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Asymptotics of the L^2 norm of derivatives of OPUC

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Communicated by Leonid Golinskii

We dedicate this paper in fond memory of Franz Peherstorfer from whom we learned so much

Abstract

We show that for many families of OPUC, one has $\|\varphi_n'\|_2/n \to 1$, a condition we call normal behavior. We prove that this implies $|\alpha_n| \to 0$ and that it holds if $\sum_{n=0}^{\infty} |\alpha_n| < \infty$. We also prove it is true for many sparse sequences. On the other hand, it is often destroyed by the insertion of a mass point.

Keywords: Orthogonal polynomials; Derivative asymptotics

1. Introduction

While there is a considerable literature on asymptotics of orthogonal polynomials (see [10,11,25,26,30,33]) including recent works, issues of behavior of derivatives are much less studied (but see [6,7,12,13,15,18,21,22,24,36]). In many of these papers, higher derivatives automatically obey analogs of the first derivative result. That is not clear in our context. Here, we will focus on one question about orthogonal polynomials on the unit circle (OPUC). Let Φ_n, φ_n be the monic and normalized orthogonal polynomials for a nontrivial probability measure $d\mu$ on $\partial\mathbb{D} = \{z \in \mathbb{C} \mid |z| = 1\}$ and $\{\alpha_n\}_{n=0}^{\infty}$ its Verblunsky coefficients — here and below, we follow the notation of [25,26]. As usual, if P_n is a polynomial of degree n, P_n^* is the reflected polynomial

$$P_n^*(z) = z^n \overline{P_n(1/\bar{z})}. \tag{1.1}$$

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The key notion we study in this paper is:

Definition. Let μ be a nontrivial probability measure on $\partial\mathbb{D}$. We say μ has *normal* L^2-derivative behavior (is *normal*, for short) if and only if

$$\left\| \frac{\varphi_n'}{n} \right\| \equiv \left(\int \frac{|\varphi_n'(e^{i\theta})|^2}{n^2} \, d\mu(\theta) \right)^{1/2} \to 1 \tag{1.2}$$

as $n \to \infty$. $\|\cdot\|$ will always be used for the $L^2(\partial\mathbb{D}, d\mu)$ norm.

We note at the start that

Proposition 1.1. *One always has*

$$\left\| \frac{\varphi_n'}{n} \right\|^2 = 1 + \left\| \frac{(\varphi_n^*)'}{n} \right\|^2. \tag{1.3}$$

In particular, normality is equivalent to

$$\lim_{n\to\infty} \left\| \frac{(\varphi_n^*)'}{n} \right\|^2 = 0 \tag{1.4}$$

and it is always true that

$$\left\| \frac{\varphi_n'}{n} \right\| \geq 1. \tag{1.5}$$

Remarks. 1. This relation on L^2 norms should be compared with the opposite bound on $L^\infty(\partial\mathbb{D})$, which is Bernstein's inequality (discussed further in Section 2),

$$\left\| \frac{P_n'}{n} \right\|_\infty \leq \|P_n\|_\infty \tag{1.6}$$

for any polynomial of degree n.

2. By (1.10), we also have

$$\left\| \frac{(\varphi_n^*)'}{n} \right\| = \left\| z\frac{\varphi_n'}{n} - \varphi_n \right\|. \tag{1.7}$$

Proof. Let P_n be a general degree n polynomial

$$P_n(z) = \sum_{j=0}^{n} c_j z^j. \tag{1.8}$$

We claim that

$$nP_n(z) = zP_n'(z) + [(P_n^*)']^*(z) \tag{1.9}$$

where the outer * on the last term is the one suitable for degree $n-1$ polynomials.

Accepting (1.9) for the moment, we apply it to φ_n to get

$$z\varphi_n' = n\varphi_n - [(\varphi_n^*)']^*. \tag{1.10}$$

Since the last term is of degree $n-1$, it is orthogonal to φ_n, so

$$\|z\varphi_n'\|^2 = \|n\varphi_n\|^2 + \|[(\varphi_n^*)']^*\|^2. \tag{1.11}$$

Since multiplication by z and * on degree $n-1$ polynomials preserve norms, and since $\|\varphi_n\| = 1$, (1.11) says

$$\|\varphi_n'\|^2 = n^2 + \|(\varphi_n^*)'\|^2 \tag{1.12}$$

which is (1.3).

To prove (1.9), we note that

$$zP_n'(z) = \sum_{j=0}^{n} jc_j z^j \tag{1.13}$$

while

$$P_n^* = \sum_{j=0}^{n} \bar{c}_j z^{n-j} \tag{1.14}$$

so

$$(P_n^*)' = \sum_{j=0}^{n} (n-j)\bar{c}_j z^{(n-1)-j} \tag{1.15}$$

which applying the * for degree $n-1$ polynomials becomes

$$((P_n^*)')^* = \sum_{j=0}^{n} (n-j)c_j z^j. \tag{1.16}$$

(1.13) plus (1.16) imply (1.9) (which also follows by suitable manipulation of $\varphi_n^*(z) = z^n\overline{\varphi_n(1/\bar{z})}$). □

This result shows the naturalness of the normality condition.

One motivation for our study comes from the theory of Sobolev polynomials [2,3]. Recall that, given a measure $d\mu$, one fixes $\lambda > 0$ and considers the Sobolev inner products

$$\langle f, g\rangle_{S,n} = \int \overline{f(e^{i\theta})} g(e^{i\theta})\, d\mu(\theta) + \frac{\lambda}{n^2} \int \overline{f'(e^{i\theta})} g'(e^{i\theta})\, d\mu(\theta) \tag{1.17}$$

with $' = d/dz$ on polynomials. One defines

$$\sigma_n = \min\{\|P\|_{S,n} \mid P(z) = z^n + \cdots\}$$

and S_n is the unique minimizer. Clearly, by the minimum properties of Φ_n and S_n,

$$\|\Phi_n\|^2 + \lambda\|\Phi_{n-1}\|^2 \le \sigma_n^2 \le \|\Phi_n\|_{S,n}^2. \tag{1.18}$$

Proposition 1.2. *Suppose that*

(a) *μ has normal derivative behavior.*
(b) *μ is in the Szegő class.*

Then

(i)

$$\lim_{n\to\infty} \frac{\sigma_n^2}{\|\Phi_n\|^2} = 1 + \lambda. \tag{1.19}$$

(ii)

$$\lim_{n\to\infty} \|S_n - \Phi_n\|_{S,n}^2 = 0. \tag{1.20}$$

(iii) *On compact subsets of* $\mathbb{C} \setminus \overline{\mathbb{D}}$,

$$\frac{S_n}{\Phi_n} \to 1 \tag{1.21}$$

uniformly.

Proof. (i) Since μ is in the Szegő class, $\|\Phi_{n-1}\|/\|\Phi_n\| \to 1$. Moreover, normal derivative behavior implies

$$\frac{\|\Phi_n'\|}{n\|\Phi_n\|} = \frac{\|\varphi_n'\|}{n\|\varphi_n\|} \to 1 \tag{1.22}$$

so (1.18) says

$$1 + \lambda \le \liminf \frac{\sigma_n^2}{\|\Phi_n\|^2} \le \limsup \frac{\sigma_n^2}{\|\Phi_n\|^2} \le 1 + \lambda$$

proving (i).

(ii) Since S_n minimizes $\|\cdot\|_{S,n}$, in $\langle , \rangle_{S,n}$ inner product, $S_n \perp \Phi_n - S_n$, so

$$\|\Phi_n\|_{S,n}^2 = \|S_n\|_{S,n}^2 + \|S_n - \Phi_n\|_{S,n}^2. \tag{1.23}$$

By (1.22),

$$\frac{\|\Phi_n\|_{S,n}^2}{\|\Phi_n\|^2} \to 1 + \lambda \tag{1.24}$$

so, by (1.19),

$$\frac{\|\Phi_n\|_{S,n}^2 - \|S_n\|_{S,n}^2}{\|\Phi_n\|^2} \to 0. \tag{1.25}$$

Since the Szegő condition implies $\|\Phi_n\|^2$ has a nonzero limit, we get (1.20) from (1.23).

(iii) (1.20) implies $\|S_n - \Phi_n\|^2 \to 0$. Thus, $\|S_n^* - D^{-1}\|^2 \to 0$ (where D is the Szegő function), so $DS_n^* \to 1$ in $H^2\left(\frac{d\theta}{2\pi}\right)$, and so uniformly on compact subsets of $\mathbb{D}$, $S_n^* \to D^{-1}$. Since $\Phi_n^* \to D^{-1}$, we get $S_n^*/\Phi_n^* \to 1$, which implies (1.21) uniformly on compact subsets of $\mathbb{C} \setminus \overline{\mathbb{D}}$. □

Remark. Our proof of (1.19) relied only on normal derivatives and $|\alpha_n| \to 0$, as does (1.25).

While this was an initial motivation, we will study normality for its own sake and not mention this motivation again. Here is a summary of the remainder of this paper. In Section 2, we recall some relevant background and state some general results. In Sections 3–6, we relate normality to asymptotics of Verblunsky coefficients and of the a.c. weight. Section 3 provides a necessary condition by proving that normality implies $\alpha_n \to 0$. Sufficient conditions appear in Sections 4–6. Section 4 shows

$$\sum_{n=0}^{\infty} |\alpha_n| < \infty \tag{1.26}$$

implies normality. Section 5 proves if $d\mu = w\frac{d\theta}{2\pi}$ (i.e., $d\mu_s = 0$), w obeys a Szegő condition, and for a nonzero constant,

$$w(\theta) \leq r \tag{1.27}$$

then one has normality. This result, of course, shows that (1.26) implies normality, but in Section 4, we will prove much more than L^2 convergence of $(\varphi_n^*)'/n$ to zero.

Sections 6–8 provide illuminating examples. In particular, Section 6 discusses some examples with sparse Verblunsky coefficients and provides examples of normal derivative behavior where the corresponding measure is purely singular continuous, and so, non-Szegő. Sections 7 and 8 provide many examples where inserting a mass point destroys normality and one where it does not. In particular, they show that the Szegő class is not a subclass of the normal measures either. Section 8 analyzes a "canonical" weight with algebraic singularities on the circle. This analysis is extended further in Sections 9 and 10, even when the weight is unbounded. Section 11 explores $\|(\varphi_n^*)'/n\|_2$ when $d\mu$ has an isolated mass point — we will show it diverges exponentially!

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2. Generalities

In this section, we begin with a brief discussion regarding some well-known facts about derivatives of orthogonal polynomials that illuminate the issues central to this paper and then discuss two equivalent conditions for normality.

As already noted, Bernstein [8] has an $L^\infty(\partial\mathbb{D})$ inequality in the opposite direction of our inequality L^2 in (1.5) (but our L^2 inequality is only for φ_n; Bernstein's is for all polynomials).

Theorem 2.1 (*Bernstein's Inequality*). *For any polynomials, P_n, of degree n, we have for all* $e^{i\theta} \in \partial\mathbb{D}$

$$|P_n'(e^{i\theta})| \leq n \sup_{z\in\partial\mathbb{D}} |P_n(z)|. \tag{2.1}$$

Remarks. 1. $P_n(z) = z^n$ provides an example with equality.
2. Szegő has a proof of a few lines, found, for example, in [25,33].

We can say more if we know something about the zeros of P_n. The following has been called Lucas's theorem, the Gauss–Lucas theorem, and Grace's theorem:

Theorem 2.2. *The zeros of P_n' lie in the convex hull of the zeros of P_n and — unless the zeros of P_n lie in a line — all zeros of P_n' not at degenerate zeros of P_n lie in the interior of that convex hull.*

Theorem 2.3 (*Turán's Inequality [35]; See also [6]*). *Let P_n have degree n with all zeros in $\overline{\mathbb{D}}$. Then for all* $e^{i\theta} \in \partial\mathbb{D}$,

$$|P_n'(e^{i\theta})| \geq \frac{n}{2}|P_n(e^{i\theta})|. \tag{2.2}$$

Proofs. The proofs are closely related and rely on the fact that if P_n has zeros at $\{z_j\}_{j=1}^n$, then for $z \notin \{z_j\}_{j=1}^n$,

$$\frac{P_n'(z)}{P_n(z)} = \sum_{j=1}^n \frac{1}{z - z_j}. \tag{2.3}$$

Suppose first that all zeros of P_n lie in $\{w \mid \operatorname{Re} w \le 0\}$ and $\operatorname{Re} z_0 \ge 0$ with $z_0 \notin \{z_j\}_{j=1}^n$. Then, by (2.3),

$$\operatorname{Re}\left[\frac{P_n'(z_0)}{P_n(z_0)}\right] = \sum_j \frac{(\operatorname{Re} z_0 - \operatorname{Re} z_j)}{|z_0 - z_j|^2}. \tag{2.4}$$

This is strictly positive if either $\operatorname{Re} z_0 > 0$ or at least one $\operatorname{Re} z_j < 0$. This shows the zeros of P' not among the $\{z_j\}_{j=1}^n$ lie in $\{\operatorname{Re} w \le 0\}$ and in $\{\operatorname{Re} w < 0\}$ if some z_j has $\operatorname{Re} z_j < 0$. This plus Euclidean motions imply Theorem 2.2.

As for Theorem 2.3, we note that if $|w| < 1$, then

$$\operatorname{Re}\left(\frac{1}{1-w}\right) = \frac{1 - \operatorname{Re} w}{1 + |w|^2 - 2\operatorname{Re} w} \ge \frac{1 - \operatorname{Re} w}{2 - 2\operatorname{Re} w} = \frac{1}{2}. \tag{2.5}$$

Thus, by (2.3), if all $z_j \in \overline{\mathbb{D}}$,

$$\operatorname{Re} \frac{e^{i\theta} P_n'(e^{i\theta})}{P_n(e^{i\theta})} = \sum_{j=1}^n \operatorname{Re}\left[\frac{1}{1 - e^{i\theta} z_j}\right] \ge \frac{n}{2} \tag{2.6}$$

by (2.5), proving (2.2). □

(2.3) is also the key to:

Theorem 2.4. *Let* $d\mu$ *be a nontrivial probability measure on* $\partial\mathbb{D}$ *and let* $\{\zeta_j^{(n)}\}_{j=1}^n$ *be the zeros of* $\varphi_n(z; d\mu)$. *Then*

$$\left\| \frac{\varphi_n'}{n} \right\|^2 = \frac{1}{n^2} \sum_{j,k=1}^n \frac{1}{1 - \overline{\zeta_j^{(n)}} \zeta_k^{(n)}} \tag{2.7}$$

$$= \iint \frac{1}{1 - z\bar{w}} \, d\nu_n(z) \, d\nu_n(w) \tag{2.8}$$

$$= 1 + \sum_{j=1}^\infty \left| \int z^j \, d\nu_n(z) \right|^2 \tag{2.9}$$

where $d\nu_n$ *is the zero counting measure, that is,*

$$d\nu_n = \frac{1}{n} \sum_{j=1}^n \delta_{\zeta_j^{(n)}}. \tag{2.10}$$

Proof. By the Bernstein–Szegő approximation (see [25, Thm. 1.7.8]),

$$\left\| \frac{\varphi_n'}{n} \right\|^2 = \frac{1}{n^2} \int \left| \frac{\varphi_n'}{\varphi_n} \right|^2 \frac{d\theta}{2\pi} \tag{2.11}$$

$$= \frac{1}{n^2} \sum_{j,k=1}^n \int_{z=e^{i\theta}} \frac{1}{\bar{z} - \bar{\zeta}_j^{(n)}} \frac{1}{z - \zeta_k^{(n)}} \frac{d\theta}{2\pi} \tag{2.12}$$

by (2.3).

For $a, b \in \mathbb{D}$,

$$\int \frac{1}{e^{-i\theta} - a} \frac{1}{e^{i\theta} - b} \frac{d\theta}{2\pi} = \frac{1}{2\pi i} \oint \frac{1}{\frac{1}{z} - a} \frac{1}{z - b} \frac{dz}{z} = \frac{1}{1 - ab} \tag{2.13}$$

since $(1 - az)^{-1}(z - b)^{-1}$ has a pole only at $z = b$.

Plugging (2.13) into (2.12) proves (2.7). (2.8) is a rewriting of (2.7), and since $d\nu_n$ is supported on a compact subset of $\mathbb{D}$, we can expand $(1 - z\bar{w})^{-1} = \sum_{j=0}^{\infty} z^j \bar{w}^j$, proving (2.9). □

Remark. (2.13) was used by Szegő [34]; see [25, Eq. (2.1.30)].

Notice that (2.9) provides another proof that $\|\frac{\varphi_n'}{n}\| \geq 1$ and shows that if $d\mu$ has normal derivative behavior, then $d\nu_n$ converges to a measure with zero positive moments (which also follows from Theorem 3.1), but fast enough to have all the moments in ℓ^1, so that the series in the right-hand side of (2.9) converges for each n. Since the right-hand side of (2.7) is greater than or equal to $\frac{1}{n^2} \sum_{k=1}^{n} \frac{1}{1 - |\zeta_k^{(n)}|^2}$, we see that if μ has normal behavior, zeros of φ_n cannot approach the unit circle too fast, at least, not faster than n^{-2}.

As a final formula for $\|\frac{\varphi_n'}{n}\|$, we define

$$f_n(z) = \frac{1}{n} \frac{K_{n-1}(z)}{|\varphi_n(z)|^2} \tag{2.14}$$

where, as usual, K is the CD kernel (see [25, Sect. 3.2] or [28]),

$$K_{n-1}(z) = \sum_{j=0}^{n-1} |\varphi_j(z)|^2. \tag{2.15}$$

By the Bernstein–Szegő approximation for $j \leq n$,

$$\int \left| \frac{\varphi_j}{\varphi_n} \right|^2 \frac{d\theta}{2\pi} = 1 \tag{2.16}$$

so

$$\int f_n(e^{i\theta}) \frac{d\theta}{2\pi} = 1. \tag{2.17}$$

Our f_n is very close to the function, I_n, of Golinskii–Khrushchev [14] defined by

$$I_n(z) = \frac{K_n(z)}{|\varphi_n(z)|^2} = 1 + nf_n(z). \tag{2.18}$$

If for $w \in \mathbb{D}$,

$$P_w(z) = \frac{1 - |w|^2}{|z - w|^2} \tag{2.19}$$

is the Poisson kernel, then Golinskii–Khrushchev [14] prove that

$$nf_n(z) = \sum_{j=1}^{n} P_{\zeta_j^{(n)}}(z). \tag{2.20}$$

(We note that $nf_n(z)$ is the weight of $\tilde{K}_{n-1}(z)\,d\tilde{\mu}_n$ where $\tilde{\mu}_n$ is the Bernstein–Szegő approximation, so (2.20) is related to ideas of Simon [29].)

Theorem 2.5. *We have that*

$$\left\| \frac{\varphi_n'}{n} \right\|^2 = \frac{1}{2} + \frac{1}{2} \int f_n^2(e^{i\theta}) \frac{d\theta}{2\pi}. \tag{2.21}$$

In particular, normality is equivalent to

$$\lim_{n\to\infty} \int f_n^2(e^{i\theta}) \frac{d\theta}{2\pi} = 1. \tag{2.22}$$

Proof. By (2.20),

$$\int |nf_n(e^{i\theta})|^2 \frac{d\theta}{2\pi} = \sum_{j,k=1}^{n} \int P_{\zeta_j^{(n)}}(e^{i\theta}) P_{\zeta_k^{(n)}}(e^{i\theta}) \frac{d\theta}{2\pi}. \tag{2.23}$$

Since

$$P_a(e^{i\theta}) = \frac{(1-|a|^2)e^{i\theta}}{(e^{i\theta}-a)(1-\bar{a}e^{i\theta})} \tag{2.24}$$

we have that

$$\int P_a(e^{i\theta}) P_b(e^{i\theta}) \frac{d\theta}{2\pi} = \frac{1}{2\pi i} \oint \frac{(1-|a|^2)(1-|b|^2)z}{(z-a)(1-\bar{a}z)(z-b)(1-\bar{b}z)}\, dz \tag{2.25}$$

$$= -1 + \frac{1}{1-\bar{a}b} + \frac{1}{1-a\bar{b}} \tag{2.26}$$

by residue calculus.

Thus, by (2.24),

$$n^2 \int |f_n(e^{i\theta})|^2 \frac{d\theta}{2\pi} = -n^2 + 2 \sum_{j,k=1}^{n} \frac{1}{1-\overline{\zeta_j^{(n)}}\zeta_k^{(n)}} \tag{2.27}$$

$$= -n^2 + 2\|\varphi_n'\|^2 \tag{2.28}$$

by (2.7). (2.28) is equivalent to (2.21). □

Golinskii–Khrushchev [14] prove (their Proposition 6.6) if $d\mu$ has an everywhere nonzero weight

$$\|f_n - 1\|_{L^1(d\theta/2\pi)} \to 0. \tag{2.29}$$

We see normality is equivalent to $\|f_n\|_{L^2(d\theta/2\pi)} \to 1$.

We also note that if

$$b_n(z) = \frac{\varphi_n(z)}{\varphi_n^*(z)} \tag{2.30}$$

is the Blaschke product of zeros and $\eta_n(\theta)$ is defined by

$$b_n(e^{i\theta}) = e^{i\eta_n(\theta)} \tag{2.31}$$

then, as shown in [14],

$$nf_n(e^{i\theta}) = \frac{d}{d\theta}\eta_n(\theta). \tag{2.32}$$

In connection with these formulas, we note that there has been considerable literature on asymptotics of $K_n(e^{i\theta})$ (see the review in [28]) and that $|\varphi_n(e^{i\theta})|^2/K_n(e^{i\theta})$ has also been studied (see [9] and references therein).

Finally, we note that (2.17) shows $\int f_n^2(e^{i\theta})\frac{d\theta}{2\pi} \geq 1$, so (2.22) provides yet another proof of (1.5).

3. Normality implies Nevai class

In this section, we prove that

Theorem 3.1. *If μ is a probability measure on $\partial\mathbb{D}$ with normal derivative behavior, then μ is in Nevai class, that is,*

$$\alpha_n \to 0 \tag{3.1}$$

as $n \to \infty$.

Proof. By Szegő recursion (4.5) with $\rho_n = (1 - |\alpha_n|^2)^{1/2}$,

$$\rho_n(\varphi_{n+1}^*)' = (\varphi_n^*)' - \alpha_n\varphi_n - \alpha_n z\varphi_n' \tag{3.2}$$

so, using $|\alpha_n| < 1$, $|\rho_n| \leq 1$,

$$|\alpha_n|\frac{\|\varphi_n'\|}{n} \leq \frac{|\alpha_n|}{n} + \frac{\|(\varphi_n^*)'\|}{n} + \frac{\|(\varphi_{n+1}^*)'\|}{n}. \tag{3.3}$$

By Proposition 1.1, the right-hand side of (3.3) → 0 if we have normal derivative behavior. Since we also have $\|\varphi_n'\|/n \to 1$, (3.3) implies (3.1). □

This shows in particular that any measure with normal derivative behavior must be supported on the whole circle. The converse is certainly not true; see Section 7. However, one does have the following, which is of interest because of the examples in Section 11.

Theorem 3.2. *If μ is a regular measure on $\partial\mathbb{D}$, then*

$$\lim_{n\to\infty} \|\varphi_n'\|_\infty^{1/n} = \lim_{n\to\infty} \|\varphi_n'\|_2^{1/n} = 1. \tag{3.4}$$

Remark. Regularity means $\lim(\rho_1 \cdots \rho_n)^{1/n} = 1$ and $\operatorname{supp}(d\mu) = \partial\mathbb{D}$. There are many equivalent forms (see [27,32]).

Proof. Regularity implies (see [17,27,32]) that

$$\|\varphi_n\|_\infty^{1/n} \to 1. \tag{3.5}$$

Thus, by Bernstein's inequality (Theorem 2.1) and $n^{1/n} \to 1$, we have

$$\limsup \|\varphi_n'\|_\infty^{1/n} \leq 1. \tag{3.6}$$

Since $d\mu$ is a probability measure,

$$\|\varphi_n'\|_2 \leq \|\varphi_n'\|_\infty. \tag{3.7}$$

By (1.5),

$$\liminf \|\varphi_n'\|_2 \geq 1. \tag{3.8}$$

(3.6)–(3.8) imply (3.4). □

Remark. We will see, however, that under the assumptions of Theorem 3.2, $\|\varphi_n'\|$ can grow faster than any positive power of n.

4. Baxter weights

Recall that Baxter's theorem (see [25, Ch. 6]) says that (1.26) holds if and only if $d\mu_s = 0$, $\inf w > 0$, and the Fourier coefficients of w lie in ℓ^1. Here we will deal directly only with (1.26). Recall $\|\cdot\|_\infty$ is the $L^\infty\left(\partial\mathbb{D}, \frac{d\theta}{2\pi}\right)$ norm.

Theorem 4.1. *If (1.26) holds, then as $n \to \infty$,*

$$\left\| \frac{(\varphi_n^*)'}{n} \right\|_\infty \to 0. \tag{4.1}$$

In particular, μ has normal derivative behavior.

We will actually prove a stronger result:

Theorem 4.2. *Suppose that μ is a probability measure on $\partial\mathbb{D}$ and that*

(a)

$$\sup_n \|\varphi_n\|_\infty < \infty. \tag{4.2}$$

(b)

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=0}^{n-1} (j+1)|\alpha_j| = 0. \tag{4.3}$$

(c) *The Szegő condition holds, that is,*

$$\sum_{j=0}^{\infty} |\alpha_j|^2 < \infty. \tag{4.4}$$

Then (4.1) holds and μ is normal.

Remark. One might guess that (4.3) implies (4.4), but it does not. If

$$\alpha_n = \begin{cases} (j+1)^{-1/2} & n = 2^{j^2},\ j = 1, 2, \dots \\ 0 & n \neq 2^{j^2},\ \text{any } j = 1, 2, \dots \end{cases}$$

then (4.3) holds but (4.4) does not. The corresponding measure has normal behavior; see Theorem 6.1.

Proof of Theorem 4.1 given Theorem 4.2. By Szegő recursion,

$$\Phi_{n+1}^*(z) = \Phi_n^*(z) - z\alpha_n \Phi_n(z) \tag{4.5}$$

so using $\|\Phi_n\|_\infty = \|\Phi_n^*\|_\infty$, we see

$$\|\Phi_{n+1}\|_\infty \leq (1 + |\alpha_n|)\|\Phi_n\|_\infty \leq e^{|\alpha_n|}\|\Phi_n\|_\infty. \tag{4.6}$$

Thus,

$$\sup_n \| \Phi_n \|_\infty \le \mathrm{e}^{\sum_{j=0}^{\infty} |\alpha_j|} < \infty. \tag{4.7}$$

By (1.26), $\inf \| \Phi_n \|_2 > 0$, so (4.2) holds. Fix $J > 0$. Then

$$\frac{1}{n} \sum_{j=0}^{n-1} (j+1)|\alpha_j| \le \frac{1}{n} \sum_{j=0}^{J} (j+1)|\alpha_j| + 2 \sum_{J+1}^{\infty} |\alpha_j|.$$

So

$$\limsup \frac{1}{n} \sum_{j=0}^{n-1} (j+1)|\alpha_j| \le 2 \sum_{J+1}^{\infty} |\alpha_j| \tag{4.8}$$

goes to zero as $J \to \infty$, proving (4.3).

As is well known, (1.26) implies (4.4) since

$$\sum_{j=0}^{J} |\alpha_j|^2 \le \left(\sum_{j=0}^{J} |\alpha_j| \right)^2. \tag{4.9}$$

Thus, Theorem 4.2 implies Theorem 4.1. □

Proof of Theorem 4.2. By Bernstein's inequality (see Theorem 2.1), (4.2) implies that

$$\sup_n \left\| \frac{\Phi_n'}{n} \right\|_\infty \le \sup_n \| \Phi_n \|_\infty \equiv A < \infty \tag{4.10}$$

since $\| \Phi_n \|_\infty = \| \Phi_n \|_2 \| \varphi_n \|_\infty \le \| \varphi_n \|_\infty$. Thus, by (4.5),

$$\| (\Phi_{j+1}^*)' - (\Phi_j^*)' \|_\infty \le |\alpha_j| [\| \Phi_j' \|_\infty + \| \Phi_j \|_\infty] \le |\alpha_j| (j+1) A \tag{4.11}$$

so

$$\frac{1}{n} \| (\Phi_n^*)' \|_\infty \le A \frac{1}{n} \sum_{j=0}^{n} (j+1)|\alpha_j| \tag{4.12}$$

goes to zero by (4.3).

Since $\| (\varphi_n^*)' \|_\infty = \| (\Phi_n^*)' \|_\infty / \| \Phi_n \|$ and $\inf \| \Phi_n \| > 0$ by (4.4), (4.12) implies (4.1). □

5. Bounded Szegő weights

We say a measure μ is weakly equivalent to the Lebesgue measure if the Szegő condition, $\int \log(w(\theta)) \frac{\mathrm{d}\theta}{2\pi} > -\infty$, holds and there exist $0 < r < \infty$ so that

$$\mathrm{d}\mu \le r \frac{\mathrm{d}\theta}{2\pi} \tag{5.1}$$

equivalently, $\mathrm{d}\mu_\mathrm{s} = 0$ and w obeys (1.27); equivalently, with $\| \cdot \| = L^2(\mathrm{d}\mu)$ norm and $\| \cdot \|_{(0)} = L^2(\frac{\mathrm{d}\theta}{2\pi})$ norm,

$$\| f \|^2 \le r \| f \|_{(0)}^2. \tag{5.2}$$

In this section, we prove the following result, which is not only simple but whose proof illuminates why normality is sometimes true and also how it might fail.

Theorem 5.1. *If* $\mathrm{d}\mu$ *obeys the Szegő condition and* (5.1)*, it has normal derivative behavior.*

Proof. Since $\mathrm{d}\mu$ obeys the Szegő condition, (1.4) is equivalent to

$$\lim_{n\to\infty} \left\| \frac{(\Phi_n^*)'}{n} \right\|^2 = 0. \tag{5.3}$$

On the other hand, since $\mathrm{d}\mu_s = 0$, by Theorem 2.4.6 of [25], we have that in $\|\cdot\|_{(0)}$,

$$\Phi_n^* \to D^{-1}. \tag{5.4}$$

Thus, if

$$\Phi_n^*(\mathrm{e}^{\mathrm{i}\theta}) = \sum_{j=0}^{n} c_j^{(n)} \mathrm{e}^{\mathrm{i}j\theta} \tag{5.5}$$

then, for suitable d_j with

$$\sum_{j=0}^{\infty} |d_j|^2 < \infty \tag{5.6}$$

as $n \to \infty$, we have that

$$c_j^{(n)} \to d_j. \tag{5.7}$$

Then

$$\left\| \frac{(\Phi_n^*)'}{n} \right\|^2 \le r \left\| \frac{(\Phi_n^*)'}{n} \right\|_{(0)}^2 = r \sum_{j=0}^{n} \left(\frac{j}{n}\right)^2 |c_j^{(n)}|^2. \tag{5.8}$$

Let $P_{>J}$ be the projection in $L^2\left(\partial\mathbb{D}, \frac{\mathrm{d}\theta}{2\pi}\right)$ onto the span of $\{\mathrm{e}^{\mathrm{i}j\theta}\}_{j=J+1}^{\infty}$. Then

$$\begin{aligned} \|P_{>J}\Phi_n^*\|_{(0)} &\le \|P_{>J}D^{-1}\|_{(0)} + \|P_{>J}(\Phi_n^* - D^{-1})\|_{(0)} \\ &\le \|P_{>J}D^{-1}\|_{(0)} + \|\Phi_n^* - D^{-1}\|_{(0)} \end{aligned} \tag{5.9}$$

so

$$\lim_{J\to\infty} \limsup_{n\to\infty} \|P_{>J}\Phi_n^*\|_{(0)} = 0. \tag{5.10}$$

Fix J and note that for $n > J$,

$$\text{LHS of (5.8)} \le r\left(\frac{J}{n}\right)^2 \sum_{j=0}^{J} |c_j^{(n)}|^2 + r \sum_{j=J+1}^{n} |c_j^{(n)}|^2$$

so, for any J,

$$\limsup_{n\to\infty} \left\| \frac{(\Phi_n^*)'}{n} \right\|^2 \le r \lim_{n\to\infty} \|P_{>J}\Phi_n^*\|_{(0)}^2.$$

Taking $J \to \infty$ and using (5.10) implies (5.3). □

Remark. As a consequence of Theorem 5.1, we can conclude that bounded Jacobi-type weights also exhibit the normal behavior of derivatives. These are weights of the form

$$w(z) = g(z) \prod_{j=1}^{k} |z - a_j|^{\alpha_j}, \qquad |a_j| = 1, \quad \alpha_j > 0, \; j = 1, \dots, k$$

where g is a bounded, and bounded away from 0, integrable function on $\partial\mathbb{D}$. For further results on such weights, see Theorem 10.1.

6. Sparse Verblunsky coefficients

On the basis of what we have seen so far, one might guess that normal derivative behavior implies a Szegő condition or at least lots of a.c. spectrum. Here we will see that there are examples with normal derivative and with non-Szegő behavior and purely singular continuous spectrum.

Definition. Let $0 < N_1 < N_2 < \cdots$ and $\{\beta_j\}_{j=0}^{\infty} \in \mathbb{D}^{\infty}$. The associated sparse sequence is the Verblunsky coefficients

$$\alpha_j = \begin{cases} \beta_k & \text{if } j = N_k - 1 \text{ for } k = 1, 2, \dots \\ 0 & \text{otherwise.} \end{cases} \tag{6.1}$$

Our main result in this section is:

Theorem 6.1. *Suppose*

$$\limsup_{k\to\infty} \frac{N_k}{N_{k+1}} < 1, \qquad \lim_{j\to\infty} \beta_j = 0. \tag{6.2}$$

Then the corresponding measure for the associated Verblunsky coefficients has normal L^2-derivative behavior.

Example 6.2. Let $N_j = j!$. If $\beta_j \in \ell^2$, $d\mu$ has purely a.c. spectrum, and if $\beta_j \to 0$ but $\beta_j \notin \ell_2$, then $d\mu$ has purely s.c. spectrum (see [26, Sect. 12.5]). In particular, if $\beta_j = (j+1)^{-1/2}$, then $d\mu$ is non-Szegő, purely singular continuous, and normal. □

Lemma 6.3. *Let $\{x_j\}_{j=1}^{\infty}$ be a sequence of nonnegative real numbers. Suppose, for $\gamma_j \geq 0$ and $\theta_j \geq 0$, we have*

$$x_{j+1} \leq \gamma_j + \theta_j x_j. \tag{6.3}$$

(a) *If*

$$\sup \theta_j = \theta < 1, \qquad \sup \gamma_j = \gamma < +\infty \tag{6.4}$$

then

$$\limsup x_j \leq (1-\theta)^{-1}\gamma. \tag{6.5}$$

(b) *If*

$$\limsup \theta_j < 1, \qquad \lim \gamma_j = 0 \tag{6.6}$$

then

$$\lim x_j = 0. \tag{6.7}$$

Proof. (a) Define y_j by

$$y_1 = x_1, \qquad y_{j+1} = \gamma + \theta y_j. \tag{6.8}$$

By induction, $x_j \le y_j$, so

$$\limsup x_j \le \limsup y_j. \tag{6.9}$$

By (6.8), if $z_j = y_j - (1-\theta)^{-1}\gamma$, then

$$z_{j+1} = \theta z_j \tag{6.10}$$

so $z_j \to 0$ and $y_j \to (1-\theta)^{-1}\gamma$. Thus, (6.9) implies (6.5).

(b) Fix N_0 so $\sup_{j\ge N_0} \theta_j = \theta < 1$. By using (a) for $\{j \mid j \ge N \ge N_0\}$, we see that for $N \ge N_0$, $\limsup x_j \le (1-\theta)^{-1} \sup_{j\ge N} \gamma_j$. So $\lim \gamma_j = 0$ implies (6.7). □

Proof of Theorem 6.1. Let

$$\eta_j(z) = \varphi^*_{N_j}(z) \tag{6.11}$$

and let $\sigma_j = (1 - |\beta_j|^2)^{1/2}$. By Szegő recursion, for $k < N_{j+1} - N_j$,

$$\varphi^*_{N_j+k} = \eta_j \tag{6.12}$$

so by Szegő recursion,

$$\sigma_{j+1}\eta_{j+1}(z) = \eta_j(z) - \beta_{j+1} z^{(N_{j+1}-N_j)} \eta_j^*(z) \tag{6.13}$$

and

$$\sigma_{j+1}\|\eta'_{j+1}\| \le \|\eta'_j\| + |\beta_{j+1}|(N_{j+1} - N_j) + |\beta_{j+1}|\|(\eta_j^*)'\|. \tag{6.14}$$

Since $1 + \|(\varphi_n^*)'/n\|^2 \le (1 + \|(\varphi_n^*)'/n\|)^2$, (1.3) implies that

$$\|(\eta_j^*)'\| \le N_j + \|\eta'_j\| \tag{6.15}$$

so (6.14) becomes

$$\sigma_{j+1}\|\eta'_{j+1}\| \le \|\eta'_j\| + |\beta_{j+1}|N_{j+1} + |\beta_{j+1}|\,\|\eta'_j\|. \tag{6.16}$$

Letting

$$x_j = \frac{\|\eta'_j\|}{N_j}, \qquad \theta_j = \frac{1+|\beta_{j+1}|}{\sigma_{j+1}} \frac{N_j}{N_{j+1}}, \qquad \gamma_j = \frac{|\beta_{j+1}|}{\sigma_{j+1}} \tag{6.17}$$

(6.16) becomes

$$x_{j+1} \le \gamma_j + \theta_j x_j. \tag{6.18}$$

By the lemma, $x_j \to 0$. By (6.12),

$$\sup_{N_j \le n < N_{j+1}} \frac{\|(\varphi_n^*)'\|}{n} = x_j \tag{6.19}$$

so $x_j \to 0$ implies (1.4), which is normality. □

Remark. One can also approach Theorem 6.1 through the function f_n of (2.14), Theorem 2.5, and

$$\frac{1}{n}\sum_{k=1}^{n}\prod_{j=k}^{n}\frac{1-|\alpha_{j-1}|}{1+|\alpha_{j-1}|}\le f_n(z)\le\frac{1}{n}\sum_{k=1}^{n}\prod_{j=k}^{n}\frac{1+|\alpha_{j-1}|}{1-|\alpha_{j-1}|},\quad z\in\mathbb{T}$$

which follows from the bounds

$$\frac{1-|\alpha_n|}{1+|\alpha_n|}\le\left|\frac{\varphi_n(z)}{\varphi_{n+1}(z)}\right|^2\le\frac{1+|\alpha_n|}{1-|\alpha_n|},\quad z\in\mathbb{T}.$$

7. Addition of mass points

Our goal here is to prove that if μ has a reasonable a.c. weight at a point in $\partial\mathbb{D}$ and we add a mass point at that point, then the resulting measure is nonnormal. By rotation covariance, we can suppose the point is $1\in\partial\mathbb{D}$. The discussion below was motivated by consideration of $(1-\gamma)\frac{d\theta}{2\pi}+\gamma\delta_1$, where everything is explicit (see [25, Example 1.6.3]), and a direct calculation (from [25, Eq. (1.6.6)]) shows that $\|(\varphi_n^*)'\|/n\to\frac{1}{2}\gamma^{1/2}(1-\gamma)^{-1/2}$, which is not zero, so (1.4) fails.

Given a probability measure μ on $\partial\mathbb{D}$, we define for $t>0$,

$$\nu_t=(1+t)^{-1}(\mu+t\delta_1). \tag{7.1}$$

Let $\Phi_n(z;t)$, $\varphi_n(z;t)$, $\alpha_n(t)$ be the monic and normalized OPs and Verblunsky coefficient for ν_t (for $t\ge 0$) and its CD kernel

$$K_n(z,w;t)=\sum_{j=0}^{n}\varphi_j(z;t)\overline{\varphi_j(w;t)}. \tag{7.2}$$

It is a result of Geronimus [11] (see [28] for a proof and a list of rediscoverers!) that

$$\Phi_n(z;t)=\Phi_n(z;0)-t\Phi_n(1;t)K_{n-1}(z,1;0) \tag{7.3}$$

$$\Phi_n(1;t)=\frac{\Phi_n(1;0)}{1+tK_{n-1}(1,1;0)}. \tag{7.4}$$

Lemma 7.1. *If $\{x_n\}_{n=1}^{\infty}$ are strictly positive and $x_n/x_{n+1}\to 1$, then $x_n/\sum_{j=1}^{n-1}x_j\to 0$.*

Proof.

$$\limsup\left(\frac{x_n}{\sum_{j=1}^{n-1}x_j}\right)\le\limsup\left(\frac{x_n}{\sum_{k=1}^{K}x_{n-k}}\right)=\limsup\left(\frac{1}{\sum_{k=1}^{K}\frac{x_{n-k}}{x_n}}\right)=\frac{1}{K}.$$

Since K is arbitrary, the limit is 0. □

Proposition 7.2. *Let $\|\cdot\|_t$ be the $L^2(\mathrm{d}\nu_t)$ norm in the framework of mass point perturbations. Then*

$$\frac{\|\Phi_n(\cdot;t)\|_t^2}{\|\Phi_n(\cdot;t=0)\|_{t=0}^2} = \frac{1}{1+t}\left[\frac{1+tK_n(1,1;t=0)}{1+tK_{n-1}(1,1;t=0)}\right]. \tag{7.5}$$

If $\alpha_n(t=0)\to 0$, then

$$\lim_{n\to\infty} \text{LHS of (7.5)} = \frac{1}{1+t}. \tag{7.6}$$

Proof. Since $K_{n-1}(z,1;0)$ is a polynomial of degree $n-1$ in z, it is μ-orthogonal to $\Phi_n(z;0)$. Since $\int |K_{n-1}(z,1;0)|^2\,\mathrm{d}\mu = K_{n-1}(1,1;0)$ by the reproducing property, we conclude, by (7.3), that

$$\|\Phi_n(\cdot;t)\|_{t=0}^2 = \|\Phi_n(\cdot;0)\|_{t=0}^2 + t^2|\Phi_n(1;t)|^2 K_{n-1}(1,1;0).$$

Thus, by (7.1),

$$\begin{aligned}(1+t)\|\Phi_n(\cdot;t)\|_t^2 &= \|\Phi_n(\cdot;0)\|_{t=0}^2 + t|\Phi_n(1;t)|^2[1+tK_{n-1}(1,1;0)] \\ &= \|\Phi_n(\cdot;0)\|_{t=0}^2\left[1+\frac{t|\varphi_n(1;t=0)|^2}{1+tK_{n-1}(1,1;0)}\right]\end{aligned} \tag{7.7}$$

by (7.4) and $\varphi_n = \Phi_n/\|\Phi_n\|$. This proves (7.5).

By Szegő recursion,

$$\left|\rho_n\frac{\varphi_{n+1}^*(\mathrm{e}^{\mathrm{i}\theta})}{\varphi_n^*(\mathrm{e}^{\mathrm{i}\theta})} - 1\right| \le |\alpha_n|$$

so, if $\alpha_n\to 0$, $|\varphi_n(\mathrm{e}^{\mathrm{i}\theta})|/|\varphi_{n+1}(\mathrm{e}^{\mathrm{i}\theta})|\to 1$, and so Lemma 7.1 implies $(1+tK_n)/(1+tK_{n-1})\to 1$, showing (7.6). □

The following will provide many examples of ν_t's which are not normal.

Theorem 7.3. *Suppose μ obeys:*

(a) *μ is Nevai, that is,*

$$\lim_{n\to\infty}\alpha_n(0) = 0. \tag{7.8}$$

(b) *For some $C_1, C_2 > 0$ and all n,*

$$C_1 \le |\varphi_n(1;0)| \le C_2. \tag{7.9}$$

(c)

$$\lim_{n\to\infty}\frac{1}{n}|(\varphi_n^*)'(1;0)| = 0. \tag{7.10}$$

Then for $t>0$,

$$\begin{aligned}\sqrt{1+t}\,\frac{C_1^3}{2C_2^2} &\le \liminf_{n\to\infty}\frac{1}{n}|(\varphi_n^*)'(1;t)| \\ &\le \limsup_{n\to\infty}\frac{1}{n}|(\varphi_n^*)'(1;t)| \le \sqrt{1+t}\,\frac{C_2^3}{2C_1^2}\end{aligned} \tag{7.11}$$

and, in particular, for all $t>0$, ν_t does not have normal behavior of derivatives.

Moreover, if

(d)

$$\lim_{n\to\infty} \frac{\|(\varphi_n^*)'(\cdot;0)\|_{t=0}}{n} = 0 \tag{7.12}$$

then

$$\lim_{n\to\infty} \frac{\|(\varphi_n^*)'(\cdot;t)\|_{t=0}}{n} = 0. \tag{7.13}$$

If, in addition to (a)–(d)*,*

(e)

$$\lim_{n\to\infty} |\varphi_n(1;0)| = C \neq 0 \tag{7.14}$$

then

$$\lim_{n\to\infty} \frac{1}{n}\|(\varphi_n^*)'(1;t)\|_t = \frac{C}{2}\sqrt{1+t}. \tag{7.15}$$

Remark. (7.11) only holds for $t > 0$ so that (7.10) is not recovered from (7.11) when $t = 0$.

Proof. Write

$$q_n = \frac{\|\Phi_n(\cdot;t)\|_t}{\|\Phi_n(\cdot;t=0)\|_{t=0}}. \tag{7.16}$$

Then (7.3) and (7.4) imply

$$q_n(\varphi_n^*)'(z;t) = \zeta_{1,n}(z;t) + \zeta_{2,n}(z;t) + \zeta_{3,n}(z;t) \tag{7.17}$$

where

$$\zeta_{1,n}(z;t) = (\varphi_n^*)'(z;0) \tag{7.18}$$

$$\zeta_{2,n}(z;t) = -\frac{t\varphi_n(1;0)}{1+tK_{n-1}(1,1;0)} \sum_{j=0}^{n-1} z^{n-j}\varphi_j(1;0)(\varphi_j^*)'(z;0) \tag{7.19}$$

$$\zeta_{3,n}(z;t) = -\frac{t\varphi_n(1;0)}{1+tK_{n-1}(1,1;0)} \sum_{j=0}^{n-1} (n-j)z^{n-j-1}\varphi_j(1;0)\varphi_j^*(z;0) \tag{7.20}$$

where we used (with $(\ )^*$, the * appropriate for degree n polynomials)

$$(\varphi_j(z;0))_n^* = z^{n-j}\varphi_j^*(z;0) \tag{7.21}$$

and the Leibniz rule to get $\zeta_{2,n}$ and $\zeta_{3,n}$.

By (7.10),

$$\frac{1}{n}|\zeta_{1,n}(1;t)| \to 0 \tag{7.22}$$

as $n \to \infty$. By (7.9),

$$\frac{tC_1}{1+nC_2^2 t} \le \frac{t|\varphi_n(1;0)|}{1+tK_{n-1}(1,1;0)} \le \frac{tC_2}{1+nC_1^2 t}. \tag{7.23}$$

Thus,

$$\frac{1}{n}|\zeta_{2,n}(1;t)| \le \frac{tC_2}{n(1+nC_1^2 t)} \sum_{j=0}^{n-1} C_2 j \left| \frac{(\varphi_j^*)'(1;0)}{j} \right| \to 0 \tag{7.24}$$

by (7.10).

At $z = 1$, the sum, S_n, in (7.20) is bounded by

$$C_1^2 \sum_{j=0}^{n-1} (n-j) \le S_n \le C_2^2 \sum_{j=0}^{n-1} (n-j) = C_2^2 \frac{n(n+1)}{2} \tag{7.25}$$

(7.6), (7.17), (7.22) and (7.24) imply

$$\limsup_{n\to\infty} \frac{1}{n} |(\varphi_n^*)'(1;t)| \le \sqrt{1+t} \limsup_{n\to\infty} \frac{1}{n} |\zeta_{3,n}(1;t)| \tag{7.26}$$

and similarly for lim infs (with $\le$ replaced by $\ge$), (7.20), (7.25) and (7.23) then imply (7.11).

Since

$$\liminf \frac{1}{n} \|(\varphi_n^*)'(\cdot;t)\|_t \ge \sqrt{\frac{t}{1+t}} \liminf \frac{1}{n} |(\varphi_n^*)'(1;t)| \tag{7.27}$$

(7.11) implies nonnormality.

Now suppose (7.12) holds. Then

$$\frac{1}{n} \|\zeta_{1,n}(\cdot;t)\|_{t=0} \to 0. \tag{7.28}$$

By (7.9), (7.19) and (7.23),

$$\frac{1}{n} \|\zeta_{2,n}(\cdot;t)\|_{t=0} \le \frac{C_2^2}{n^2 C_1^2} \sum_{j=1}^{n-1} j \left(\frac{\|(\varphi_j^*)'(\cdot;0)\|_{t=0}}{j} \right) \to 0 \tag{7.29}$$

by (7.12).

In the same way, since $z^{-j}\varphi_j^*(z;0) = \overline{\varphi_j(z;0)}$ on $\partial\mathbb{D}$ are orthogonal,

$$\frac{1}{n} \|\zeta_{3,n}(\cdot,t)\|_{t=0} \le \frac{C_2^2}{n^2 C_1^2} \left(\sum_{j=0}^{n-1} |n-j|^2 \right)^{1/2} \to 0 \tag{7.30}$$

proving (7.13).

Finally, if (e) also holds, we note first that, by (7.13), one has equality in (7.27) with lim inf replaced by inf. The existence of the limit if (7.14) yields

$$\lim_{n\to\infty} \frac{1}{n} |(\varphi_n^*)'(1;t)| = \sqrt{1+t} \frac{C}{2} \tag{7.31}$$

by the arguments that led to (7.11). □

Example 7.4. If μ obeys Baxter's condition, (a)–(d) of Theorem 7.3 hold, since (a) is trivial, (b) is Baxter's theorem, (c) and (d) follow from (4.1). Thus, whenever Baxter's condition holds for μ, all ν_t are nonnormal. In many cases, (e) also holds. □

There are also local conditions on the weight that imply (b) and (c), following ideas of Freud [10], Badkov [5], Golinskii [13], and Nevai [22]:

Theorem 7.5. *Let μ obey the Szegő condition so that for some $\varepsilon > 0$, $\mu_s(\{e^{i\theta} \mid |\theta| < \varepsilon\}) = 0$, and with weight, w, obeys*

(i) *For some $\delta > 0$, $\delta < w(e^{i\theta}) < \delta^{-1}$ if $|\theta| < \varepsilon$.*
(ii)

$$\sup_{|\varphi|<\varepsilon} \int_{|\theta|<\varepsilon} \left| \frac{w(\theta) - w(\varphi)}{\theta - \varphi} \right|^2 d\theta < \infty. \tag{7.32}$$

Then (a)–(c) *of Theorem 7.3 hold and every ν_t associated to μ via (7.1) is nonnormal.*

Remark. (7.32) is Freud's condition [10]. (b) has been proven under a weaker and close-to-optimal condition by Badkov [5], namely,

$$\int_0^1 \left[\sup_{\substack{\theta,\varphi\in(-\varepsilon,\varepsilon)\\ |\theta-\varphi|<\delta}} |w(\theta) - w(\varphi)| \right] \Big/ \delta \, d\delta < \infty \tag{7.33}$$

see also [31]. It is possible that by combining Badkov [5] with Nevai [22], one can also prove (c) under this condition.

Proof. (a) follows from the fact that μ obeys the Szegő condition. (b) is from Freud. (c) follows from [22] who proves, under these conditions, that

$$|n^{-1}\varphi_n'(1) - \varphi_n(1)| = o(1). \tag{7.34}$$

From this, it is easy to see that

$$|(\varphi_n^*)'(1)| = |n\varphi_n(1) - \varphi_n'(1)|. \quad \square \tag{7.35}$$

8. Circular Jacobi measures and their perturbations

The circular Jacobi measure and polynomials are the measure defined for a real with $a > -\frac{1}{2}$ by

$$d\mu_a(\theta) = w_a(\theta)\frac{d\theta}{2\pi}, \qquad w_a(\theta) = \frac{\Gamma^2(a+1)}{\Gamma(2a+1)}|1 - e^{i\theta}|^{2a} \tag{8.1}$$

and the normalized polynomials

$$\varphi_n(z; d\mu_a) = \frac{(a)_n}{\sqrt{n!(2a+1)_n}} {}_2F_1(-n, a+1; -n+1-a; z) \tag{8.2}$$

$$\varphi_n^*(z; d\mu_a) = \frac{(a+1)_n}{\sqrt{n!(2a+1)_n}} {}_2F_1(-n, a; -n-a; z) \tag{8.3}$$

where, as usual, $(s)_n = s(s+1)\cdots(s+n-1)$ is the Pochhammer symbol and ${}_2F_1$ the hypergeometric function. These are due to Witte–Forrester [37] and appear as Example 8.2.5 of Ismail [16]. As in the last section,

$$d\nu_{a,t} = \frac{d\mu_a + t\delta_1}{1+t}. \tag{8.4}$$

Here we will discuss three facts:

$$(1) \quad \frac{1}{n}(\varphi_n^*)'(1; d\nu_{a,t}) \sim n^{2a} \tag{8.5}$$

$$(2) \quad \left\| \frac{(\varphi_n^*)'(\cdot; d\mu_a)}{n} \right\|_{L^2(d\mu_a)}^2 = \frac{a^2}{(2a+1)n} \tag{8.6}$$

$$(3) \quad \lim_n \left\| \frac{(\varphi_n^*)'(\cdot; d\nu_{a,t})}{n} \right\|_{L^2(d\mu_a)} = 0. \tag{8.7}$$

These have the following consequences:

(a) $\|(\varphi_n^*)'\|/n$ can grow as any power n^a for measures in the Nevai class.
(b) $d\mu_a$ is normal for any a (for $a \geq 0$, this follows from Theorem 3.1 but is new for $-\frac{1}{2} < a < 0$).
(c) For $-\frac{1}{2} < a < 0$, $d\nu_{a,t}$ is normal, showing that inserting a mass point at a singular point for the weight may not destroy normality.

Facts (1)–(3) are consequences of explicit calculations that follow.

Proposition 8.1. *For $k \in \mathbb{Z}$,*

$$\gamma_k(a) \equiv \int_0^{2\pi} e^{ik\theta} w_a(\theta) \frac{d\theta}{2\pi} = (-1)^k \frac{\Gamma^2(a+1)}{\Gamma(k+a+1)\Gamma(-k+a+1)}. \tag{8.8}$$

Proof. We begin by noting that $\gamma_{-k}(a) = \gamma_k(a)$ since $w_a(\theta)$ is even under $\theta \to -\theta$. (8.8) clearly holds for $k = 0$ since $w_a(\theta)$ is a unit weight.

Since

$$\frac{w_{a+1}(\theta)}{w_a(\theta)} = \frac{a+1}{2(2a+1)}\left(2 - z - \frac{1}{z}\right) \tag{8.9}$$

where $z = e^{i\theta}$, we get

$$\gamma_k(a+1) = \frac{a+1}{2(2a+1)}(2\gamma_k(a) - \gamma_{k+1}(z) - \gamma_{k-1}(a)). \tag{8.10}$$

For $k = 0$, this implies, using $\gamma_1 = \gamma_{-1}$,

$$1 = \gamma_0(a+1) = \frac{a+1}{2(2a+1)}(2 - 2\gamma_1(a)) \tag{8.11}$$

proving (8.8) for $k = 1$. From (8.10), by

$$\gamma_{k+1}(a) = 2\gamma_k(a) - \gamma_{k-1}(a) - \frac{2(2a+1)}{a+1}\gamma_k(a+1) \tag{8.12}$$

and induction, we get (8.8) in general. □

Thus, for any real polynomial, $P(z) = \sum_{j=0}^{n} d_j z^j$, we get

$$\|P(z)\|^2_{L^2(\mathrm{d}\mu_a)} \equiv \int |P(z)|^2 w_a(z) \frac{\mathrm{d}\theta}{2\pi} = \Gamma(a+1)^2 \sum_{j,m=0}^{n} d_j d_m \frac{(-1)^{k+1}}{\Gamma(k-m+a+1)\Gamma(m-k+a+1)}. \tag{8.13}$$

The polynomials that we are most interested in are ${}_2F_1(-n, a+1; -n-a; z)$ since

$$(\varphi_n^*)'(z; \mathrm{d}\mu_a) = \frac{(a+1)_n}{\sqrt{n!(2a+1)_n}} \frac{an}{a+n} {}_2F_1(-n+1, a+1; -n-a+1; z). \tag{8.14}$$

So we note that

$$Q_n(z) \equiv {}_2F_1(-n, a+1; -n-a; z) \equiv \sum_{k=0}^{n} c_k z^k \tag{8.15}$$

where, by the definition of ${}_2F_1$ [4],

$$c_k = \frac{(-n)_k (a+1)_k}{(-n-a)_k k!}. \tag{8.16}$$

Proposition 8.2. *For $n \geq 0$ and $k = 0, 1, \ldots, n$,*

(a) $$\sum_{m=0}^{n} (-1)^m \frac{c_m}{\Gamma(k-m+a+1)\Gamma(m-k+a+1)} = (-1)^k \frac{n!}{\Gamma(a+1)\Gamma(n+a+1)} \tag{8.17}$$

(b) $$\sum_{m=0}^{n} (-1)^m \frac{m c_m}{\Gamma(k-m+a+1)\Gamma(m-k+a+1)} = (-1)^k \frac{(k+(2k-n)a)n!}{\Gamma(a+1)\Gamma(n+a+1)}. \tag{8.18}$$

As a consequence, for Q_n in (8.15),

$$\|Q_n\|^2_{L^2(\mathrm{d}\mu_a)} = \frac{n!}{(a+1)_n} Q_n(1). \tag{8.19}$$

Proof. Formulas (8.17) and (8.18) can be directly verified by a computer algebra system, such as Mathematica. They can also be proven using Zeilberger's algorithm [38,39], implemented as a Mathematica package [23], which establishes recurrence relations for the left-hand side of each identity. For instance, [23] finds the following relations for

$$F(n, m, k) = \frac{(-1)^m c_m}{\Gamma(a+k-m+1)\Gamma(a-k+m+1)}$$

that can be verified by dividing both sides by $F(n, m, k)$ and checking the resulting rational equation: if Δ_m is the forward difference operator in m, then

$$\begin{aligned}
&-(n+1)(2a+n+2)(-k+n+1)F(n,m,k)\\
&\quad+(a+n+1)(-2ak+3an+4a-2kn-4k+2n^2+7n+6)F(n+1,m,k)\\
&\quad-(a+n+1)(a+n+2)(a-k+n+2)F(n+2,m,k)\\
&\quad=\Delta_m\big(F(n,m,k)R_1(n,m,k)\big)
\end{aligned}\tag{8.20}$$

and

$$\begin{aligned}
&-(k+1)(a-k+n)F(n,m,k)+(-an+2k^2-2kn+4k-3n+2)F(n,m,k+1)\\
&\quad+(a+k+2)(k-n+1)F(n,m,k+2)\\
&\quad=\Delta_m\big(F(n,m,k)R_2(n,m,k)\big)
\end{aligned}\tag{8.21}$$

with

$$R_1(n,m,k)=\frac{am(n+1)(a-k+m)(a-m+n+1)}{(-m+n+1)(-m+n+2)}$$

$$R_2(n,m,k)=\frac{(2a+1)m(a-k+m)(a-m+n+1)}{(-a-k+m-2)(-a-k+m-1)}.$$

Summing (8.20) over m from 0 to $n+2$, and (8.21) over m from 0 to n we conclude that

$$y_{n,k}\coloneqq\sum_{m=0}^{n}F(n,m,k)$$

satisfies the following recurrence relations:

$$\begin{aligned}
&(n+1)(2a+n+2)(-k+n+1)y_{n,k}\\
&\quad-(a+n+1)(-2ak+3an+4a-2kn-4k+2n^2+7n+6)y_{n+1,k}\\
&\quad+(a+n+1)(a+n+2)(a-k+n+2)y_{n+2,k}=0
\end{aligned}\tag{8.22}$$

and

$$\begin{aligned}
&(1+k)(a-k+n)y_{n,k}-(2+4k+2k^2-3n-an-2kn)y_{n,k+1}\\
&\quad-(2+a+k)(1+k-n)y_{n,k+2}=0
\end{aligned}\tag{8.23}$$

with initial conditions

$$y_{0,0}=\frac{1}{\Gamma^2(a+1)},\qquad y_{1,0}=-y_{1,1}=\frac{1}{\Gamma(a+1)\Gamma(a+2)}.\tag{8.24}$$

It is straightforward to check that the right-hand side in (8.17),

$$s_{n,k}=(-1)^k\frac{n!}{\Gamma(a+1)\Gamma(a+n+1)}$$

also verifies (8.22)–(8.24). This yields (8.17).

Finally, by (8.13) and (8.17),

$$\begin{aligned}
\frac{\|Q_n\|^2_{L^2(\mathrm{d}\mu_a)}}{\Gamma^2(a+1)}&=\sum_{k=0}^{n}(-1)^k c_k\sum_{m=0}^{n}(-1)^m\frac{c_m}{\Gamma(k-m+a+1)\Gamma(m-k+a+1)}\\
&=\frac{n!}{\Gamma(a+1)\Gamma(n+a+1)}\sum_{k=0}^{n}c_k
\end{aligned}$$

which proves (8.19). □

From (8.14), (8.15), and the well-known formula for the hypergeometric function with the unit argument (see [1, Eq. (15.1.20)]), we obtain

Theorem 8.3.

$$\left\| \frac{(\varphi_n^*)'(z; \mathrm{d}\mu_a)}{n} \right\|^2_{L^2(\mathrm{d}\mu_a)} = \frac{a^2}{(2a+1)n}. \tag{8.25}$$

In particular, all $\mathrm{d}\mu_a$, $a > -\frac{1}{2}$, *are normal.*

Next, we turn to the $\nu_{a,t}$. By (8.2) and (8.14), we see that

$$\frac{1}{n}(\varphi_n^*)'(1; \mathrm{d}\mu_a) = \frac{a}{2a+1}\varphi_n(1; \mathrm{d}\mu_a) \tag{8.26}$$

and that, in the sense of the ratio approaching a fixed nonzero, a-dependent constant,

$$\varphi_n(1; \mathrm{d}\mu_a) \sim n^a \tag{8.27}$$

so that

$$K_{n-1}(1, 1; \mathrm{d}\mu_a) \sim n^{2a+1}. \tag{8.28}$$

By (7.3) and (8.26), we obtain

$$\frac{1}{n}\Phi_n^*(1; \mathrm{d}\nu_{a,t}) = \Phi_n(1; \mathrm{d}\mu_a)\left[\frac{a}{2a+1} - \frac{n+1}{2n}\frac{tK_{n-1}(1,1;\mathrm{d}\mu_a)}{1+tK_{n-1}(1,1;\mathrm{d}\mu_a)}\right]. \tag{8.29}$$

So, by (8.28),

$$\frac{1}{n}(\Phi_n^*)'(1; \mathrm{d}\nu_{a,t}) \sim n^a. \tag{8.30}$$

In particular, $\|(\varphi_n^*)'(\cdot; \mathrm{d}\nu_{a,t})/n\|_{\nu_{a,t}} \geq O(n^a)$, proving at least arbitrary power growth for suitable a.

Finally, we turn to estimating $\|(\varphi_n^*)'(\cdot; \mathrm{d}\nu_{a,t})\|_{L^2(\mathrm{d}\mu_a)}$. By (7.3) and (7.4) with $q_n^2 \to 1/(1+t)$, we have

$$q_n\varphi_n(z; \mathrm{d}\nu_{a,t}) = \varphi_n(z; \mathrm{d}\mu_a) - \frac{t\varphi_n(1; \mathrm{d}\mu_a)}{1+tK_{n-1}(1,1;\mathrm{d}\mu_a)}K_{n-1}(1, z; \mathrm{d}\mu_a). \tag{8.31}$$

By the CD formula,

$$K_{n-1}(1, z; \mathrm{d}\mu_a) = \varphi_n(1; \mathrm{d}\mu_a)\left[\frac{\varphi_n^*(z; \mathrm{d}\mu_a) - \varphi_n(z; \mathrm{d}\mu_a)}{1-z}\right] \tag{8.32}$$

$$= \frac{(a+1)_n}{n!(1-z)}\left({}_2F_1(-n, a; -n-a; z) - \frac{a}{a+n}{}_2F_1(-n, a+1; -n-a+1; z)\right) \tag{8.33}$$

$$= \frac{a+1}{n!}\frac{n}{a+n}\left[P_n(z) - \frac{1}{n}zP_n'(z)\right] \tag{8.34}$$

where

$$P_n(z) = {}_2F_1(-n, a+1; -n-a+1; z). \tag{8.35}$$

In the above, (8.33) comes from (8.32), (8.2) and (8.3); and (8.28) from relations on ${}_2F_1$.

Using this and letting

$$\delta_n = \frac{2a+1}{a} \frac{t K_{n-1}(1,1;\mathrm{d}\mu_a)}{1 + t K_{n-1}(1,1;\mathrm{d}\mu_a)} \tag{8.36}$$

(so $\delta_n \to (2a+1)/a$), we obtain

$$q_n \varphi_n(z;\mathrm{d}\nu_{a,t}) = \varphi_n(z;\mathrm{d}\mu_a) + \delta_n \left(z \frac{\varphi_n'(z;\mathrm{d}\mu_a)}{n} - \varphi_n(z;\mathrm{d}\mu_a) \right). \tag{8.37}$$

This plus (1.10) yields

$$q_n \frac{(\varphi_n^*)'(z;\mathrm{d}\nu_{a,t})}{n} = \frac{(\varphi_n^*)'(z;\mathrm{d}\mu_a)}{n} \left(1 - \frac{\delta_n}{n}\right) - \delta_n \frac{z(\varphi_n^*)''(z;\mathrm{d}\mu_a)}{n^2}. \tag{8.38}$$

By the proven normality of $\mathrm{d}\mu_a$ (Theorem 8.3), the first term on the right-hand side of (8.38) has an $L^2(\mathrm{d}\mu_a)$ norm going to zero, so we focus on the second. By the explicit formula for $\varphi_n^*(z;\mu_a)$,

$$(\varphi_n^*)''(z;\mathrm{d}\mu_a) = a\sqrt{\frac{a+1}{2(2a+1)}}\sqrt{n(n-1)}\varphi_{n-2}(z;\mathrm{d}\mu_{a+1}). \tag{8.39}$$

Thus, we need

Proposition 8.4.

$$\|\varphi_n(z;\mathrm{d}\mu_{a+1})\|^2_{L^2(\mathrm{d}\mu_a)} = 1 + \frac{2n}{2a+3}. \tag{8.40}$$

Proof. By (8.2),

$$\begin{aligned} \varphi_n(z;\mathrm{d}\mu_{a+1}) &= \frac{(a+1)_n}{\sqrt{n!(2a+3)_n}} P_n(z) \\ P_n(z) &= {}_2F_1(-n, a+2; -n-a; z) = \sum_{k=0}^n \widehat{c}_k z^k \end{aligned} \tag{8.41}$$

where

$$\widehat{c}_k = \frac{(-n)_k (a+2)_k}{(-n-a)_k k!} = \left(1 + \frac{k}{a+1}\right) c_k \tag{8.42}$$

with c_k given in (8.16). Thus, from (8.13) it follows that

$$\|P_n\|^2_{L^2(\mathrm{d}\mu_a)} = \Gamma^2(a+1)(S_1 + S_2 + S_3) \tag{8.43}$$

where

$$S_1 = \sum_{k=0}^n \sum_{m=0}^n \frac{(-1)^{k+m} c_k c_m}{\Gamma(k-m+a+1)\Gamma(m-k+a+1)} \tag{8.44}$$

$$S_2 = \frac{2}{a+1} \sum_{k=0}^n \sum_{m=0}^n \frac{(-1)^{k+m} k c_k c_m}{\Gamma(k-m+a+1)\Gamma(m-k+a+1)} \tag{8.45}$$

$$S_3 = \frac{1}{(a+1)^2} \sum_{k=0}^{n} \sum_{m=0}^{n} \frac{(-1)^{k+m} k c_k m c_m}{\Gamma(k-m+a+1)\Gamma(m-k+a+1)}. \tag{8.46}$$

The first sum has been computed in Proposition 8.2 and Theorem 8.3:

$$S_1 = \frac{1}{\Gamma^2(a+1)} \|Q_n\|^2_{L^2(\mathrm{d}\mu_a)} = \frac{(2a+2)_n}{\Gamma^2(n+a+1)} n!$$

where Q_n is defined in (8.15). On the other hand, by (8.18),

$$\begin{aligned} S_2 &= \frac{2}{a+1} \sum_{k=0}^{n} (-1)^k k c_k \sum_{m=0}^{n} \frac{(-1)^m c_m}{\Gamma(k-m+a+1)\Gamma(m-k+a+1)} \\ &= \frac{2(n!)}{\Gamma(a+2)\Gamma(n+a+1)} \sum_{k=0}^{n} k c_k = \frac{2(n!)}{\Gamma(a+2)\Gamma(n+a+1)} Q_n'(1). \end{aligned}$$

Using the formula for the derivatives of the hypergeometric function,

$$Q_n'(z) = \frac{(a+1)n}{a+n} {}_2F_1(-n+1, a+2; -n-a+1; z) \tag{8.47}$$

$$Q_n''(z) = \frac{(a+1)(a+2)n(n-1)}{(a+n)(a+n-1)} {}_2F_1(-n+2, a+3; -n-a+2; z) \tag{8.48}$$

and [1, Eq. (15.1.20)], we conclude that

$$S_2 = \frac{2(n!)}{\Gamma(a+1)\Gamma(n+a+1)} \frac{n}{n+a} \frac{(2a+3)_{n-1}}{(a+1)_{n-1}}.$$

Analogously,

$$\begin{aligned} S_3 &= \frac{1}{(a+1)^2} \sum_{k=0}^{n} (-1)^k k c_k \sum_{m=0}^{n} \frac{(-1)^m m c_m}{\Gamma(k-m+a+1)\Gamma(m-k+a+1)} \\ &= \frac{n!}{(a+1)^2 \Gamma(a+1)\Gamma(n+a+1)} \sum_{k=0}^{n} k c_k (k + (2k-n)a) \\ &= \frac{n!}{(a+1)^2 \Gamma(a+1)\Gamma(n+a+1)} \left((2a+1) \sum_{k=0}^{n} k^2 c_k - na \sum_{k=0}^{n} k c_k \right) \\ &= \frac{n!}{(a+1)^2 \Gamma(a+1)\Gamma(n+a+1)} ((2a+1)(z Q_n'(z))'(1) - na Q_n'(1)) \\ &= \frac{n!}{(a+1)^2 \Gamma(a+1)\Gamma(n+a+1)} ((2a+1) Q_n''(1) + (2a+1-na) Q_n'(1)). \end{aligned}$$

Using (8.47)–(8.48), we obtain

$$S_3 = \frac{n(2n+2a+1)\Gamma(n+2a+2)(n!)}{\Gamma(2a+4)\Gamma^2(n+a+1)}.$$

Hence,

$$S_1 + S_2 + S_3 = \frac{(2n+2a+3)\Gamma(n+2a+3)(n!)}{\Gamma(2a+4)\Gamma^2(n+a+1)}$$

(8.40) now follows from (8.41) and (8.43). □

By (8.39) and (8.40), we obtain

$$\left\| \frac{\varphi_n^*(\cdot\,; d\mu_a)''}{n^2} \right\|^2_{L^2(d\mu_a)} = a^2 \frac{a+1}{2(2a+1)} \frac{n(n-1)}{n^4} \left(1 + \frac{2n-4}{2a+3}\right) = O\left(\frac{1}{n}\right) \tag{8.49}$$

so

Theorem 8.5. *For $a > -\frac{1}{2}$, $\|\varphi_n^*(\cdot\,; d\nu_{a,t})'/n\|_{L^2(d\mu_a)} \to 0$. In particular, for $-\frac{1}{2} < a < 0$, $d\nu_{a,t}$ is normal (and for $a \geq 0$, it is not normal).*

9. Multiplicative perturbations of the weight

In the preceding section, we saw that the circular Jacobi weight, even in the unbounded case, where $-\frac{1}{2} < a < 0$, is normal. In this section and the next, we extend this to other cases. A key tool will be (2.21). Here we will prove a general result about perturbations of weights:

Theorem 9.1. *Let $d\mu$ be a measure on $\partial\mathbb{D}$ satisfying the Nevai condition (3.1), and g is a Lipschitz continuous, strictly positive function on $\partial\mathbb{D}$. Then normality of $d\mu$ implies normality of $g\,d\mu$.*

The proof depends on a preliminary result.

Proposition 9.2. *Let $d\mu$ be a measure on $\partial\mathbb{D}$ satisfying the Nevai condition (3.1), and g is a continuous and nonvanishing function on $\partial\mathbb{D}$ so that $g\,d\mu$ also obeys (3.1). Then*

$$\lim_{n\to\infty} \frac{K_{n-1}(z, z; g\,d\mu)}{K_{n-1}(z, z; d\mu)} = \frac{1}{g(z)} \tag{9.1}$$

uniformly on $\partial\mathbb{D}$.

Proof. Under the assumption of Nevai's condition, uniformly on $\partial\mathbb{D}$ for any fixed $m \in \mathbb{N}$,

$$\lim_{n\to\infty} \frac{K_{n+m-1}(z, z; d\mu)}{K_{n-1}(z, z; d\mu)} = 1$$

by Corollary 9.4.3 of [26].

If $g(z) = |P(z)|^2$, then by the extremal properties of the CD kernel where $\deg(P) = m$,

$$\frac{K_{n-1}(z, z; g\,d\mu)}{K_{n+m-1}(z, z; d\mu)} g(z) \leq 1$$

so that

$$\limsup_{n\to\infty} \frac{K_{n-1}(z, z; g\,d\mu)}{K_{n-1}(z, z; d\mu)} \leq \frac{1}{g(z)}.$$

Using the monotonicity of the kernel and the $\|\cdot\|_\infty$-density of $\{|P(z)|^2\}$ in the nonnegative functions, we can extend this inequality to any continuous and nonvanishing function g. Finally, reversing the role of $d\mu$ and $g\,d\mu$, we obtain (9.1). □

Proof of Theorem 9.1. By Theorem 2 of [20],

$$\lim_{n\to\infty} \frac{|\varphi_n(z; g\,\mathrm{d}\mu)|^2}{|\varphi_n(z; \mathrm{d}\mu)|^2} = 1 \tag{9.2}$$

uniformly on $\partial\mathbb{D}$. By Lemma 9.3, $g\,\mathrm{d}\mu$ also obeys the Nevai condition, so Proposition 9.2 is applicable. Thus, using (9.2),

$$\lim_{n\to\infty} \frac{f_n(z; g\,\mathrm{d}\mu)}{f_n(z; \mathrm{d}\mu)} = 1. \tag{9.3}$$

By (2.21), $g\,\mathrm{d}\mu$ is normal if and only if $\mathrm{d}\mu$ is. □

Lemma 9.3. *If $\alpha_n(\mu) \to 0$ and (9.2) holds, then $\alpha_n(g\,\mathrm{d}\mu) \to 0$.*

Proof. By the Szegő recursion formula, for any measure, ν,

$$\rho_n \frac{\varphi^*_{n+1}(z; \mathrm{d}\nu)}{\varphi^*_n(z; \mathrm{d}\nu)} - 1 = -\alpha_n(\mathrm{d}\nu) z \frac{\varphi_n(z; \mathrm{d}\nu)}{\varphi^*_n(z; \mathrm{d}\nu)}. \tag{9.4}$$

Since $z\varphi_n/\varphi^*_n$ is a nontrivial Blaschke product, there are points $z_0 \in \partial\mathbb{D}$ so that the right side is positive and equal to $|\alpha_n|$. Thus,

$$|\alpha_n(\mathrm{d}\nu)| = \rho_n \sup_{z\in\partial\mathbb{D}} \frac{|\varphi^*_{n+1}(z; \mathrm{d}\nu)|}{|\varphi^*_n(z; \mathrm{d}\nu)|} - 1 \tag{9.5}$$

(9.2) plus (9.5) completes the proof. □

10. Algebraic singularities

In this section, we prove

Theorem 10.1. *Let w_0 be the weight*

$$w_0(z) = \prod_{k=1}^{m} |z - \zeta_k|^{2a_k} \tag{10.1}$$

where $\zeta_1, \ldots, \zeta_m \in \partial\mathbb{D}$ are distinct and each $a_k > -\frac{1}{2}$. Let g be a nonvanishing Lipschitz continuous function on $\partial\mathbb{D}$. Then $g w_0(\mathrm{e}^{\mathrm{i}\theta}) \frac{\mathrm{d}\theta}{2\pi}$ is a normal measure on $\partial\mathbb{D}$.

Proposition 10.2. *Let $\mathcal{F}_n(x) = \min\{n^2, |1 - \cos x|^{-1}\}$, $x \in (-\pi, \pi)$. Then for $k = 1, \ldots, m$ and for a sufficiently small $\delta > 0$, there exists $C \in (0, 1)$, not depending on n or k, such that for $\varphi_n(z) = \varphi_n(z; w_0(z)|\mathrm{d}z|)$,*

$$C \le \frac{|\varphi_n(\zeta_k \mathrm{e}^{\mathrm{i}x})|^2}{\mathcal{F}_n^{a_k}(x)} \le C^{-1}, \quad -\delta < x < \delta. \tag{10.2}$$

Proof. Obviously, it is sufficient to establish an analogous bound for the monic orthogonal polynomials Φ_n. Fix $k \in \{1, \ldots, m\}$, $\mathcal{B}_k \stackrel{\text{def}}{=} \{z \in \mathbb{C} \mid |z - \zeta_k| \le \delta\}$. For $z = \zeta_k \mathrm{e}^{\mathrm{i}x}$, $-\delta < x < \delta$, define $t_n(z) = nx/2 \in \mathbb{R}$. From Theorem 1.4 of [19], it follows that

$$|\Phi_n(z)|^2 = \frac{\pi}{2} \frac{|\mathcal{H}(a_k; t_n(z))|^2}{|z - \zeta_k|^{2a_k}} \left(1 + O\left(\frac{1}{n}\right)\right) \tag{10.3}$$

where the $O(1/n)$ term is uniform in $(-\delta, \delta)$. $\mathcal{H}$, analytic in a punctured neighborhood of the origin, is defined by

$$\mathcal{H}(a;t) \stackrel{\text{def}}{=} \begin{cases} e^{-2\pi i a} t^{1/2}(iJ_{a+1/2}(t) + J_{a-1/2}(t)) & \text{if } t \text{ is in the second quadrant} \\ t^{1/2}(iJ_{a+1/2}(t) + J_{a-1/2}(t)) & \text{otherwise} \end{cases} \tag{10.4}$$

and J_ν is the Bessel function of the first kind. In particular,

$$|\Phi_n(\zeta_k e^{ix})|^2 = \frac{\pi}{2^{1+a_k}} \frac{|t_n|(J^2_{a_k+1/2}(|t_n|) + J^2_{a_k-1/2}(|t_n|))}{(1-\cos x)^{a_k}} \left(1 + O\left(\frac{1}{n}\right)\right), \quad -\delta < x < \delta.$$

Since the zeros of $J^2_{a+1/2}$ and $J^2_{a-1/2}$, $a > -\frac{1}{2}$, interlace, we have

$$J^2_{a+1/2}(t) + J^2_{a-1/2}(t) > 0, \quad \text{for } t > 0. \tag{10.5}$$

On the other hand, from the asymptotic formula [1, Eq. (9.2.1)], we obtain that

$$\lim_{t\to+\infty} t(J^2_{a+1/2}(t) + J^2_{a-1/2}(t)) = \frac{2}{\pi}$$

and we conclude that for $\delta_1 > 0$, there exists $C_1 = C_1(a, \delta_1) \in (0, 1)$ such that

$$C_1 \le t(J^2_{a+1/2}(t) + J^2_{a-1/2}(t)) \le C_1^{-1}, \quad \text{for } t \in (\delta_1, +\infty).$$

In particular, for

$$F_n(x) = \frac{t_n(J^2_{a_k+1/2}(t_n) + J^2_{a_k-1/2}(t_n))}{(1-\cos x)^{a_k}}, \quad t_n = \frac{nx}{2}$$

we have

$$\frac{C_1}{(1-\cos x)^{a_k}} \le F_n(x) \le \frac{C_1^{-1}}{(1-\cos x)^{a_k}}, \quad x > \frac{2\delta_1}{n}. \tag{10.6}$$

On the other hand, for $x \in [0, 2\delta_1/n]$,

$$F_n(x) = n^{2a_k} \frac{x^{2a_k} 2^{1-4a_k}}{(1-\cos x)^{a_k}} \left(\left(\frac{t_n}{2}\right)^2 G^2_{a_k+1/2}(t_n) + G^2_{a_k-1/2}(t_n)\right)$$

where $G_a(z) = (2/z)^a J_a(z) \to 0$ when $z \to 0$. Taking into account (10.5), we conclude that there exists $C_2 = C_2(\beta, \delta_1) \in (0, 1)$ such that

$$C_2 n^{2\beta} \le F_n(x) \le C_2^{-1} n^{2\beta}, \qquad x \in \left[0, \frac{2\delta_1}{n}\right]. \tag{10.7}$$

Combining (10.6) and (10.7), we obtain (10.2). □

Corollary 10.3. *For the weight given in (10.1), the sequence f_n is uniformly bounded on $\partial\mathbb{D}$. In particular,*

$$\lim_n \frac{1}{2\pi} \int_0^{2\pi} f_n^2(e^{i\theta})\, d\theta = 1$$

so that the generalized circular Jacobi measure, $w_0 \frac{d\theta}{2\pi}$, is normal.

Remark. Observe that normality of this measure for $a_k \geq 0$ follows from Theorem 5.1. So this result is new for the negative values of a_k, when the weight is unbounded.

Proof. That the measure is Nevai class follows from Rakhmanov's theorem. The first assertion follows from (10.2) and the fact that for $a > -\frac{1}{2}$,

$$\frac{1}{n}\frac{\sum_{k=0}^{n-1}}{\mathcal{F}_k^a(x)}\mathcal{F}_n^a(x)$$

is uniformly bounded on $\mathbb{R}$. The second assertion is a consequence of (2.29) and Theorem 2.5. □

Thus, Theorem 10.1 follows from Theorem 9.1.

11. Isolated mass points

In this section, we will consider a situation where μ has a gap in its essential spectrum containing an isolated mass point at $z_0 \in \partial\mathbb{D}$. Of course, since $\alpha_n \to 0$ implies $\mathrm{supp}(\mathrm{d}\mu) = \partial\mathbb{D}$ (see [25, Thm. 4.3.5]), Theorem 3.1 implies μ is not normal. What we want to show is that, in fact, $\|\varphi_n'\|$ always grows exponentially in this setting. The intuition is: since $\varphi_n(z_0)$ decreases exponentially while $\varphi_n(z)$ grows exponentially for z near z_0, $\varphi_n'(z_0)$ must be very large. The only surprise is that the result is very general and the proof simple. Here are the results:

Theorem 11.1. *Let μ have a gap in its essential spectrum and z_0 a mass point in this gap. Then for some $A, C > 0$,*

$$|\varphi_n'(z_0)| \geq A\mathrm{e}^{Cn}. \tag{11.1}$$

In particular,

$$\|\varphi_n'\| \geq A\mu(\{z_0\})^{1/2}\mathrm{e}^{Cn}. \tag{11.2}$$

Theorem 11.2. *Let μ have a gap in its essential spectrum, $\mathfrak{e}$, and $z_0 \notin \mathfrak{e}$ a mass point. Suppose μ is regular. Then*

$$\lim_{n\to\infty} |\varphi_n'(z_0)|^{1/n} = \exp(G_{\mathfrak{e}}(z_0)) \tag{11.3}$$

where $G_{\mathfrak{e}}$ is the logarithmic potential of $\mathfrak{e}$. In particular,

$$\liminf \|\varphi_n'\|^{1/n} \geq \exp(G_{\mathfrak{e}}(z_0)). \tag{11.4}$$

Remarks. 1. Regularity was defined by Stahl–Totik [32] (see [27]) and means

$$\lim_{n\to\infty} (\rho_0 \cdots \rho_{n-1})^{1/n} = C(\mathfrak{e}) \tag{11.5}$$

where $C(\mathfrak{e})$ is the logarithmic capacity. It holds, for example, if the equilibrium measure for $\mathfrak{e}$ is $\frac{\mathrm{d}\theta}{2\pi}$ absolutely continuous and $\mathrm{d}\mu = w\frac{\mathrm{d}\theta}{2\pi} + \mathrm{d}\mu_{\mathrm{s}}$ with $\{\theta \mid w(\theta) > 0\} = \mathfrak{e}$ up to sets of measure zero (see [32,27]).

2. These results on $\|\varphi_n\|^{1/n}$ should be compared with Theorem 3.2.

We will prove both of these theorems from the following elegant formula:

Theorem 11.3. *Let μ have a gap in its essential spectrum with z_0 an isolated point of μ. Let ψ_n be the second kind polynomials. Then there is an ℓ^2 sequence, $\tilde{\eta}_n$, so that*

$$\varphi_n'(z_0) = (2z_0\mu(\{z_0\}))^{-1}\psi_n(z_0) + \tilde{\eta}_n. \tag{11.6}$$

Proof. Let $d\nu$ be the measure for which ψ_n are the first kind polynomials and φ_n the second kind polynomials (i.e., $\alpha_n(d\nu) = -\alpha_n(d\mu)$). Then (see [25, Prop. 3.2.8]) for $z \in \mathbb{D}$,

$$\varphi_n(z) = \int (\psi_n(e^{i\theta}) - \psi_n(z))\left[\frac{e^{i\theta} + z}{e^{i\theta} - z}\right] d\nu(\theta). \tag{11.7}$$

By analyticity, since $z_0 \notin \operatorname{supp}(d\nu)$, this holds for z in a neighborhood of z_0. Using $F_{d\nu}(z) = F_{d\mu}(z)^{-1}$, we conclude

$$\eta_n(z) \equiv \varphi_n(z) + F(z)^{-1}\psi_n(z) = \int \psi_n(e^{i\theta})\left[\frac{e^{i\theta} + z}{e^{i\theta} - z}\right] d\nu(\theta). \tag{11.8}$$

Thus, $\eta_n(z) \in \ell_2$ and is analytic near z_0, so $\tilde{\eta}_n \equiv \eta_n'(z_0) \in \ell_2$ by a Cauchy estimate.

Near z_0,

$$F(z) = \frac{2z_0\mu(\{z_0\})}{z_0 - z} + O(1) \tag{11.9}$$

so

$$F^{-1}(z_0) = 0 \qquad \frac{d}{dz}F^{-1}(z)\bigg|_{z=z_0} = -(2z_0\mu(\{z_0\}))^{-1} \tag{11.10}$$

which leads to (11.6). □

Proof of Theorem 11.1. By [26, Thm. 10.14.2],

$$|\varphi_n(z_0)| \le A_0 e^{-Cn} \tag{11.11}$$

for some A_0, C. By [25, (3.2.33)],

$$|\psi_n(z_0)| \ge A_0^{-1} e^{Cn}. \tag{11.12}$$

Thus, (11.6) implies (11.1). □

Proof of Theorem 11.2. Let $d\nu$ be the measure for which ψ_n are the first kind OPUC. Then $d\nu$ is regular and $z_0 \notin \operatorname{supp}(d\nu)$. It follows, since then z_0 is also not in the convex hull of $\operatorname{supp}(d\nu)$, that (see [32,27])

$$\lim_{n\to\infty} \|\psi_n(z_0)\|^{1/n} = e^{G_{\mathfrak{e}}(z_0)} \tag{11.13}$$

(11.6) completes the proof. □

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