). □

Since we want to make the poles of $\mathcal{J}$ as far from $\mathcal{S}_+$ as possible, we define $\vec{w}$ to be the point on $\mathbb{G}$ whose coordinates $(w_1, \ldots, w_\ell)$ have points $\tilde{\zeta}_1, \ldots, \tilde{\zeta}_\ell$ in $\tilde{C}_j^+$ with $\mathbf{x}^\sharp(\tilde{\zeta}_j) = w_j$ and

$$\left|\tilde{\zeta}_j\right| = \max_{\zeta \in \tilde{C}_j^+} |\zeta|. \tag{8.24}$$

Definition Let $d\nu$ be the measure in $\mathcal{T}_{\mathfrak{e}}$ associated to $\vec{w}$. For any $d\mu$ in the Szegő class, we define the *Jost function*, $u(z; \mu)$, by

$$u(z; \mu) = \mathcal{J}(z; \mu, \nu). \qquad (8.25)$$

For $\vec{y} \in \mathbb{G}$, we use $u(z; \vec{y})$ for $u(z; \mu_{\vec{y}})$.

$u(z; \mu)$ will play a major role in the later papers of this series. $u(z; \vec{y})$ will concern us in the rest of this paper. We begin by noting:

Theorem 8.6 *There is a neighborhood, N, of $\overline{\mathcal{F}}$ (closure and neighborhood in $\mathbb{C}$) so that each $u(\,\cdot\,; \vec{y})$ is analytic in N, and u is uniformly bounded on N and in $\vec{y} \in \mathbb{G}$.*

Proof Obvious from (8.23), which says that

$$u(z; \vec{y}) = \varphi(\vec{y}) \varphi(\vec{w})^{-1} \frac{\tilde{\Theta}(z; \vec{y})}{\tilde{\Theta}(z; \vec{w})}, \qquad (8.26)$$

and the fact that the ratio of $\tilde{\Theta}$'s has poles only at $\{\gamma(\tilde{\zeta}_j)\}_{\gamma \in \Gamma, j=1,\dots,\ell}$. $\qquad \square$

Proposition 8.7 *For $z \in \mathbb{R}$, $u(z; \vec{y}) > 0$.*

Proof Follows from (8.26) and the facts that φ is positive and that Θ is positive on $\mathbb{R}$. $\qquad \square$

9 Jost Solutions

One big benefit of the covering map formalism is that it provides explicit information about solutions of (8.1) for J in the isospectral torus and, thereby, of ground states, spectral theorist's Green's function, etc. We begin by moving the Weyl solutions, (8.3), to $\mathbb{D}$:

Definition For $z \in \mathbb{D}$, the *Weyl solution* is defined by

$$W_n(z) = \langle \delta_n, (\mathbf{x}(z) - J)^{-1} \delta_1 \rangle. \qquad (9.1)$$

For the case $\mathfrak{e} = [-2, 2]$, this function is studied in Sect. 13.9 of [61]. The proof of Proposition 13.9.3 of [61] is purely algebraic and immediately extends to our context:

Theorem 9.1 *Suppose J is the Jacobi matrix of any OPRL with $\sigma_{\mathrm{ess}}(J) = \mathfrak{e}$ and define M by*

$$M(z) = -m(\mathbf{x}(z)) \qquad (9.2)$$

for $z \in \mathbb{D}$. Let $J^{(n)}$ be the n-times stripped Jacobi matrix, that is,

$$a_j^{(n)} = a_{n+j}, \qquad b_j^{(n)} = b_{n+j}, \qquad (9.3)$$

and denote by $M^{(n)}$ its m-function on $\mathbb{D}$. Then

$$W_n(z) = M(z)\big(a_1 M^{(1)}(z)\big) \cdots \big(a_{n-1} M^{(n-1)}(z)\big). \tag{9.4}$$

Definition Suppose $d\mu$ lies in the Szegő class. Then the *Jost solution* is defined for $z \in \mathbb{D}$ by

$$u_n(z) = u(z; \mu) W_n(z). \tag{9.5}$$

We focus here on the case $d\mu = d\mu_{\vec{y}}$ for $\vec{y} \in \mathbb{G}$, in which case we use the notation $u_n(z; \vec{y})$.

Theorem 9.2 *For $n \geq 1$, we have*

$$u_n(z; \vec{y}) = a_n^{-1} B(z)^n u\big(z; U^n(\vec{y})\big). \tag{9.6}$$

If (9.6) is used to define u_n for all $n \in \mathbb{Z}$, then:

(i) *u_n solves (8.1) with z replaced by $\mathbf{x}(z)$.*
(ii) *There is a neighborhood, N, of $\overline{\mathcal{F}}$ so that $u_n(z; \vec{y})$ has an analytic continuation to N and is real analytic in $\vec{y}$.*
(iii) *$B(z)^{-n} u_n(z; \vec{y})$ is almost periodic in n. Indeed, uniformly for $z \in N$ and $\vec{y} \in \mathbb{G}$, it is real analytic quasiperiodic.*

Proof By (7.19),

$$a_{j+1} M^{(j)}(z) = B(z) \frac{u(z; U^{j+1}(\vec{y}))}{u(z; U^j(\vec{y}))}. \tag{9.7}$$

Thus, by (9.4),

$$a_n W_n(z) = \prod_{j=0}^{n-1} a_{j+1} M^{(j)}(z) = B(z)^n \frac{u(z; U^n(\vec{y}))}{u(z; \vec{y})}, \tag{9.8}$$

which is (9.6).

By (9.6), we have

$$a_{n+j} u_{n+j}(z; \vec{y}) = a_n B(z)^j u_n\big(z; U^j(\vec{y})\big) \tag{9.9}$$

for all $n, j \in \mathbb{Z}$. Since W_n solves (8.1) with z replaced by $\mathbf{x}(z)$ for $n \geq 1$, u_n does also, and then by (9.9), u_n is a solution for all $n \in \mathbb{Z}$. (ii) is immediate from Theorem 8.6. (iii) is then immediate from (9.6) and the fact that under $\tilde{\mathfrak{A}}$, U is transformed to multiplication (by $\mathfrak{A}(\infty)^{-1}$) on Γ^*. $\qquad\square$

For $x \in \mathfrak{e}$, define u_n^+ by picking $\mathbf{z}(x) \in \overline{\mathcal{F}}$ with $\operatorname{Im} \mathbf{z} \geq 0$, so that $\mathbf{x}(\mathbf{z}(x)) = x$, and by letting

$$u_n^+(x; \vec{y}) = u_n\big(\mathbf{z}(x); \vec{y}\big), \tag{9.10}$$

where we take boundary values of u_n on the right-hand side. We also let

$$u_n^-(x; \vec{y}) = \overline{u_n^+(x; \vec{y})} = u_n\big(\overline{\mathbf{z}(x)}; \vec{y}\big). \tag{9.11}$$

Theorem 9.3 *Define the Wronskian,* $\mathrm{Wr}(f, g)$, *of two solutions of* (8.1) *by*

$$\mathrm{Wr}(f, g) = a_n(f_{n+1}g_n - f_n g_{n+1}) \tag{9.12}$$

(which is n-independent). Then, if $x = \mathbf{x}(\mathbf{z}(x))$ *and*

$$\mathrm{Im}\,\mathbf{z} > 0, \qquad x \in \mathfrak{e}^{\mathrm{int}}, \tag{9.13}$$

we have

$$\mathrm{Wr}\big(u_\cdot^+(x; \vec{y}), u_\cdot^-(x; \vec{y})\big) = 2\pi i \big|u\big(\mathbf{z}(x); \vec{y}\big)\big|^2 w_{\vec{y}}(x), \tag{9.14}$$

where $w_{\vec{y}}$ *is the weight in the spectral measure,* $d\mu_{\vec{y}}$.

Proof Since $W_0 = a_0^{-1}$,

$$\mathrm{Wr}(\bar{W}, W) = a_0\big(\bar{W}_1 a_0^{-1} - W_1 a_0^{-1}\big) = -2i\,\mathrm{Im}\,W_1. \tag{9.15}$$

Taking into account that

$$\frac{1}{\pi}\,\mathrm{Im}\,m(x_0 + i0) = w(x_0) \tag{9.16}$$

and

$$\mathrm{Wr}(\bar{c}\bar{f}, cf) = |c|^2\mathrm{Wr}(\bar{f}, f), \tag{9.17}$$

we get (9.14).   □

Recall ([66, Chap. 3]) that the transfer matrix, $T_n(z)$, for a Jacobi matrix updates solutions of (8.1) via

$$\begin{pmatrix} u_{n+1} \\ a_n u_n \end{pmatrix} = T_n(z) \begin{pmatrix} u_1 \\ a_0 u_0 \end{pmatrix} \tag{9.18}$$

and is given by

$$T_n(z) = \begin{pmatrix} p_n(z) & -q_n(z) \\ a_n p_{n-1}(z) & -a_n q_{n-1}(z) \end{pmatrix}, \tag{9.19}$$

where q_n are the second kind polynomials. As usual, if we want to indicate the underlying point in the isospectral torus, we write $T_n(z; \vec{y})$.

Theorem 9.4 *There is a constant C so that uniformly for* $\vec{y} \in \mathbb{G}$ *and* $x \in \mathfrak{e}^{\mathrm{int}}$,

$$\big\|T_n(x; \vec{y})\big\| \leq C\,\mathrm{dist}(x, \mathbb{R} \setminus \mathfrak{e})^{-1/2}. \tag{9.20}$$

Remark This result is used in Proposition 7.2 of [11].

Proof Let $U_n(x; \vec{y})$ be the matrix

$$U_n(x; \vec{y}) = \begin{pmatrix} u_{n+1}^+(x; \vec{y}) & u_{n+1}^-(x; \vec{y}) \\ a_n u_n^+(x; \vec{y}) & a_n u_n^-(x; \vec{y}) \end{pmatrix}. \tag{9.21}$$

Then

$$T_n(x; \vec{y})U_0(x; \vec{y}) = U_n(x; \vec{y}), \tag{9.22}$$

so

$$T_n(x; \vec{y}) = U_n(x; \vec{y})U_0(x; \vec{y})^{-1}, \tag{9.23}$$

and, since 2×2 matrices obey $\|C^{-1}\| = |\det(C)|^{-1}\|C\|$,

$$\left\| T_n(x; \vec{y}) \right\| \leq \left| \det\left(U_0(x; \vec{y}) \right) \right|^{-1} \left\| U_0(x; \vec{y}) \right\| \left\| U_n(x; \vec{y}) \right\|. \tag{9.24}$$

As u_n is uniformly bounded in n, $\vec{y}$, and $x \in \mathfrak{e}$, and $\det(U_0)$ is the Wronskian, by (9.14),

$$\left\| T_n(x; \vec{y}) \right\| \leq C w_{\vec{y}}(x)^{-1},$$

which yields (9.20), given (6.6) and (6.25). □

Corollary 9.5 *Uniformly in n, $x \in \mathfrak{e}^{\mathrm{int}}$, and $\vec{y} \in \mathbb{G}$,*

$$C_1 \mathrm{dist}(x, \mathbb{R} \setminus \mathfrak{e}) \leq \left| p_n(x) \right|^2 + \left| p_{n-1}(x) \right|^2 \leq C_2 \mathrm{dist}(x, \mathbb{R} \setminus \mathfrak{e})^{-1}$$

for suitable constants C_1 and C_2.

Proof Immediate from (9.19), $\det(T_n) = 1$ (so $\|T_n^{-1}\| = \|T_n\|$), and (9.20), recalling that the a_n's are bounded and bounded away from 0. □

Next we look at band edges where $\|T_n\|$ can diverge. We begin with a critical fact about the outer edges:

Theorem 9.6 *There are positive, finite constants C_1 and C_2, so that uniformly in n and $\vec{y}$,*

$$C_1 \leq u_n^+(\beta_{\ell+1}; \vec{y}) \leq C_2, \tag{9.25}$$

$$C_1 \leq (-1)^n u_n^+(\alpha_1; \vec{y}) \leq C_2. \tag{9.26}$$

Remark (9.25) says, in the language of [27], that each whole-line $\tilde{J}_{\vec{y}}$ has a regular ground state (see [27, Example 1.5]). It implies critical Lieb–Thirring bounds for perturbations of $\tilde{J}_{\vec{y}}$ in $(\beta_{\ell+1}, \infty)$. Inequalities (9.26) imply similar bounds for $(-\infty, \alpha_1)$.

Proof $B(z)/z$ is positive at $z = 0$ and real and nonvanishing on $(-1, 1)$, so $B(x) > 0$ on $(0, 1]$ and $B(x) < 0$ on $[-1, 0)$. Since $|B(\pm 1)| = 1$, we conclude that

$$B(\pm 1) = \pm 1. \tag{9.27}$$

Since $u(x; \vec{y})$ is bounded, strictly positive, and continuous in x and $\vec{y}$ for $x \in [-1, 1]$ and $\vec{y} \in \mathbb{G}$, (9.6) implies (9.25)–(9.26). □

$u_n^+(x)$ is real at $x \in \{\alpha_j, \beta_j\}_{j=1}^{\ell+1}$, so $u_n^- = u_n^+$ and the Jost solutions are not linearly independent. The following gives a second solution which grows linearly in n.

Theorem 9.7 *Uniformly in $\vec{y} \in \mathbb{G}$ and $z \in \partial\mathcal{F} \cap \overline{\mathbb{D}}$,*

(i)
$$\left| \frac{\partial u_n^+(\mathbf{x}(z); \vec{y})}{\partial z} \right| \le C\big(|n| + 1\big), \tag{9.28}$$

(ii)
$$\liminf_{n \to \infty} \left| \frac{1}{n} \frac{\partial u_n^+(\mathbf{x}(z); \vec{y})}{\partial z} \right| > 0, \tag{9.29}$$

(iii) *At $\mathbf{x}(z) \in \{\alpha_j, \beta_j\}_{j=1}^{\ell+1}$,*

$$v_n(\vec{y}) = \frac{\partial u_n^+}{\partial z}\big(\mathbf{x}(z); \vec{y}\big) \tag{9.30}$$

is a solution of (8.1) *with z replaced by $\mathbf{x}(z)$, linearly independent of $u_n^+(\mathbf{x}(z); \vec{y})$.*

Proof (i), (ii) By (9.6),

$$\frac{\partial u_n^+(\mathbf{x}(z); \vec{y})}{\partial z} = a_n^{-1} n B^{n-1}(z) B'(z) u\big(z; U^n(\vec{y})\big)$$

$$+ a_n^{-1} B^n(z) \frac{\partial}{\partial z} u\big(z; U^n(\vec{y})\big). \tag{9.31}$$

Since $u(z; U^n(\vec{y}))$ and $\frac{\partial}{\partial z}(u(z; U^n(\vec{y})))$ are uniformly bounded in $\vec{y}$ and n, and $B'(e^{i\theta}) > 0$ for all θ, (9.28)–(9.29) are immediate.

(iii) $u_n^+(\mathbf{x}(z))$ obeys (8.1) with z replaced by $\mathbf{x}(z)$. Since $\mathbf{x}'(z) = 0$ at points with $\mathbf{x}(z) \in \{\alpha_j, \beta_j\}_{j=1}^{\ell+1}$, we see that v_n also solves (8.1). Since u_n^+ is bounded and v_n is not, they are linearly independent. $\qquad\square$

Corollary 9.8 *For $z \in \mathfrak{e}$,* (8.1) *has no solution which belongs to ℓ^2 at $+\infty$ or at $-\infty$.*

Remark This result is used in [11].

Proof If $z \in \mathfrak{e}^{\text{int}}$, this follows from the fact that $\|T_n(z)^{-1}\|$ is bounded, and for $z \in \partial\mathfrak{e}$, it follows from Theorem 9.7. $\qquad\square$

Corollary 9.9 *If $x \in \{\alpha_j, \beta_j\}_{j=1}^{\ell+1}$, then*

$$\big\|T_n(x)\big\| \le C\big(1 + |n|\big). \tag{9.32}$$

Proof Let

$$\tilde{U}_n(x) = \begin{pmatrix} u_{n+1}^+(x) & v_{n+1}(x) \\ a_n u_n^+(x) & a_n v_n(x) \end{pmatrix}. \tag{9.33}$$

As in (9.23),

$$T_n(x) = \tilde{U}_n(x)\tilde{U}_0(x)^{-1}. \tag{9.34}$$

Since u^+, v are independent, $\tilde{U}_0$ is invertible and, clearly, $\|\tilde{U}_n\| \le C(1 + |n|)$. $\qquad\square$

The bound (9.20) diverges as x approaches a point in $\{\alpha_j, \beta_j\}_{j=1}^{\ell+1}$. Since $\|T_n\|$ is not bounded at these points, this must be so. However, we are heading towards a uniform (on $\mathfrak{e}$) $O(n)$ bound. As a starting point, we need to know more about the right side of (9.14) than the crude bound from (6.6).

Proposition 9.10 *For any $\vec{y} \in \mathbb{G}$ and fixed $x_0 \in \{\alpha_j, \beta_j\}_{j=1}^{\ell+1}$,*

$$\lim_{\substack{x \to x_0 \\ x \in \mathfrak{e}^{\mathrm{int}}}} |x - x_0|^{-1/2} \big|u\big(\mathbf{z}(x); \vec{y}\big)\big|^2 w_{\vec{y}}(x) \tag{9.35}$$

exists, is finite and nonvanishing, and is continuous in $\vec{y}$. In (9.35), $\mathbf{z}(x)$ is the point obeying (9.13) with $\mathbf{z} \in \overline{\mathcal{F}} \cap \mathbb{C}_+$ and $x = \mathbf{x}(\mathbf{z}(x))$.

Remark This result is subtle because $|u(\mathbf{z}(x); \vec{y})|^2$ can vanish at x_0. In that case, $w_{\vec{y}}(x)$ has $O(\sqrt{|x - x_0|}^{-1})$ asymptotics rather than $O(\sqrt{x - x_0})$ asymptotics.

Proof By the explicit formula for $w_{\vec{y}}$ ((6.10) and (6.22)), each factor in $w_{\vec{y}}(x)|x - x_0|^{-1/2}$ is continuous in x and $\vec{y}$ except for the $|x - \pi(y_j)|^{-1}$ factor with the $\pi(y_j)$ closest to x_0. There is a canceling factor in $|u(\mathbf{z}(x); \vec{y})|^2$, so the limit exists and is continuous in $\vec{y}$. $\qquad\qquad\square$

Theorem 9.11 *There is a constant C so that uniformly in $x \in \mathfrak{e}$ and $\vec{y} \in \mathbb{G}$,*

$$\big\|T_n(x; \vec{y})\big\| \le C\big(|n| + 1\big). \tag{9.36}$$

Proof For each $x_0 \in \{\alpha_j, \beta_j\}_{j=1}^{\ell+1}$, we prove (9.36) in the half-band starting at x_0. Form a matrix $\tilde{U}_n$ like (9.21) but with u^- replaced by $\tilde{v} = (u^- - u^+)/|x - x_0|^{1/2}$. As we have seen, $\tilde{v}$ has a limit as x approaches x_0 from $\mathfrak{e}^{\mathrm{int}}$. By the Wronskian calculation, $\det(U_n)$ (which is n-independent) is bounded as x approaches x_0.

Finally, writing $(u^- - u^+)$ as the integral of a derivative and using (9.28), we get

$$|\tilde{U}_n| \le C\big(|n| + 1\big), \tag{9.37}$$

so

$$\|\tilde{U}_n\| \le C\big(|n| + 1\big), \tag{9.38}$$

which implies (9.36). $\qquad\qquad\square$

Finally, we want to note that (9.6) implies a result about the Jost solutions used in [64].

Theorem 9.12 *For any compact interval $I \subset \mathfrak{e}^{\mathrm{int}}$, we can write*

$$u_n^+(x) = e^{in\theta(x)} f_n(x), \tag{9.39}$$

where $\theta'(x) = \pi \rho_{\mathfrak{e}}(x)$ and f_n is real analytic in x with derivatives uniformly bounded in n.

Proof Let $\mathbf{z}(x) \in \overline{\mathcal{F}} \cap \mathbb{C}_+$ with $\mathbf{x}(\mathbf{z}(x)) = x$, and write

$$B\big(\mathbf{z}(x)\big) = e^{i\theta(x)}.$$

By (9.6), u_n^+ has the form (9.39), and we see that f_n has the required properties. By the calculation that led to (4.61), we get the expression for θ'. $\qquad\square$

In [64], this was used to prove that for any $d\mu_{\vec{y}}$, with $\vec{y} \in \mathbb{G}$,

$$\frac{1}{n} K_n(x,x) \to \frac{\rho_{\mathfrak{e}}(x)}{w_{\vec{y}}(x)} \tag{9.40}$$

uniformly on I, where K_n is the CD kernel (see [63] for definition and background on the classical work of Máté–Nevai–Totik and Totik on limits like (9.40)). Using the calculations in [5] and identifying u_n^+ as a multiple of the Deift–Simon eigenfunctions of [5], one sees that (9.40) implies that:

Proposition 9.13 *Uniformly on compact subsets of $\mathfrak{e}^{\mathrm{int}} \times \mathbb{G}$,*

$$\frac{1}{n} \sum_{k=0}^{n-1} \mathrm{Re}\big[u_k^+(x;\vec{y})^2\big] \to 0 \ \text{as } n \to \infty. \tag{9.41}$$

Remark That is, $(\mathrm{Re}\, u_n^+)^2$ and $(\mathrm{Im}\, u_n^+)^2$ have the same average.

10 Bounds on (Spectral Theorist's) Green's Function

For $\vec{y} \in \mathbb{G}$, let $J_{\vec{y}}$ be the associated Jacobi matrix and $\tilde{J}_{\vec{y}}$ the associated full-line Jacobi matrix. The (spectral theorist's) Green's functions are defined by

$$G_{nm}(z) = \langle \delta_n, (J_{\vec{y}} - z)^{-1}\delta_m\rangle, \quad n,m = 1,2,\ldots, \tag{10.1}$$

$$\tilde{G}_{nm}(z) = \langle \delta_n, (\tilde{J}_{\vec{y}} - z)^{-1}\delta_m\rangle, \quad n,m \in \mathbb{Z}. \tag{10.2}$$

We will use $G_{nm}(z;\vec{y})$ when we need $\vec{y}$ to be explicit. It is unfortunate that "Green's function" is used both for these objects and for $G_{\mathfrak{e}}(z)$, the potential theorist's Green's function, but both names are ubiquitous. $G_{\mathfrak{e}}$ will appear below in our discussions of G_{nm}.

The analogs of G_{nm} and $\tilde{G}_{nm}$ for $-d^2/dx^2$ on $L^2(0,\infty)$ or $L^2(-\infty,\infty)$ are given by

$$\tilde{G}(x,y;E) = \frac{e^{-\kappa|x-y|}}{2\kappa}, \tag{10.3}$$

$$G(x,y;E) = \frac{e^{-\kappa x_>}\sinh(\kappa x_<)}{\kappa}, \tag{10.4}$$

where $E = -\kappa^2$, $x_> = \max(x, y)$, $x_< = \min(x, y)$. The bounds

$$\left|\tilde{G}(x, y; E)\right| \le (2\kappa)^{-1}, \tag{10.5}$$

$$\left|G(x, y; E)\right| \le x_< \tag{10.6}$$

play important roles in the analysis of bound states of Schrödinger operators with short-range potentials. Here we find analogs of these bounds for Jacobi matrices in the isospectral torus for z in $\mathbb{R} \setminus \mathfrak{e}$. These bounds were used in [36] to obtain bounds on perturbations of $J_{\vec{y}}$ and $\tilde{J}_{\vec{y}}$.

We need to begin by defining $u_n^{\pm}(x; \vec{y})$ for $x \in \mathbb{R} \setminus \mathfrak{e}$. Define $\mathbf{z}(x)$ to be the unique point in $[\bigcup_{j=1}^{\ell} C_j^+] \cup (-1, 1)$ with

$$\mathbf{x}\big(\mathbf{z}(x)\big) = x. \tag{10.7}$$

Then we define

$$u_n^+(x; \vec{y}) = a_n^{-1} B\big(\mathbf{z}(x)\big)^n u\big(\mathbf{z}(x); U^n(\vec{y})\big), \tag{10.8}$$

$$u_n^-(x; \vec{y}) = a_n^{-1} B\big(\mathbf{z}(x)\big)^{-n} \overline{u\big(1/\overline{\mathbf{z}(x)}; U^n(\vec{y})\big)}. \tag{10.9}$$

u_n^+ is the analytic continuation of u_n^+ as defined for $x \in \mathfrak{e}$ in the last section if we keep $x \in \overline{\mathbb{C}}_+$. So is u_n^-, since u_n^- is defined on the lower lip of the cuts, so continuing in $\overline{\mathbb{C}}_+$ brings us to the second sheet and $1/\overline{\mathbf{z}(x)}$. The $-n$ in u_n^- comes from

$$\overline{B\big(1/\overline{\mathbf{z}(x)}\big)} = B\big(\mathbf{z}(x)\big)^{-1}. \tag{10.10}$$

Since

$$\left|B\big(\mathbf{z}(x)\big)\right| = e^{-G_{\mathfrak{e}}(x)}, \tag{10.11}$$

$u_n^{\pm}$ decays exponentially as $n \to \pm\infty$ and grows exponentially as $n \to \mp\infty$. It follows that $u_n^{\pm}$ must have constant phase (and perhaps we should redefine them to be real). Indeed, the phase is constant on each gap and u^+ and u^- have opposite phases.

Theorem 10.1 *For $x \in \mathbb{R} \setminus \mathfrak{e}$ and $n \le m$, we have*

$$\tilde{G}_{nm}(x) = \frac{u_n^-(x) u_m^+(x)}{\mathrm{Wr}(x)}, \tag{10.12}$$

where

$$\mathrm{Wr}(x) = a_n\big(u_{n+1}^+(x) u_n^-(x) - u_{n+1}^-(x) u_n^+(x)\big). \tag{10.13}$$

Uniformly in $x \in \mathbb{R} \setminus \mathfrak{e}$ and $\vec{y} \in \mathbb{G}$, we have

$$\left|\tilde{G}_{nm}(x)\right| \le C e^{-G_{\mathfrak{e}}(x)|n-m|} \mathrm{dist}(x, \mathfrak{e})^{-1/2}. \tag{10.14}$$

Proof (10.12)–(10.13) is a standard formula for $\tilde{G}$ in terms of any solutions decaying at $\pm\infty$. As x runs through $\mathbb{R} \setminus \mathfrak{e}$, $\mathbf{z}(x)$ runs through $[\bigcup_{j=1}^{\ell} C_j^+] \cup (-1, 1) \setminus \{0\}$, and $u(\mathbf{z}(x); \vec{y})$ is uniformly bounded there. So, by (10.11), we get (10.14) from

$$\left|\mathrm{Wr}(x)\right| \geq C \, \mathrm{dist}(x, \mathfrak{e})^{1/2}. \tag{10.15}$$

This is trivial, except near the points $|x| = \infty$ and $x \in \{\alpha_j, \beta_j\}_{j=1}^{\ell+1}$, since $\mathrm{Wr}(x)$ is nonvanishing and continuous away from those points.

Since u is regular at $z = 0$ and $z = \infty$, the dominant term in (10.8)–(10.9) is the $B(z)$ term. In

$$\mathrm{Wr}(x) = a_0\left(u_1^+ u_0^- - u_1^- u_0^+\right),$$

the dominant term is $B(z)^{-1}$ in u_1^-, so

$$\left|\mathrm{Wr}(x)\right| \sim C\left|\mathbf{z}(x)\right|^{-1} \sim C|x| \geq C|x|^{1/2}, \tag{10.16}$$

and (10.15) holds near $|x| = \infty$.

Near points $x_0 \in \{\alpha_j, \beta_j\}_{j=1}^{\ell+1}$, we are looking at the Wronskian of two solutions u^+ and u^- which approach each other. Thus, we get a Wronskian which goes to zero as $(\mathbf{z}(x) - \mathbf{z}(x_0))$ times the Wronskian of u^+ and du^-/dz. We have already seen in the last section that these are two linearly independent solutions, so their Wronskian is nonzero and thus, near $x_0 \in \{\alpha_j, \beta_j\}_{j=1}^{\ell+1}$, for some $C > 0$,

$$\mathrm{Wr}(x) \geq C\left|\mathbf{z}(x) - \mathbf{z}(x_0)\right| = C \cdot O\left(|x - x_0|^{1/2}\right), \tag{10.17}$$

proving (10.15). □

As x approaches a point $x_0 \in \{\alpha_j, \beta_j\}_{j=1}^{\ell+1}$ from $\mathbb{R} \setminus \mathfrak{e}$, typically (i.e., except for special values of $\vec{y}$, n, and m), $\tilde{G}_{nm}(x) \to \infty$. For $G_{nm}(x)$, this is normally not true, which is why one expects bounds in this case that are not divergent at x_0. However, there are special values of $\vec{y}$ for which this is not the case. One sees this for $n = m = 1$, since

$$G_{11}(z) = m(z), \tag{10.18}$$

the m-function of (6.3). m is meromorphic on $\mathcal{S}$, so it normally has a finite value at x_0 but might have a pole there.

Definition Fix $\vec{y} \in \mathbb{G}$. A point $x_0 \in \{\alpha_j, \beta_j\}_{j=1}^{\ell+1}$ is said to be a *resonance* if and only if it is a pole of $m_{\vec{y}}(z)$ (in the sense of poles on $\mathcal{S}$ which means $(x - x_0)^{-1/2}$ divergence since x_0 is a branch point). Otherwise, we say x_0 is *nonresonant*.

It is easy to see that resonances are equivalent to $u_0^+(x_0) = 0$.

Theorem 10.2 *Fix $\vec{y} \in \mathbb{G}$. For $n, m \geq 1$ and $x \in \mathbb{R} \setminus \mathfrak{e}$, we have*

$$G_{nm}(x) = \tilde{G}_{nm}(x) - \tilde{G}_{0n}(x)\tilde{G}_{0m}(x)\tilde{G}_{00}(x)^{-1}. \tag{10.19}$$

Suppose x_0 is nonresonant for $\vec{y}$. Let I be the open interval with one end at x_0 and the other determined by $I \cap \mathfrak{e} = \emptyset$ and:

(i) *If $x_0 \in \{\alpha_1, \beta_{\ell+1}\}$, then $|I| = 1$.*
(ii) *If x_0 is an edge of an internal gap and $J_{\vec{y}}$ has no eigenvalue in that gap, then the other end of I is the middle of the gap.*
(iii) *If x_0 is an edge of an internal gap and $J_{\vec{y}}$ has an eigenvalue in the gap, then the other end of I is halfway between x_0 and that eigenvalue.*

Then for all $n, m \geq 1$ and $x \in I$, we have

$$|G_{nm}(x)| \leq C \min(n, m), \tag{10.20}$$

$$|G_{nm}(x)| \leq C|x - x_0|^{-1/2} \tag{10.21}$$

for some constant C.

Proof Since, for $x \in \mathbb{R} \setminus \mathfrak{e}$ fixed,

$$u_0^+ u_n^- - u_0^- u_n^+ \equiv q_n \tag{10.22}$$

vanishes at $n = 0$, $q_n(x) = C(x) p_{n-1}(x)$ for some constant (depending on x). Thus, by the standard formula for G_{nm},

$$G_{nm} = \frac{p_{n-1} u_m^+}{\mathrm{Wr}(u^+, p_{\cdot -1})}, \quad 1 \leq n \leq m, \tag{10.23}$$

we get

$$G_{nm}(x) = q_n(x) u_m^+(x) \widetilde{\mathrm{Wr}}(x)^{-1}, \tag{10.24}$$

where $\widetilde{\mathrm{Wr}}$ is the Wronskian of u^+ and q.

By (10.22),

$$\widetilde{\mathrm{Wr}}(x) = u_0^+(x) \mathrm{Wr}(x), \tag{10.25}$$

so (10.24) becomes

$$G_{nm}(x) = \frac{u_0^+(x) u_n^-(x) u_m^+(x) - u_0^-(x) u_n^+(x) u_m^+(x)}{u_0^+(x) \mathrm{Wr}(x)}. \tag{10.26}$$

The first term in (10.26) is, by (10.12), $\tilde{G}_{nm}(x)$. If we note that (also by (10.12))

$$\tilde{G}_{0n}(x) \tilde{G}_{0m}(x) \tilde{G}_{00}(x)^{-1} = \frac{u_0^-(x) u_n^+(x) u_m^+(x)}{u_0^+(x) \mathrm{Wr}(x)}, \tag{10.27}$$

we see that the second term in (10.26) is the second term in (10.19), so we have proven (10.19).

As we noted above, x_0 nonresonant implies that $u_0^+(x_0) \neq 0$. Thus, (10.26) shows that

$$\sup_{\substack{n,m \geq 1 \\ x \in I}} |G_{nm}(x)| \leq C \mathrm{Wr}(x)^{-1}, \tag{10.28}$$

which, by (10.15), proves (10.21).

We claim first that (10.20) is implied by

$$\left| q_n(x) B\big(\mathbf{z}(x)\big)^n \right| \le C n |x - x_0|^{1/2}. \tag{10.29}$$

For, by (10.24) and (10.15),

$$\left| G_{nm}(x) \right| \le C \left| u_m^+(x) B\big(\mathbf{z}(x)\big)^{-m} \right| \left| B\big(\mathbf{z}(x)\big) \right|^{m-n} \left| q_n(x) B\big(\mathbf{z}(x)\big)^n \right| \left| x - x_0 \right|^{-1/2}, \tag{10.30}$$

so (10.29) together with $|B(\mathbf{z}(x))| \le 1$ and the fact that $|u_m^+(x) B(\mathbf{z}(x))^{-m}|$ is bounded implies (10.20).

Next, note that because of definitions (10.8)–(10.9) and the constancy of the phase of $u_n^\pm$, we see that for all n,

$$q_n(x_0) = 0. \tag{10.31}$$

Define

$$h_n(z) = q_n\big(\mathbf{x}(z)\big) B(z)^n. \tag{10.32}$$

By (10.31) and $|\mathbf{z}(x) - \mathbf{z}(x_0)| = O(|x - x_0|^{1/2})$, (10.29) follows from

$$\sup_{z \in \mathbf{z}(I)} \left| \frac{dh_n(z)}{dz} \right| \le C n. \tag{10.33}$$

h_n is built out of u's, which have bounded derivatives, and $B(z)^n$, which has a derivative bounded by Cn, so (10.33) holds.   □

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