THE HILBERT TRANSFORM OF A MEASURE

By

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Abstract. Let $\mathfrak{e}$ be a homogeneous subset of $\mathbb{R}$ in the sense of Carleson. Let μ be a finite positive measure on $\mathbb{R}$ and $H_\mu(x)$ its Hilbert transform. We prove that if $\lim_{t\to\infty} t|\mathfrak{e}\cap\{x \mid |H_\mu(x)| > t\}| = 0$, then $\mu_s(\mathfrak{e}) = 0$, where μ_s is the singular part of μ.

1 Introduction

This is a paper about the Hilbert transform of a measure defined as follows. The Stieltjes transform (also called Borel transform or Markov function) of a finite (positive) measure μ is defined on $\mathbb{C}_+ = \{z \mid \operatorname{Im} z > 0\}$ by

$$F_\mu(z) = \int \frac{d\mu(x)}{x-z}. \tag{1.1}$$

For Lebesgue a.e. $x \in \mathbb{R}$,

$$F_\mu(x+i0) = \lim_{\varepsilon\downarrow 0} F_\mu(x+i\varepsilon) \tag{1.2}$$

exists. The Hilbert transform is given by

$$H_\mu(x) = \frac{1}{\pi} \operatorname{Re} F_\mu(x+i0). \tag{1.3}$$

A result of Loomis [8] is that for a universal constant C ($\|\mu\| \equiv \mu(\mathbb{R})$),

$$|\{x : |H_\mu(x)| \geq t\}| \leq \frac{C\|\mu\|}{t}. \tag{1.4}$$

This was earlier proven for the a.c. case by Kolmogorov (attributed by Zygmund [16]). For finite point measures, Boole [1] proved (and Loomis rediscovered)

$$|\{x : \pm H_\mu(x) \geq t\}| = \frac{\|\mu\|}{\pi t}. \tag{1.5}$$

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We note that (1.5) was extended by Hruščëv–Vinogradov [7] to all singular measures; see also [5, 10].

Remark. We do not need an explicit value of C in (1.4). Davis [3, 4] has shown the optimal constant in (1.4) is $C = 1$.

In distinction, for a.c. measures $d\mu = f\,dx$, we have

$$\lim_{t\to\infty} t|\{x : |H_{f\,dx}(x)| \geq t\}| = 0. \tag{1.6}$$

This follows from the facts that if $f \in L^2$, then $H_{f\,dx} \in L^2$ (indeed, $\|H_{f\,dx}\|_2 = \|f\|_2$), that $L^2 \cap L^1$ is dense in L^1, that (1.6) is trivial if $H_{f\,dx}$ is L^2, and that for any $\theta \in [0, 1]$,

$$\begin{aligned} &|\{x : |f(x) + g(x)| > t\}| \\ &\qquad \leq |\{x : |f(x)| > \theta t\}| + |\{x : |g(x)| > (1-\theta)t\}|. \end{aligned} \tag{1.7}$$

From (1.5), (1.6), and (1.7), one sees that

$$\lim_{t\to+\infty} \pi t|\{x : \pm H_\mu(x) \geq t\}| = \|\mu_{\rm s}\|, \tag{1.8}$$

where

$$d\mu = f\,dx + d\mu_{\rm s} \tag{1.9}$$

is the Lebesgue decomposition of μ (i.e., $\mu_{\rm s}$ is singular).

In fact, a more general statement holds: for any finite complex measure μ, the measures

$$\tfrac{1}{2}\pi t \chi_{\{x:\ |H_\mu(x)|\geq t\}}\,dx$$

converge in the $*$-weak topology to the measure $|d\mu_{\rm s}|$; see (5.4) or [10].

One can rephrase this. We recall that weak-L^1 is defined by (this is not a norm!) setting

$$\|f\|_{1,w} \equiv \sup_t t|\{x : |f(x)| \geq t\}| \tag{1.10}$$

and

$$L^1_w = \{f : \|f\|_{1,w} < \infty\}, \tag{1.11}$$

so (1.4) says $H_\mu \in L^1_w$. We also define

$$L^1_{w;0} = \bigl\{f \in L^1_w : \lim_{t\to\infty} t|\{x : |f(x)| \geq t\}| = 0\bigr\}. \tag{1.12}$$

Then (1.8) implies that

$$H_\mu \in L^1_{w;0} \Leftrightarrow \mu_s(\mathbb{R}) = 0. \tag{1.13}$$

Our main goal is to provide a local version of this theorem for special sets singled out by Carleson [2].

Definition. We say that a compact set $\mathfrak{e} \subset \mathbb{R}$ is **homogeneous** (with homogeneity constant δ) if there is $\delta > 0$ such that for all $x \in \mathfrak{e}$ and $0 < a < \operatorname{diam}(\mathfrak{e})$,

$$|\mathfrak{e} \cap (x-a, x+a)| \geq 2\delta a. \tag{1.14}$$

Given a function f, we use $f \restriction \mathfrak{e}$ to denote the function $f\chi_\mathfrak{e}$ with $\chi_\mathfrak{e}$ the characteristic function of $\mathfrak{e}$. The purpose of this paper is to prove

Theorem 1.1. *Let $\mathfrak{e}$ be homogeneous and let μ be a measure on $\mathbb{R}$ such that $H_\mu \restriction \mathfrak{e} \in L^1_{w;0}$. Then*

$$\mu_{\mathrm{s}}(\mathfrak{e}) = 0. \tag{1.15}$$

Remarks. 1. There is an analog for measures on $\partial\mathbb{D} = \{z \in \mathbb{C} : |z| = 1\}$.

2. The Hilbert transform can be defined if μ, rather than being finite, satisfies $\int (1+|x|)^{-1}\, d\mu < \infty$. Indeed, H_μ can be defined up to an additive constant if $\int (1+|x|^2)^{-1}\, d\mu(x) < \infty$. Theorem 1.1 extends to both these cases.

3. It follows from the arguments in Section 2 that a converse to Theorem 1.1 holds and that $H_\mu \restriction \mathfrak{e} \in L^1_{w;0}$ if and only if $H_{\mu\restriction\mathfrak{e}} \in L^1_{w;0}$. Thus, we have a three-fold equivalence,

$$H_\mu \restriction \mathfrak{e} \in L^1_{w;0} \Leftrightarrow H_{\mu\restriction\mathfrak{e}} \in L^1_{w;0} \Leftrightarrow \mu_{\mathrm{s}}(\mathfrak{e}) = 0. \tag{1.16}$$

There is a special case that is both important and one motivation for this work. Recall from [9] the following

Definition. A finite measure μ on $\mathbb{R}$ is called **reflectionless** on $\mathfrak{e} \subset \mathbb{R}$, where $\mathfrak{e}$ is compact and of strictly positive Lebesgue measure, if and only if $H_\mu \restriction \mathfrak{e} = 0$.

There has been an explosion of recent interest about reflectionless measures due to work of Remling [12]. Clearly, the zero function lies in $L^1_{w;0}$, so we have

Corollary 1.2. *Let $\mathfrak{e}$ be homogeneous, and let μ be a measure on $\mathbb{R}$ which is reflectionless on $\mathfrak{e}$. Then* (1.15) *holds.*

This result is not new. For cases where $\operatorname{supp}(\mu) \subset \mathfrak{e}$, it is due to Sodin–Yuditskii [15], with some extensions due to Gesztesy–Zinchenko [6]. Recently, Poltoratski–Remling [11] have proven a stronger result than Corollary 1.2—instead of requiring that $\mathfrak{e}$ be homogeneous, they only need that for all $x_0 \in \mathfrak{e}$,

$$\limsup_{a\downarrow 0} \frac{|\mathfrak{e}\cap(x_0-a,x_0+a)|}{2a} > 0. \tag{1.17}$$

If (1.17) holds for all $x_0 \in \mathfrak{e}$, we call $\mathfrak{e}$ **weakly homogeneous**, following [11].

The property of being reflectionless is not robust in that changing μ off $\mathfrak{e}$ usually destroys the reflectionless property. As we see in Section 2, having $H_\mu \restriction \mathfrak{e}$ in $L^1_{w;0}$ is robust and explains one reason we sought this result.

Our proof is quite different from [11]. We note, however, that our proof, like the one in [11], is essentially a real variable proof (we go into the complex plane but use no contour integrals), while the earlier work of [15, 6] is a complex variable argument.

We mention that Corollary 1.2 (and so Theorem 1.1) does not hold for arbitrary $\mathfrak{e}$. Nazarov–Volberg–Yuditskii [9] have examples of reflectionless measures on their supports where (1.17) fails and that have a singular component.

We mention another special case of Theorem 1.1.

Corollary 1.3. *Let $\mathfrak{e}$ be a homogeneous set in $\mathbb{R}$. Let μ be a measure on $\mathbb{R}$ such that there is a set A satisfying*

(i) $|A| = 0$,

(ii) $\mu(\mathbb{R}\setminus A) = 0$,

(iii) *A is closed and $A \subset \mathfrak{e}$.*

Suppose $H_\mu \restriction \mathfrak{e} \in L^1_{w;0}$. Then $\mu = 0$.

We need a strengthening of this special case.

Theorem 1.4. *Let $\mathfrak{e}$ be a homogeneous set in $\mathbb{R}$. There is a constant C_1 depending only on $\mathfrak{e}$ such that for any measure μ obeying* (i)–(iii) *of Corollary 1.3, we have*

$$\mu(\mathfrak{e}) \le C_1 \liminf_{t\to\infty} t|\{x\in\mathfrak{e} : |H_\mu(x)| \ge t\}|. \tag{1.18}$$

Remarks. 1. In fact, C_1 is only δ-dependent; explicitly, one can take

$$C_1 = \frac{1536\pi^3}{\delta^2}. \tag{1.19}$$

We have made no attempt to optimize this constant and, indeed, have made choices to simplify the arithmetic. The δ^{-2} may be optimal; it certainly seems that δ^{-1} is not possible.

2. There is also a strengthening of Theorem 1.1 of this same form.

We can say more about weakly homogeneous sets, that is, ones that obey (1.17), and thereby illuminate and limit Theorem 1.1.

Theorem 1.5. *Let $\mathfrak{e}$ be a compact weakly homogeneous set and μ a measure on $\mathbb{R}$ such that $H_\mu \restriction \mathfrak{e} \in L^1_{w;0}$. Then for all $x_0 \in \mathfrak{e}$,*

$$\mu(\{x_0\}) = 0, \tag{1.20}$$

that is, μ has no pure point masses in $\mathfrak{e}$.

Theorem 1.6. *There exists a weakly homogeneous set $\mathfrak{e}$ containing the classical Cantor set, such that if μ is the conventional Cantor measure, $H_\mu \restriction \mathfrak{e} \in L^1_{w;0}$. In particular, Theorem 1.1 does not extend to weakly homogeneous sets.*

While the gap between homogeneous and weakly homogeneous sets is not large, we can extend Theorem 1.1 to partly fill it in. We call a set $\mathfrak{e}$ **non-uniformly homogeneous** if it is closed and

$$\liminf_{a\downarrow 0}\, (2a)^{-1} |\mathfrak{e} \cap (x-a, x+a)| > 0 \tag{1.21}$$

for all $x \in \mathfrak{e}$.

Theorem 1.7. *Let $\mathfrak{e}$ be non-uniformly homogeneous and let μ be a measure on $\mathbb{R}$ such that $H_\mu \restriction \mathfrak{e} \in L^1_{w;0}$. Then*

$$\mu_{\mathrm{s}}(\mathfrak{e}) = 0. \tag{1.22}$$

In fact, we obtains this from a stronger result. We emphasize that in the next theorem, $\mathfrak{e}$ is not assumed closed.

Theorem 1.8. *Let $\mathfrak{e}$ be a Borel set in $\mathbb{R}$ and μ a finite measure such that $H_\mu \restriction \mathfrak{e} \in L^1_{w;0}$. Then*

$$\mu_{\mathrm{s}}\big(\{x \in \mathfrak{e} : \liminf_{a\downarrow 0}\, (2a)^{-1} |\mathfrak{e} \cap (x-a, x+a)| > 0\}\big) = 0. \tag{1.23}$$

This is to be compared with the result of Poltoratski–Remling [11] that if $\mathfrak{e}$ is Borel and $H_\mu \restriction \mathfrak{e} = 0$, then

$$\mu_{\mathrm{s}}\big(\{x \in \mathfrak{e} : \limsup_{a\downarrow 0}\, (2a)^{-1} |\mathfrak{e} \cap (x-a, x+a)| > 0\}\big) = 0, \tag{1.24}$$

and the statement (which follows from our proof of Theorem 1.5) that if μ_{pp} is the pure point part of μ, then if $H_\mu \restriction \mathfrak{e} \in L^1_{w;0}$,

$$\mu_{\mathrm{pp}}\big(\{x \in \mathfrak{e} : \limsup_{a\downarrow 0}\, (2a)^{-1} |\mathfrak{e} \cap (x-a, x+a)| > 0\}\big) = 0.$$

Moreover, we note that the example in Theorem 1.6 shows that in Theorem 1.8, we cannot replace (1.23) by (1.24).

In Section 2, we reduce the proof of Theorem 1.1 to proving Theorem 1.4. In Section 3, we prove Theorem 1.4. In proving Theorem 1.4, we first show that if $[a, b]$ is an interval on which $|F_\mu(x+i0)| \geq t$, then $|F_\mu(x+i(b-a))| \geq t/8\pi^2$. Then we use this to prove that on most of $[a-(b-a), a]$ and $[b, b+(b-a)]$, $|F_\mu(x+i0)|$ is a significant fraction of t, which is the key to the proof. In Section 4, we prove Theorems 1.5 and 1.6. In Section 5, we prove Theorem 1.8, and so Theorem 1.7.

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2 Reduction to Theorem 1.4

In this section, we show that Theorem 1.4 implies Theorem 1.1.

Proposition 2.1. *Let μ have the form* (1.9). *Then for any set $\mathfrak{e} \subset \mathbb{R}$,*

$$H_\mu \restriction \mathfrak{e} \in L^1_{w;0} \Leftrightarrow H_{\mu_s} \restriction \mathfrak{e} \in L^1_{w;0}. \tag{2.1}$$

In particular, we need only prove Theorem 1.1 for purely singular measures to get it for all measures.

Remark. This shows the advantage of working with $L^1_{w;0}$. Purely singular measures are never reflectionless (for $|\{x : F_\mu(x+i0) = 0\}| = 0$ and thus, $\operatorname{Im} F_\mu(x+i0) > 0$ a.e. on $\mathfrak{e}$ if $H_\mu \restriction \mathfrak{e} = 0$).

Proof. By (1.7) with $\theta = \frac{1}{2}$, $L^1_{w;0}$ is a vector space. Since $H_\mu - H_{\mu_s} = H_{f\,d\mu} \in L^1_{w;0}$, by (1.6), we get (2.1) immediately. □

Proposition 2.2. *Let $\mathfrak{e}$ be a closed set. Let μ be a measure with $\mu(\mathfrak{e}) = 0$. Then*

$$H_\mu \restriction \mathfrak{e} \in L^1_{w;0}. \tag{2.2}$$

Proof. Let $\mu_m = \mu \restriction \{x : \operatorname{dist}(x, \mathfrak{e}) \geq m^{-1}\}$. Then for $x \in \mathfrak{e}$,

$$H_{\mu_m}(x) = \frac{1}{\pi} \int \frac{d\mu_m(y)}{y-x}, \tag{2.3}$$

so

$$\|H_{\mu_m} \restriction \mathfrak{e}\|_\infty \leq \frac{m}{\pi} \|\mu_m\|; \tag{2.4}$$

hence $H_{\mu_m} \in L^1_{w;0}$.

By (1.7) with $\theta = \frac{1}{2}$, for any m,

$$\limsup_{t\to\infty} t|\{x \in \mathfrak{e} : |H_\mu(x)| \geq t\}| \leq 2 \limsup_{t\to\infty} t|\{x \in \mathfrak{e} : |H_{\mu-\mu_m}(x)| \geq t\}| \leq 2C\|\mu - \mu_m\|, \tag{2.5}$$

where C is the constant in (1.4).

Since (2.5) holds for all m and $\|\mu - \mu_m\| \to 0$ (since $\mu(\mathfrak{e}) = 0$), we conclude $H_\mu \restriction \mathfrak{e} \in L^1_{w;0}$. □

Proposition 2.3. *Let $\mathfrak{e}$ be a closed set. Let $\nu = \mu \restriction \mathfrak{e}$, that is, $\nu(A) = \mu(\mathfrak{e} \cap A)$. Then*

$$H_\nu \restriction \mathfrak{e} \in L^1_{w;0} \Leftrightarrow H_\mu \restriction \mathfrak{e} \in L^1_{w;0}. \tag{2.6}$$

In particular, it suffices to prove Theorem 1.1 for purely singular measures supported on $\mathfrak{e}$.

Proof. Let $\eta = \mu - \nu$. By Proposition 2.2,

$$H_\mu \restriction \mathfrak{e} - H_\nu \restriction \mathfrak{e} = H_\eta \restriction \mathfrak{e} \in L^1_{w;0}. \tag{2.7}$$

Since $L^1_{w;0}$ is a vector space, (2.7) implies (2.6). □

Proof of Theorem 1.1 given Theorem 1.4. By Proposition 2.3, we can suppose μ is purely singular and supported by $\mathfrak{e}$. Thus, there exists $A_\infty \subset \mathfrak{e}$ with $|A_\infty| = 0$, so $\mu(\mathbb{R} \setminus A_\infty) = 0$.

By regularity of measures, we can find $A_n \subset A_{n+1} \subset \cdots \subset A_\infty$ with each A_n closed, and so

$$\mu(A_\infty \setminus A_n) \to 0. \tag{2.8}$$

Define $\mu_n = \mu \restriction A_n$ and $\nu_n = \mu - \mu_n$. By (1.7) with $\theta = \frac{1}{2}$, $H_\mu \restriction \mathfrak{e} \in L^1_{w;0}$, and (1.4),

$$\limsup_{t\to\infty} t|\{x \in \mathfrak{e} : |H_{\mu_n}(x)| \geq t\}| \leq 2 \limsup_{t\to\infty} t|\{x \in \mathfrak{e} : |H_{\nu_n}(x)| \geq t\}| \leq 2C\mu(A_\infty \setminus A_n). \tag{2.9}$$

Now A_n satisfies (i)–(iii) for μ_n, so by (1.18),

$$\mu(A_n) = \mu_n(\mathfrak{e}) \leq 2CC_1\mu(A_\infty \setminus A_n). \tag{2.10}$$

As $n \to \infty$, $\mu(A_n) \to \mu_s(\mathfrak{e})$ while, by (2.8), $\mu(A_\infty \setminus A_n) \to 0$. So $\mu_s(\mathfrak{e}) = 0$. □

3 Proof of Theorem 1.4

Throughout this section, where we prove Theorem 1.4 and so complete the proof of Theorem 1.1, we suppose $\mathfrak{e}$ is homogeneous with homogeneity constant δ, and μ is a measure for which there exists $A \subset \mathfrak{e}$ satisfying covditions (i)–(iii) of Corollary 1.3. In particular, since μ is singular, for a.e. $x \in \mathbb{R}$,

$$F_\mu(x + i0) = \pi H_\mu(x). \tag{3.1}$$

We consider F_μ throughout.

The key is to prove that for all large t,

$$\left|\left\{x \in \mathfrak{e} : |F_\mu(x + i0)| > \frac{\delta}{128\pi^2} t\right\}\right| \geq \frac{\delta}{24} |\{x : |F_\mu(x + i0)| > t\}|. \tag{3.2}$$

We do this by showing that if I is an interval in $\mathbb{R} \setminus A$ where $|F_\mu(x+i0)| > t$, then at most points of the two touching intervals of the same size, $|F_\mu| \geq \delta t/128\pi^2$. We do this in two steps. We show that at points over I with $\operatorname{Im} z = |I|$, $F(z)$ is comparable to t and use that to control F on the touching intervals. A Vitali covering map argument then boosts that up to the full sets. We need

Proposition 3.1. *Let*

$$I = [c - a, c + a] \tag{3.3}$$

be an interval contained in

$$\{x : |F_\mu(x + i0)| \geq t\}. \tag{3.4}$$

Then

$$|F_\mu(c + a + 2ia)| \geq \frac{t}{8\pi^2}. \tag{3.5}$$

Proof. F_μ lies in weak L^1 and is bounded off a compact subset of $\mathbb{R}$. For $z \in \mathbb{C}_+$, let

$$G(z) = \sqrt{F_\mu(z)/i}. \tag{3.6}$$

Then G has locally L^1 boundary values on $\mathbb{R}$ and is bounded off a compact set, so if $z = x + iy$,

$$G(z) = \frac{1}{\pi} \int \frac{yG(\lambda + i0)\, d\lambda}{(x - \lambda)^2 + y^2}. \tag{3.7}$$

Now $\arg(G) \in [-\frac{\pi}{4}, \frac{\pi}{4}]$, so on $\mathbb{R}$,

$$\operatorname{Re} G(\lambda + i0) \geq 0. \tag{3.8}$$

On I, $\arg(G) = \pm\frac{\pi}{4}$, and so for $\lambda \in I$,

$$\operatorname{Re} G(\lambda + i0) \geq \sqrt{t/2}. \tag{3.9}$$

Thus, by (3.7), (3.8) and (3.9),

$$\begin{aligned} \operatorname{Re} G(c + a + 2ia) &\geq \frac{1}{\pi} \int_I \frac{2a \operatorname{Re} G(\lambda + i0)}{(c + a - \lambda)^2 + (2a)^2}\, d\lambda \\ &\geq \frac{1}{\pi} \frac{(2a)^2 \sqrt{t/2}}{(2a)^2 + (2a)^2} \geq \frac{1}{2\pi} \sqrt{t/2}, \end{aligned} \tag{3.10}$$

so

$$|F_\mu(c + a + 2ia)| \geq (\operatorname{Re} G(c + a + 2ia))^2 \geq \frac{t}{8\pi^2}. \tag{3.11}$$

□

Lemma 3.2. *Fix* $t_0 > 0$ *and let*

$$F_{t_0}(z) = \frac{F(z)}{1 + \frac{1}{t_0} F(z)}. \tag{3.12}$$

Then $\operatorname{Im} F_{t_0} > 0$ *on* $\mathbb{C}_+$, *and*

$$\{x : |F(x + i0)| > t_0\} = \{x : F_{t_0}(x + i0) > t_0/2\}. \tag{3.13}$$

Remark. F_{t_0} is the Stieltjes transform of a measure associated with a rank one perturbation (see, e.g., [14, Sect. 11.2]), but that plays no direct role here.

Proof. The invertible map

$$H(z) = \frac{z}{1 + z/t_0} \tag{3.14}$$

maps $\mathbb{C}_+$ to $\mathbb{C}_+$ and $(t_0, \infty) \cup \{\infty\} \cup (-\infty, -t_0)$ to $(\frac{t_0}{2}, \infty)$. □

For any $x > 0$, define

$$\Gamma_s = \{x : |F(x + i0)| > s\}. \tag{3.15}$$

Proposition 3.3. *Fix* $t > 0$ *and let*

$$t_0 = \frac{\delta}{128\pi^2}\, t. \tag{3.16}$$

Suppose

$$I = [c - a, c + a] \subseteq \Gamma_t, \tag{3.17}$$

and let

$$\tilde{I} = [c + a, c + 3a] \tag{3.18}$$

be the touching interval of the same size as I. Then

$$|\tilde{I} \setminus \Gamma_{t_0}| \leq a\delta = \frac{\delta}{2}\, |I|. \tag{3.19}$$

Proof. By the lemma for x real,

$$\chi_{\Gamma_{t_0}}(x) = 1 - \frac{1}{\pi}\, \arg(F_{t_0}(x + i0) - t_0/2), \tag{3.20}$$

which is the boundary value of a bounded harmonic function.

Let

$$z_0 = c + a + 2ia. \tag{3.21}$$

Then

$$\begin{aligned} \arg\left(F_{t_0}(z_0) - \frac{t_0}{2}\right) &= \arg\left(\frac{F(z_0) - t_0/2 - F(z_0)/2}{1 + \frac{1}{t_0}F(z_0)}\right) \\ &= \arg\left(\frac{F(z_0)/t_0 - 1}{F(z_0)/t_0 + 1}\right) = \arg\left(1 - \frac{2}{F(z_0)/t_0 + 1}\right). \end{aligned} \tag{3.22}$$

By Proposition 3.1,

$$\left|\frac{F(z_0)}{t_0}\right| \geq \frac{t}{8\pi^2 t_0} = \frac{16}{\delta} \geq 16 \tag{3.23}$$

since $\delta \leq 1$. Thus,

$$\left|\frac{2}{F(z_0)/t_0 + 1}\right| \leq \frac{2}{|F(z_0)/t_0| - 1} \leq \frac{2}{15} < 1. \tag{3.24}$$

If $|w| \leq 1$ for $w \in \mathbb{C}$, then

$$\arg(1 + w) \leq \arcsin(|w|) \leq \frac{\pi}{2}\, |w| \tag{3.25}$$

($\sin(y) \geq (2y)/\pi$ for $y \in [0, \pi/2]$ implies for $x \in [0, 1]$, $\arcsin x \leq \frac{\pi}{2}x$). By (3.22),

$$\arg\left(F_{t_0}(z_0) - \frac{t_0}{2}\right) \leq \frac{8\pi^3 t_0}{t - 8\pi^2 t_0}. \tag{3.26}$$

Thus, if $\chi_{\Gamma_{t_0}}(z)$ is the harmonic function whose boundary value is $\chi_{\Gamma_{t_0}}(x)$, we find, by (3.20), that

$$\pi(1-\chi_{\Gamma_{t_0}}(z_0)) \le \frac{8\pi^3 t_0}{t-8\pi^2 t_0}. \tag{3.27}$$

By a Poisson formula with $z_0 = x_0 + iy_0$ as in (3.21),

$$\pi(1-\chi_{\Gamma_{t_0}}(z_0)) = \int_{\mathbb{R}\setminus\Gamma_{t_0}} \frac{y_0\,d\lambda}{(\lambda-x_0)^2+y_0^2} \tag{3.28}$$

$$\ge \frac{1}{2}\frac{|\tilde{I}\setminus\Gamma_{t_0}|}{|I|}, \tag{3.29}$$

since on $\tilde{I}$, the minimum of $y_0/((\lambda-x_0)^2+y_0^2)$ is $1/(2|I|)$.

Thus, by (3.27) and (3.29),

$$|\tilde{I}\setminus\Gamma_{t_0}| \le \frac{16\pi^3 t_0}{t-8\pi^2 t_0}\,|I|. \tag{3.30}$$

Since $\frac{8\pi^2 t_0}{t} \le \frac{1}{16}$,

$$\frac{\frac{16\pi^3 t_0}{t}}{1-\frac{8\pi^2 t_0}{t}} \le \frac{256\pi^3 t_0}{15\,t} = \frac{4\pi}{15}\,\frac{\delta}{2} \le \frac{\delta}{2},$$

and (3.30) implies (3.19). □

Proposition 3.4. *Under the notation of Proposition 3.3, let*

$$I^\sharp = [c-3a, c+3a] \tag{3.31}$$

and suppose

$$\mathfrak{e}\cap I \ne \emptyset \tag{3.32}$$

and

$$a \le \operatorname{diam}(\mathfrak{e}). \tag{3.33}$$

Then

$$|\Gamma_{t_0}\cap\mathfrak{e}\cap I^\sharp| \ge \frac{\delta}{2}\,|I|. \tag{3.34}$$

Proof. Pick $x_0 \in \mathfrak{e}\cap I$. Suppose $x_0 \ge c$. If not, we pick $\tilde{I}$ to be the third of $I^\sharp$ below I instead of the choice here. By homogeneity,

$$|\mathfrak{e}\cap(x_0-a, x_0+a)| \ge 2a\delta = \delta|I| \tag{3.35}$$

and the intersection lies in $I \cup \tilde{I}$. Thus,

$$|\Gamma_{t_0} \cap \mathfrak{e} \cap I^\sharp| \geq |\mathfrak{e} \cap (x_0 - a, x_0 + a)| - |(I \cup \tilde{I}) \setminus \Gamma_{t_0}|. \tag{3.36}$$

Since $I \subset \Gamma_t \subset \Gamma_{t_0}$,

$$|(I \cup \tilde{I}) \setminus \Gamma_{t_0}| = |\tilde{I} \setminus \Gamma_{t_0}| \leq \frac{\delta}{2}\, |I| \tag{3.37}$$

by (3.19); (3.35) and (3.36) imply (3.34). □

Proof of Theorem 1.4. Suppose $\mu \neq 0$. On $\mathbb{R} \setminus A$, $F_\mu(x + i0)$ is continuous and real, so $\{x : |F_\mu(x+i0)| > t\}$ is open, and hence a countable union of maximal disjoint open intervals.

Let $I = [c - a, c + a]$ be the closure of any such interval. On $\mathbb{R} \setminus A$, $F_\mu(x)$ has

$$F'_\mu(x) = \int \frac{d\mu(x)}{(y - x)^2} > 0. \tag{3.38}$$

If $F_\mu > t$ on I, $c + a$ must be in A or else $F_\mu(c + a) < \infty$ and $F_\mu(c + a + \varepsilon) \in \Gamma_t$ for ε small (so I is not maximal). Similarly, if $F_\mu < -t$ on I, $c - a \in A$. Thus, $I \cap A \neq \emptyset$, so $I \cap \mathfrak{e} \neq \emptyset$.

Let

$$T = \frac{\pi C \|\mu\|}{\operatorname{diam}(\mathfrak{e})}, \tag{3.39}$$

where C is the constant in (1.4). Then for $t > T$, $|\Gamma_t| \leq \operatorname{diam}(\mathfrak{e})$, so $a \leq \operatorname{diam}(\mathfrak{e})$. Thus, by Proposition 3.4,

$$|\Gamma_{t_0} \cap \mathfrak{e} \cap I^\sharp| \geq \frac{\delta}{2}\, |I|. \tag{3.40}$$

Clearly, the I's and so the $(I^\sharp)^{\text{int}}$'s are an open cover of $\Gamma_t \setminus A$. Thus, by the Vitali covering theorem (see Rudin [13, Lemma 7.3]), we can find a subset of mutually disjoint $I^\sharp$'s (call them $\{I_j^\sharp\}$) such that

$$|\Gamma_t| \leq 4 \sum_j |I_j^\sharp| \leq 12 \sum_j |I_j|. \tag{3.41}$$

By the disjointness, with t_0 given by (3.16),

$$\begin{aligned} |\Gamma_{t_0} \cap \mathfrak{e}| &\geq \sum_j |I_j^\sharp \cap \Gamma_{t_0} \cap \mathfrak{e}| \\ &\geq \frac{\delta}{2} \sum_j |I_j| \qquad \text{(by (3.34))} \end{aligned}$$

$$\geq \frac{\delta}{24}\,|\Gamma_t| \qquad\qquad \text{(by (3.41))}.$$

Thus,

$$\liminf_{t\to\infty} t_0|\Gamma_{t_0}\cap \mathfrak{e}| \geq \liminf_{t\to\infty} \frac{\delta}{24}\,\frac{\delta}{128\pi^2}\,t|\Gamma_t|.$$

Therefore, by (1.8) and (3.1),

$$\liminf_{t\to\infty} t|\{x\in\mathfrak{e} : |H_\mu(x)| > t\}| \geq \frac{\delta^2}{3072\pi^2}\,\frac{2(\mu(A))}{\pi},$$

which is (1.18)/(1.19). □

4 Weakly homogeneous sets

Proof of Theorem 1.5. For $x_0\in\mathfrak{e}$ and $\varepsilon > 0$, write

$$\mu = \mu_1 + \mu_2 + \mu_3 \tag{4.1}$$

with $\mu_1 = \mu \restriction \{x_0\}$, $\mu_2 = \mu \restriction [(x_0-\varepsilon, x_0+\varepsilon)\setminus\{x_0\}]$, $\mu_3 = \mu\restriction \mathbb{R}\setminus(x_0-\varepsilon, x_0+\varepsilon)$; and by (1.7), note that

$$\begin{aligned} &|\{x\in\mathfrak{e};\ |x-x_0| < \tfrac{\varepsilon}{2} : |H_{\mu_1}(x)| > 3t\}| \\ &\qquad\leq |\{x\in\mathfrak{e} : |H_\mu(x)| > t\}| + |\{x : |H_{\mu_2}(x)| > t\}| \\ &\qquad\quad + |\{x;\ |x-x_0| < \tfrac{\varepsilon}{2} : |H_{\mu_3}(x)| > t\}|. \end{aligned} \tag{4.2}$$

By hypothesis, the first term on the right of (4.2) is $o(1/t)$. Since $|H_{\mu_3}(x)| \leq 2/\varepsilon$, the third term is $o(1/t)$. By (1.4), the second term is bounded by $C\mu((x_0-\varepsilon, x_0+\varepsilon)\setminus\{x_0\})/t$.

So long as $t > \frac{2\mu(\{x_0\})}{3\pi\varepsilon}$, the left side of (4.2) is $|\mathfrak{e}\cap(x_0 - \frac{2\mu(\{x_0\})}{3\pi t}, x_0 + \frac{2\mu(\{x_0\})}{3\pi t})|$. Thus, if

$$C(x_0) = \limsup_{s\downarrow 0}\,(2s)^{-1}|\mathfrak{e}\cap(x_0-s, x_0+s)|, \tag{4.3}$$

(4.2) implies that

$$\frac{4C(x_0)\mu(\{x_0\})}{3\pi} \leq C\mu((x_0-\varepsilon, x_0+\varepsilon)\setminus\{x_0\}) \tag{4.4}$$

for any ε. Since $\cap[(x_0 - \frac{1}{m}, x_0 + \frac{1}{m})\setminus\{x_0\}] = \emptyset$, the right side of (4.4) goes to zero as $\varepsilon\downarrow 0$, and we conclude that $\mu(\{x_0\}) = 0$. □

To prove Theorem 1.6, we need to describe some sets connected with the Cantor set. Let K_1 be the two connected closed sets $K_{1,1}, K_{1,2}$ obtained from [0, 1] by removing the middle third. At level n, we have 2^n intervals $\{K_{n,j}\}_{j=1}^{2^n}$, each with $|K_{n,j}| = 3^{-n}$ so $|K_n| = (\frac{2}{3})^n$. The Cantor set, of course, is $K_\infty = \cap K_n$. The Cantor measure is determined by

$$\mu(K_{n,j}) = \frac{1}{2^n}. \tag{4.5}$$

We order $\mathfrak{I} = \{(n, j) : n = 1, 2, \dots, j = 1, 2, \dots, 2^n\}$ with lexigraphic order and use $(n, j + 1)$ for the obvious pair if $j < 2^n$ and to be $(n + 1, 1)$ if $j = 2^n$. Similarly, $(n, j - 1)$ is $(n - 1, 2^{n-1})$ if $j = 1$.

Let E_1 be the middle closed third of $[0, 1] \setminus K_1$, so $|E_1| = 1/9$. Let E_2 be the two middle thirds of the two gaps in $K_1 \setminus K_2$; E_m has 2^{m-1} closed intervals of size $1/3^{m+1}$. There is a unique affine order preserving map of [0, 1] to $K_{n,j}$. Let $E_{n,j,m}$ be the image of E_m under this map, so $E_{n,j,m}$ has 2^{m-1} intervals, each of size $1/3^{n+m+1}$, that is,

$$|E_{n,j,m}| = 2^{m-1}/3^{n+m+1}. \tag{4.6}$$

We want to pick a positive integer $m(n, j)$ for each $(n, j) \in \mathfrak{I}$ so that

$$m(n, j + 1) > m(n, j), \tag{4.7}$$

and we define

$$k(n, j) = n + m(n, j). \tag{4.8}$$

Given such a choice, we define

$$\mathfrak{e} = K_\infty \cup \bigcup_{n,j \in \mathfrak{I}} E_{n,j,m(n,j)}. \tag{4.9}$$

Our goal will be to prove $\mathfrak{e}$ is always weakly homogeneous, and that if $m(n, j)$ grows fast enough, then $H_\mu \restriction \mathfrak{e}$ is in $L^1_{w;0}$.

Lemma 4.1. *For any choice of $m(n, j)$, $\mathfrak{e}$ is weakly homogeneous. Indeed, for any $x_0 \in \mathfrak{e}$,*

$$\limsup_{\delta \downarrow 0} (2\delta)^{-1} |\mathfrak{e} \cap (x_0 - \delta, x_0 + \delta)| \geq 1/10. \tag{4.10}$$

Proof. Let

$$\tilde{E}_{n,j} = E_{n,j,m(n,j)}. \tag{4.11}$$

If $x_0 \in \tilde{E}_{n,j}$, which is a closed interval, for all small δ, $(2\delta)^{-1}|\tilde{E}_{n,j} \cap (x_0-\delta, x_0+\delta)| = \frac{1}{2}$ or 1, depending on whether x_0 is a boundary or an interior point. So (4.10) is certainly true.

Thus, we need only consider $x_0 \in K_\infty$. Fix $x_0 \in K_\infty$. For each n, $x_0 \in K_n$, and so in K_{n,j_n} for some j_n. Let $k_n \equiv k(n, j_n)$. On level k_n, x_0 is contained in some interval $K_{k_n,\ell}$ of size 3^{-k_n}, and on one side or the other there is an interval of size 3^{-k_n-1} in $\tilde{E}_{n,j_n}$ in a touching gap. Let

$$\delta_n = \frac{5}{3}\, 3^{-k_n}. \tag{4.12}$$

Then $(x_0 - \delta_n, x_0 + \delta_n)$ contains this interval in $\tilde{E}_{n,j_n}$. Thus,

$$(2\delta_n)^{-1}|\mathfrak{e} \cap (x_0 - \delta_n, x_0 + \delta_n)| \geq \frac{3^{-k_n-1}}{2\delta_n} = \frac{1}{10}. \tag{4.13}$$

Since $\delta_n \to 0$ as $n \to \infty$, (4.10) holds. □

For each (n, j), we wish to define

$$\mu_{n,j} = \mu \restriction K_{n,j} \cup K_{n,j-1}, \quad \tilde{\mu}_{n,j} = \mu - \mu_{n,j}, \tag{4.14}$$

that is, single out the part of the Cantor measure near $K_{n,j}$, and so near $E_{n,j}$. We define

$$F_{n,j} = F_{\mu_{n,j}}, \quad \tilde{F}_{n,j} = F_{\tilde{\mu}_{n,j}}. \tag{4.15}$$

Lemma 4.2. *On* $\bigcup_{(\tilde{n},\tilde{j})\leq(n,j)} \tilde{E}_{\tilde{n},\tilde{j}}$, *we have*

$$|\tilde{F}_{n,j}| \leq 3^{k(n,j-1)}. \tag{4.16}$$

Proof. Since $\|\tilde{\mu}_{n,j}\| \leq 1$, we have

$$|\tilde{F}_{n,j}(x)| \leq \text{dist}(x, K_\infty \setminus K_{n,j-1} \cup K_{n,j}))^{-1}. \tag{4.17}$$

By construction,

$$\text{dist}(\tilde{E}_{\tilde{n},\tilde{j}}, K_\infty) = 3^{-k(\tilde{n},\tilde{j})-1}, \tag{4.18}$$

so if $(\tilde{n}, \tilde{j}) < (n, j-1)$, then for $x \in \tilde{E}_{\tilde{n},\tilde{j}}$,

$$|\tilde{F}_{n,j}(x)| \leq 3^{k(\tilde{n},\tilde{j})+1} \leq 3^{k(n,j-1)}, \tag{4.19}$$

since $m(\tilde{n}, \tilde{j}) + 1 < m(n, j-1)$ implies $k(\tilde{n}, \tilde{j}) + 1 \leq k(n, j-1)$.

On the other hand, since we have removed $K_{n,j-1} \cup K_{n,j}$,

$$\text{dist}(\tilde{E}_{n,j} \cup \tilde{E}_{n,j-1}, K_\infty \setminus (K_{n,j} \cup K_{n,j-1})) \geq 3^{-n}. \tag{4.20}$$

Thus, for x in $\tilde{E}_{n,j} \cup \tilde{E}_{n,j-1}$, we have

$$|\tilde{F}_{n,j}(x)| \leq 3^n \leq 3^{k(n,j-1)}, \tag{4.21}$$

proving (4.16) on the claimed set. □

Proof of Theorem 1.6. We construct $\mathfrak{e}$ by using the above construction where $m(n, j)$ is picked inductively, so that

$$k(n, j+1) = 3k(n, j). \tag{4.22}$$

By Lemma 4.1, $\mathfrak{e}$ is weakly homogeneous.

Let

$$3^{k(n,j-1)} < t \leq 3^{k(n,j)}. \tag{4.23}$$

Since $F_\mu = F_{n,j} + \tilde{F}_{n,j}$, by (1.7),

$$\begin{aligned} 2t|\{x \in \mathfrak{e} : |F_\mu(x)| \geq 2t\}| \leq{} & 2t|\{x : |F_{n,j}(x)| \geq t\}| \\ & + 2t|\{x \in \mathfrak{e} : |\tilde{F}_{n,j}(x)| \geq t\}|. \end{aligned} \tag{4.24}$$

By Boole's equality (1.5), the first term on the right side of (4.24) is bounded by

$$4(\mu_{n,j-1}(\mathbb{R}) + \mu_{n,j}(\mathbb{R})) \leq 4[2^{-n} + 2 \cdot 2^{-n}] = 12 \cdot 2^{-n} \tag{4.25}$$

(where we need the $2 \cdot 2^{-n}$ if $j = 1$).

By Lemma 4.2, the second term is bounded by

$$2 \cdot 3^{k(n,j)} \sum_{(\tilde{n},\tilde{j}) \geq (n,j+1)} |E_{\tilde{n},\tilde{j}}|. \tag{4.26}$$

By (4.6),

$$|\tilde{E}_{n,j}| = \frac{1}{2^{n+1}\,3} \left(\frac{2}{3}\right)^{k(n,j)}, \tag{4.27}$$

therefore, since

$$\sum_{\ell=\ell_0}^{\infty} \left(\frac{2}{3}\right)^{\ell} = 3\left(\frac{2}{3}\right)^{\ell_0}, \tag{4.28}$$

$$(4.26) \leq 3^{k(n,j)}\, 2^{-n} \left(\frac{2}{3}\right)^{k(n,j+1)}. \tag{4.29}$$

By (4.22) and $(\frac{3}{2})^3 = \frac{27}{8} > 3$, we see that

$$(4.26) \leq 2^{-n}. \tag{4.30}$$

Thus, if t obeys (4.23), then by (4.24), (4.25), and (4.30),

$$2t|\{x \in \mathfrak{e} : |F_\mu(x)| \geq t\}| \leq 13 \cdot 2^{-n}. \tag{4.31}$$

Since $n \to \infty$ as $t \to \infty$, we see that $F_\mu \restriction \mathfrak{e} \in L^1_{w;0}$. □

5 Non-uniformly homogeneous sets

Our goal in this section is to prove Theorem 1.8 and then also Theorem 1.7. For any Borel set $\mathfrak{e}$, define

$$\mathfrak{e}_n = \left\{ x \in \mathfrak{e} : \forall\, a < \frac{1}{n},\ |(x-a, x+a) \cap \mathfrak{e}| \geq \frac{2a}{n} \right\}. \tag{5.1}$$

Proposition 5.1. *Let μ be a measure with $\mu(\mathbb{R} \setminus \mathfrak{e}_n) = 0$. Suppose $H_\mu \restriction \mathfrak{e} \in L^1_{w;0}$. Then $\mu_{\mathrm{s}} = 0$.*

Proof. First note that $\mathfrak{e}_n$ is closed, for if $x_m \to x$ and $|(x_m - a, x_m + a) \cap \mathfrak{e}| \geq \frac{2a}{n}$, then for all m,

$$|(x-a, x+a) \cap \mathfrak{e}| \geq \frac{2a}{n} - 2|x - x_m|, \tag{5.2}$$

so $x \in \mathfrak{e}_n$. Applying Theorem 1.1 to $d\mu$ and compact homogeneous sets $\mathfrak{e}_n \cap [-N, N]$ for all $N \geq 1$, we get the result. □

Because $\mathfrak{e}$ is not closed, we cannot use Propositions 2.2 and 2.3 to restrict to $\mathfrak{e}_m$. Instead we need

Proposition 5.2. *Let μ and ν be two measures on $\mathbb{R}$ whose singular parts are mutually singular. Then for all $c > 0$,*

$$t|\{x : |H_\mu(x)| \geq t\} \cap \{x : |H_\nu(x)| \geq ct\}| \to 0 \tag{5.3}$$

as $t \to \infty$.

Remark. This result is essentially in Poltoratski [10] (see the last set out formula in the proof of Theorem 2 in that paper), so we only sketch the proof.

Sketch. Suppose first that $c = 1$. We begin with what is essentially Theorem 1 of [10], that for any positive measure μ, as $t \to \infty$,

$$\tfrac{1}{2}\pi t \chi_{\{x : |H_\mu(x)| \geq t\}}\, dx \xrightarrow{w} d\mu_{\mathrm{s}} \tag{5.4}$$

in the weak-$*$ topology. By (1.6) and (1.7), it suffices to prove this for $\mu = \mu_{\mathrm{s}}$. In that case, if $\mu^{(\alpha)}$ is the measure with Stieltjes transform,

$$F_\alpha(z) = \frac{F(z)}{1+\alpha F(z)}, \tag{5.5}$$

then ([5, 10])

$$\int_{-(\pi t)^{-1}}^{(\pi t)^{-1}} (d\mu_\alpha(x))\, d\alpha = \chi_{\{x:|H_\mu(x)|\geq t\}}\, dx,$$

so (5.4) follows from $d\mu_\alpha \overset{w}{\to} d\mu$ as $|\alpha| \to 0$.

By (1.8), if $\mu^{(t)}$ is the measure on the left side of (5.4), then

$$\|\mu^{(t)}\| \to \|\mu_{\mathrm{s}}\|. \tag{5.6}$$

By (5.4),

$$\mu^{(t)} - \nu^{(t)} \overset{w}{\longrightarrow} \mu_{\mathrm{s}} - \nu_{\mathrm{s}}, \tag{5.7}$$

so

$$\liminf \|\mu^{(t)} - \nu^{(t)}\| \geq \|\mu_{\mathrm{s}} - \nu_{\mathrm{s}}\| = \|\mu_s\| + \|\nu_s\| \tag{5.8}$$

by the assumed mutual singularity.

But

$$\|\mu^{(t)} - \nu^{(t)}\| = \|\mu^{(t)}\| + \|\nu^{(t)}\| - \pi(\text{lhs of (5.3)}); \tag{5.9}$$

(5.6) and (5.8) then imply (5.3) for $c = 1$.

This implies the result for $c \geq 1$ and then, by symmetry, for all $c > 0$. □

Proof of Theorem 1.8. For each n, define

$$\mu_n = \mu \restriction \mathfrak{e}_n, \quad \nu_n = \mu - \mu_n. \tag{5.10}$$

By (1.7),

$$\begin{aligned} |\{x \in \mathfrak{e} : |H_{\mu_n}(x)| \geq 2t\}| \leq{}& |\{x \in \mathfrak{e} : |H_\mu(x)| \geq t\}| \\ &+ |\{x : |H_{\mu_n}(x)| \geq 2t,\ |H_{\nu_n}(x)| \geq t\}|. \end{aligned} \tag{5.11}$$

By the hypothesis, the first term on the right is $o(1/t)$ and, by Proposition 5.2, the second is $o(1/t)$. Thus, $H_{\mu_n} \restriction \mathfrak{e} \in L^1_{w;0}$, and it follows from Proposition 5.1 that $(\mu_n)_{\mathrm{s}} = 0$, that is, $\mu_{\mathrm{s}}(\mathfrak{e}_n) = 0$.

Since

$$\bigcup_n \mathfrak{e}_n = \big\{x \in \mathfrak{e} : |\liminf_{a\downarrow 0} (2a)^{-1} |\mathfrak{e} \cap (x-a, x+a)| > 0\big\},$$

we have (1.23). □

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