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Cantor polynomials and some related classes of OPRL

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Abstract

We explore the spectral theory of the orthogonal polynomials associated to the classical Cantor measure and similar singular continuous measures. We prove regularity in the sense of Stahl–Totik with polynomial bounds on the transfer matrix. We present numerical evidence that the Jacobi parameters for this problem are asymptotically almost periodic and discuss the possible meaning of the isospectral torus and the Szegő class in this context.

MSC: 42C05; 26A30; 58J53

Keywords: Orthogonal polynomials; Cantor set; Almost periodic

1. Introduction

The past fifteen years have seen remarkable progress in the spectral theory of orthogonal polynomials but mainly where the underlying orthogonality measure is purely absolutely continuous or at least has a substantial a.c. part. Our goal in this paper is to begin the exploration of the simplest purely singular continuous cases—indeed, we will focus on the case where the underlying measure is the classical Cantor $1/3$-measure. While we have some theorems in this case, we will mainly have conjectures, discussion, and some numerical experiments.

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Our model is the Szegő class of [21,10]. There is associated to any finite gap set, $\mathfrak{e}$, an isospectral torus of almost periodic Jacobi matrices, J, with $\sigma_{\text{ess}}(J) = \mathfrak{e}$. The frequency module of the almost periodic functions is generated by the harmonic measures of parts of $\mathfrak{e}$ between two points in the complement of $\mathfrak{e}$.

Whatever the notion of Szegő class is when $\mathfrak{e}$ is the Cantor set, one expects that the Cantor measure is so regular that it should lie in this Szegő class. Thus, a test of the notion is whether the Jacobi parameters are asymptotically almost periodic. We have computed the first 100,000 a_n's and provide convincing numerical evidence that they are. We have not explored whether the corresponding Cantor polynomials i.e., the OPRL ($\equiv$ orthogonal polynomials on the real line) for the Cantor measure, have an analog of Szegő asymptotics.

The Cantor measure is one element of a family parameterized by $\delta \in (0, \frac{1}{2}]$. Let

$$\varphi_{\pm}(x) = \pm(1 + \delta x). \tag{1.1}$$

$\varphi_{\pm}$ maps $[-2, 2]$ into two disjoint intervals (if $\delta < \frac{1}{2}$) and after k iterations to 2^k intervals. There is a unique probability measure, μ_δ, that obeys

$$\int f(x) d\mu = \frac{1}{2} \int f\big(\varphi_+(x)\big)\, d\mu + \frac{1}{2} \int f\big(\varphi_-(x)\big)\, d\mu. \tag{1.2}$$

It is, up to scaling and translations, the Cantor measure when $\delta = \frac{1}{3}$ ($[-1, 1]$ is translated and scaled to $[-\frac{3}{2}, \frac{3}{2}]$). We have also done some calculations for other values of $\delta < \frac{1}{2}$ which are qualitatively similar ($\delta = \frac{1}{2}$ is the normalized Lebesgue measure on $[-2, 2]$).

In Section 2, we discuss regularity in the sense of Stahl–Totik [28] and the stronger results that transfer matrices are polynomially bounded. We use this to get lower bounds on zero spacing for the Cantor polynomials. Section 3 is the central one where we analyze the first 100,000 Jacobi parameters for Cantor polynomials supporting the idea that these parameters are asymptotically almost periodic. Sections 4 and 5 discuss what the isospectral torus might mean here while Section 6 treats perturbations of this torus. Section 7 discusses differing meaning of the dimensions of the spectrum.

Throughout, we use ideas from the modern theory of OPRL and refer the reader to Simon [27] for background. For comparison to the finite gap case, see Christiansen et al. [10] or Simon [27]. We note that there was work twenty-five years ago [2–4,7] on different aspects of self-similar measures. We also mention recent papers of Christiansen [8,9] that include some Cantor set OPRL, but positive Lebesgue measure Cantor sets rather than zero measures.

It is a pleasure to thank J. Breur and V. Totik for useful discussions. Herbert Stahl was a key figure in the ideas relevant to Section 2 of this paper. It is a pleasure to be able to dedicate this paper to his memory and to the memory of Andrei Aleksandrovich Gonchar, two giants of analytic problems connected to orthogonal polynomials and approximation theory.

2. Regularity and zero spacing

Recall that a spectral measure on $\mathbb{R}$ is called regular if and only if the Jacobi parameters obey $\lim(a_1 \dots a_n)^{1/n} = C(\mathfrak{e})$, where $\mathfrak{e} = \sigma_{\text{ess}}(\mu)$ and $C(\cdot)$ is logarithmic capacity. Regular measures for general sets were defined and studied in the book of Stahl–Totik [28]; see also the review article of Simon [26]. Regularity implies the density of zeros is the equilibrium measure of $\mathfrak{e}$ and results on the lack of exponential decay of continuum eigenfunctions. In this section, we will first remark that a result of Stahl–Totik [28] implies regularity of a variety of singular continuous measures, including the Cantor measure. We will then present a refinement of Totik [31] which proves polynomial bounds on the growth of $p_n(x)$, $x \in \mathfrak{e}$ for the Cantor case, and then show that it implies a polynomial bound on the transfer matrix and so, an inverse polynomial bound on eigenvalue spacing.

The following specializes Theorem 4.2.3 of Stahl–Totik [28]; it covers many cases of OPRL, including the Cantor measure. For the reader's convenience, we sketch the proof, which follows Stahl–Totik with some simplifications given our stronger hypotheses.

Theorem 2.1. *Let $\mathfrak{e} \subset \mathbb{R}$ be a closed set which is regular for the Dirichlet problem. Let μ be a probability measure with*

$$\lim_{\varepsilon \downarrow 0} (2\varepsilon)^{-1} \inf_{x \in \mathfrak{e}} \mu(x-\varepsilon, x+\varepsilon) = \infty \tag{2.1}$$

on $\mathfrak{e}$. Then μ is regular.

Remarks. 1. (2.1) implies that $\mathfrak{e}$ has Lebesgue measure zero, so μ is singular.
2. (2.1) is much stronger than necessary, but holds for many interesting cases like the Cantor measure.

Proof. Let $P_n(x)$ be the monic OPs for μ. By Schiefermayr's theorem [27, Corollary 5.7.7], with $\|f\|_{\mathfrak{e}} = \sup_{x \in \mathfrak{e}} |f(x)|$, we have

$$\|P_n\|_{\mathfrak{e}} \geq 2C(\mathfrak{e})^n. \tag{2.2}$$

By hypothesis, the potential theoretic Green's function, $g_{\mathfrak{e}}$, is continuous and vanishes on $\mathfrak{e}$. So for any ε, there is a δ so that

$$\mathrm{dist}(z, \mathfrak{e}) < \delta \Rightarrow g_{\mathfrak{e}}(z) \leq \log(1+\varepsilon). \tag{2.3}$$

By the Bernstein–Walsh lemma (see, e.g., [27, Theorem 5.5.14]) and (2.3),

$$\sup_{\mathrm{dist}(z,\mathfrak{e})<\delta} |P_n(z)| \leq \|P_n\|_{\mathfrak{e}} (1+\varepsilon)^n. \tag{2.4}$$

Then, by a Cauchy estimate,

$$\mathrm{dist}(z, \mathfrak{e}) < \frac{\delta}{2} \Rightarrow |P_n'(z)| \leq \frac{2}{\delta} \|P\|_{\mathfrak{e}} (1+\varepsilon)^n. \tag{2.5}$$

Let $\rho_n = \frac{\delta}{4}(1+\varepsilon)^{-n}$ and let $x_n \in \mathfrak{e}$ be such that $|P_n(x_n)| = \|P_n\|_{\mathfrak{e}}$. Then $|x_n - y| \leq \rho_n \Rightarrow |P_n(y)| \geq \frac{1}{2}\|P_n\|_{\mathfrak{e}} \geq C(\mathfrak{e})^n$. It follows that

$$\begin{aligned} \int |P_n(x)|^2 \, d\mu(x) &\geq \int_{|y-x_n| \leq \rho_n} |P_n(x)|^2 \, d\mu(x) \\ &\geq C(\mathfrak{e})^{2n} \mu(x-\rho_n, x+\rho_n) \\ &\geq C(\mathfrak{e})^{2n} (1+\varepsilon)^{-n} \frac{\delta}{2} A(\rho_n) \end{aligned}$$

where $A(\rho) = \inf_{x \in \mathfrak{e}} [\mu(x-\rho, x+\rho)/2\rho]$.

Since $\|P_n\|_{L^2(\mu)} = a_1 \dots a_n$, we have

$$(a_1 \dots a_n)^{1/n} \geq C(\mathfrak{e})(1+\varepsilon)^{-1/2} \left(\frac{\delta}{2}\right)^{1/2n} A(\rho_n)^{1/2n}$$

from which it follows that

$$\liminf(a_1 \dots a_n)^{1/n} \ge C(\mathfrak{e})(1+\varepsilon)^{-1/2}.$$

Since ε is arbitrary and $\limsup(a_1 \dots a_n)^{1/n} \le C(\mathfrak{e})$ always holds, we see that μ is regular. □

We owe to Totik (private communication, quoted with permission) a refinement of this result: under a stronger hypothesis, a stronger result, namely, a polynomial in n bound on $p_n(x)$, $x \in \mathfrak{e}$, rather than just a subexponential bound.

Theorem 2.2 ([31]). *Suppose for some $\alpha \in (0,1]$, $A > 0$, and all $z, w \in \mathbb{C}$, we have*

$$|g_{\mathfrak{e}}(z) - g_{\mathfrak{e}}(w)| \le A|z-w|^{\alpha} \tag{2.6}$$

and for $\beta < 1$, we have for all $x \in \mathfrak{e}$ that

$$\mu([x-\varepsilon, x+\varepsilon]) \ge B\varepsilon^{\beta}. \tag{2.7}$$

Then for a constant, C, and the orthonormal polynomials, p_n, we have

$$\|p_n\|_{\mathfrak{e}} = \sup_{x\in\mathfrak{e}} |p_n(x)| \le Cn^{\beta/2\alpha}. \tag{2.8}$$

Remark. This implies $\limsup |p_n(x)|^{1/n} \le 1$ for all $x \in \mathfrak{e}$ and so, regularity.

Proof. By the Bernstein–Walsh lemma [27, Theorem 5.5.14], for any polynomial, q_n, of degree n, and any z,

$$|q_n(z)| \le \|q_n\|_{\mathfrak{e}} \exp(ng_{\mathfrak{e}}(z)) \tag{2.9}$$

so if $\operatorname{dist}(z, \mathfrak{e}) \le n^{-1/\alpha}$, by (2.6),

$$|q_n(z)| \le \|q_n\|_{\mathfrak{e}} \exp(A). \tag{2.10}$$

Thus, by a Cauchy estimate (i.e., $|q_n'(z)| \le r^{-1} \sup_{|w-z|=r} |q_n(w)|$), we get a Markov-type bound

$$\operatorname{dist}(z, \mathfrak{e}) \le \tfrac{1}{2} n^{-1/\alpha} \Rightarrow |q_n'(z)| \le 2e^A n^{1/\alpha} \|q_n\|_{\mathfrak{e}}. \tag{2.11}$$

Pick $x_n \in \mathfrak{e}$ so

$$|p_n(x_n)| = \|p_n\|_e. \tag{2.12}$$

By (2.12), if $|x - x_n| \ge \varepsilon_n \equiv (4e^A n^{1/\alpha})^{-1}$, then

$$|p_n(x)| \ge \tfrac{1}{2} \|p_n\|_e \tag{2.13}$$

which implies

$$\|p_n\|_2^2 = 1 \ge (\tfrac{1}{2} \|p_n\|_e)^2 \mu(x_n - \varepsilon_n, x_n + \varepsilon_n) \ge \tfrac{1}{4} \|p_n\|_e^2 B\varepsilon_n^{\beta} \tag{2.14}$$

or

$$\|p_n\|_e \le 2B^{-1/2} \varepsilon_n^{-\beta/2} = 2 \cdot 2^{\beta/2} B^{-1/2} e^{\beta A/2} n^{\beta/2a} \tag{2.15}$$

proving (2.8). □

This goes beyond regularity in that we will see it implies a power lower bound on zero spacing. First, we need to apply a result of Jitomirskaya–Last [12,13]. We define the transfer matrix associated to Jacobi parameters, $\{a_n, b_n\}_{n=1}^{\infty}$, by

$$T_n(x) = B_n(x) \dots B_1(x), \quad n = 1, 2, \dots \tag{2.16}$$

$$B_j(x) = \frac{1}{a_j} \begin{pmatrix} x - b_j & -a_{j-1} \\ 1 & 0 \end{pmatrix} \tag{2.17}$$

where $a_0 = 1$. Then

$$T_n(x) = \begin{pmatrix} p_n(x) & q_n(x) \\ p_{n-1}(x) & q_{n-1}(x) \end{pmatrix} \tag{2.18}$$

p_n are the standard OPRL and q_n suitably normalized second kind polynomials (q_n has degree $n - 1$). Following Jitomirskaya–Last, define

$$\rho_n(x) = \sum_{j=1}^{n} |p_n(x)|^2, \qquad \eta_n(x) = \sum_{j=1}^{n} |q_n(x)|^2. \tag{2.19}$$

The m-function of the spectral measure, $d\mu$,

$$m(z) = \int \frac{d\mu(x)}{x - z}. \tag{2.20}$$

For each x, ε_n is defined by

$$\varepsilon_n = \frac{1}{2\rho_n(x)\eta_n(x)}. \tag{2.21}$$

Then Jitomirskaya proved there is a universal constant, A, so that for all $x \in \mathfrak{e} = \text{supp}(\mu)$,

$$A^{-1}|m(x + i\varepsilon_n)| \le \frac{\eta_n(x)}{\rho_n(x)} \le A|m(x + i\varepsilon_n)|. \tag{2.22}$$

This implies the following:

Theorem 2.3. *Suppose for some $x \in \mathfrak{e} = \text{supp}(d\mu)$, we have*

$$|m(x + i\varepsilon)| \le C\varepsilon^{\gamma-1} \tag{2.23}$$

where $0 < \gamma < 1$. Then for a constant, D,

$$\eta_n(x) \le D\rho_n(x)^{(2-\gamma)/\gamma}. \tag{2.24}$$

In particular, if

$$|p_n(x)| \le En^{\nu}, \tag{2.25}$$

then

$$\eta_n(x) \le D(E^2 n^{2\nu+1})^{(2-\gamma)/\gamma} \tag{2.26}$$

is polynomially bounded.

Proof. By (2.22) and (2.23),

$$\frac{\eta_n}{\rho_n} \le AC\varepsilon_n^{\gamma-1} = AC2^{1-\gamma}\rho_n^{1-\gamma}\eta_n^{1-\gamma} \tag{2.27}$$

so

$$\eta_n^\gamma \leq AC2^{1-\gamma}\rho_n^{2-\gamma} \tag{2.28}$$

implies (2.24).

Since (2.25) implies $\rho_n \leq C^2 n^{2\nu-1}$, (2.24) implies (2.26). □

In [16], Last–Simon showed that if E and E' are two successive zeros of p_n and $x \in (E, E')$, then ([16, Theorems 2.1 and 2.2]) ($T_0 = \mathbf{1}$)

$$|E - E'| \geq \left(\sum_{j=0}^{n} \|T_n(x)\|^2 \right)^{-1}. \tag{2.29}$$

Thus,

Theorem 2.4. *If (2.23) holds for all $x \in \mathfrak{e}$ (with C, γ independent of x) and $\|p_n\|_{\mathfrak{e}} \leq Dn^\nu$, then for a constant, S, and any two successive zeros, E, E', of $p_n(x)$, we have*

$$|E - E'| \geq Sn^{-\lambda}, \quad \lambda = \frac{2(2\nu+1)(2-\gamma)}{\gamma}. \tag{2.30}$$

For the Cantor measure, one has that $\mu(x - \varepsilon, \mu + \varepsilon) \leq \tilde{B}\varepsilon^\beta$, $\beta = \log 2/\log 3$ which yields (2.23) with $\gamma = (\log 3 - \log 2)/\log 2$. It is known that (2.7) holds with the same β and (2.6) for some α (see [30,24]). Thus,

Theorem 2.5. *For the Cantor polynomials, there is $\lambda > 0$ so that (2.30) holds.*

3. Cantor polynomials: numerics

As we saw, up to scaling and translation, the Cantor measure is the $\delta = \frac{1}{3}$ of the measures determined by (1.2). Mantica [18,19] found a recursive procedure for computing the Jacobi measure associated to any family of iterated linear maps and computed about 100 parameters in typical cases. Heilman et al. [11] computed about 10,000 a_n's for suitable δ. We computed 100,000, and for one calculation 200,000, for $\delta = \frac{1}{3}$ and for several other values of δ. Since the qualitative results are the same for other values of δ, we only report on $\delta = \frac{1}{3}$. The Matlab script to generate our numbers is part of the ArXiv posting for this paper. Notice that since these measures are invariant under $x \to -x$, all $b_n = 0$.

Mantica considers the orthonormal polynomials, $p_n(x)$, and expands

$$p_n(\delta x + 1) = \sum_{\ell=0}^{n} \gamma_\ell^{(n)} p_\ell(x). \tag{3.1}$$

If one defines

$$\tilde{\gamma}_\ell^{(n+1)} = \delta a_{\ell+1}\gamma_{\ell+1}^{(n)} + \gamma_\ell^{(n)} + \delta a_\ell \gamma_{\ell-1}^{(n)} - a_n \gamma^{(n)} \tag{3.2}$$

one obtains a_{n+1} from $\int p_n^2(\delta x + 1)\, d\mu(x) = 1$ via

$$a_{n+1}^2(1 - \delta^{2n+2}) = \sum_{\ell=0}^{n} (\tilde{\gamma}_\ell^{(n+1)})^2 \tag{3.3}$$

Table 3.1
Harmonic measures, aka IDS.

Gap	IDS
$(\frac{1}{27}, \frac{2}{27})$	0.170240
$(\frac{1}{9}, \frac{2}{9})$	0.287260
$(\frac{7}{27}, \frac{8}{27})$	0.387270

and (from the recursion for $p_\ell(\delta x + 1)$)

$$\gamma_\ell^{(n+1)} = (a_{n+1})^{-1}\tilde{\gamma}_\ell^{(n+1)}; \quad \ell = 0, \dots, n \tag{3.4}$$

and, by looking at the leading x^{n+1} terms,

$$\gamma_{n+1}^{(n+1)} = \delta^{n+1}. \tag{3.5}$$

One preliminary calculation we looked at is to use the fact that we have regularity

$$(a_1 \dots a_n)^{1/n} \to C(\mathfrak{e}) \tag{3.6}$$

to get the capacity of the Cantor set (this multiplies the $C(\mathfrak{e})$ in (3.6) by $\frac{1}{3}$ because of scaling) to find

$$\text{Capacity of the classical Cantor set} \cong 0.22094998647421 \quad \text{(our value)}. \tag{3.7}$$

Since we expect $O(1/n)$ errors, we should only trust the first six digits or so. This compares with the value found by Ransford–Rostand [23] in 2007 who got

$$\text{Capacity of the classical Cantor set} \cong 0.220949102 \quad \text{(RR)}. \tag{3.8}$$

With the a_n's, one can also compute the harmonic measures of the part of the Cantor measure to the left of a given gap, that is, the weight of that set in the equilibrium measure. For by regularity, this is the limit of the density of zeros, so we compute the zeros of $p_{100,000}(x)$ and count the number below a gap. We do not compute the polynomial and its zeros but use the fact that these zeros are the eigenvalues of the truncated Jacobi matrix (see [27,29]) which, as a sparse matrix, has Matlab routine for its eigenvalues. For the three largest gaps in the bottom half of the Cantor set, we find the values in Table 3.1. We use the abbreviation IDS (for "integrated density of states") for the harmonic measure of the set below the gap. These are consistent with values computed by Ransford [22].

One of our main reasons for doing the numerics is to check the following conjecture:

Conjecture 3.1. *The a_n's for the Cantor polynomials are asymptotically almost periodic with the almost periodic limit having frequency module generated by the harmonic measures of subsets of the Cantor set within gaps.*

Notice that if $a_n^{(0)}$ is the almost periodic limit, then if $a_n - a_n^{(0)} \to 0$, the IDS are the same, so by gap labeling [14,6], the values of the IDS in gaps are among the frequencies. The conjecture says they are exactly the generators of the frequency module.

To see if the numerics support the conjecture, we computed the discrete Fourier transform, $\widehat{a}$, of the a_n's for $1 \le n \le 100{,}000$, normalizing frequencies to run from 0 to 1 and normalizing $|\widehat{a}|^2$ by dividing by $\sum_{n=1}^{100{,}000} |\widehat{a}_n|^2$. Fig. 3.1 shows this plot (with the large value at $n = 0$ not plotted).

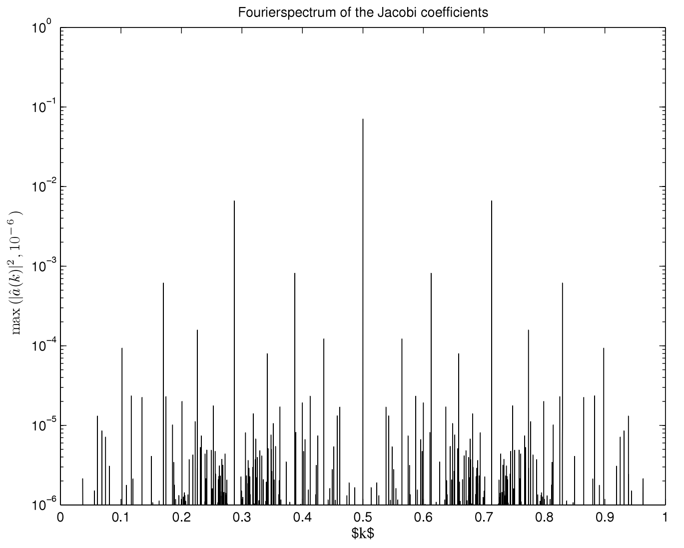

Fig. 3.1. Power spectrum of $|\widehat{a}_n|^2$.

Table 3.2
Top twelve peaks in the normalized power spectrum.

Normalized k	Normalized $\lvert\widehat{a}^{(k)}\rvert^2$
0	0.538035030339233
0.500000000000000	0.070682567302743
0.287260000000000	0.006615921068324
0.712740000000000	0.006615921068324
0.387270000000000	0.000814035809854
0.612730000000000	0.000814035809854
0.170240000000000	0.000612664851182
0.829760000000000	0.000612664851182
0.170250000000000	0.000180011375447
0.829750000000000	0.000180011375447
0.226330000000000	0.000157561287264
0.773670000000000	0.000157561287264

If the a_n's had no structure, one would expect all the normalized $|\widehat{a}_n|^2$ to be about 10^{-6}. As Fig. 3.1 and Table 3.2 show, there are a few anomalously large peaks (the y-axis is logarithmic) as one would expect for an almost periodic function. Table 3.2 shows the top twelve peaks.

Notice that other than the largest peaks at 0 and $\frac{1}{2}$ (which is the IDS in the gap $(\frac{1}{3}, \frac{2}{3})$), the next six peaks are at the largest gaps listed in Table 3.2 and the symmetric points (since $k(1-x) = 1 - k(x)$). Fig. 3.2 shows the IDS and the power spectrum superimposed, showing that the peaks are precisely at the flat parts.

As a sign of the fact that most of the power spectrum is in a few frequencies, only 62 of the 100,000 are larger than the mean of 10^{-5}. Fig. 3.3 shows the fit of those 62 frequency contributions against the values of a_n for $40{,}000 < n \leq 40{,}050$. We emphasize the parameters are chosen on all 100,000 a_n's, not just the 50 shown.

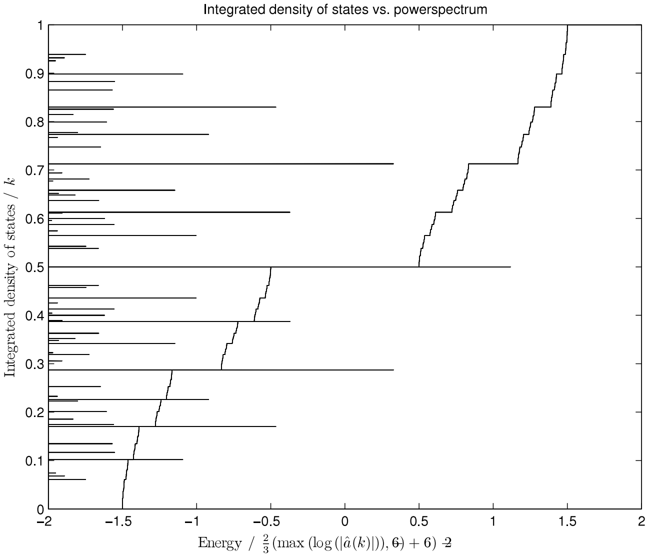

Fig. 3.2. Power spectrum vs IDS.

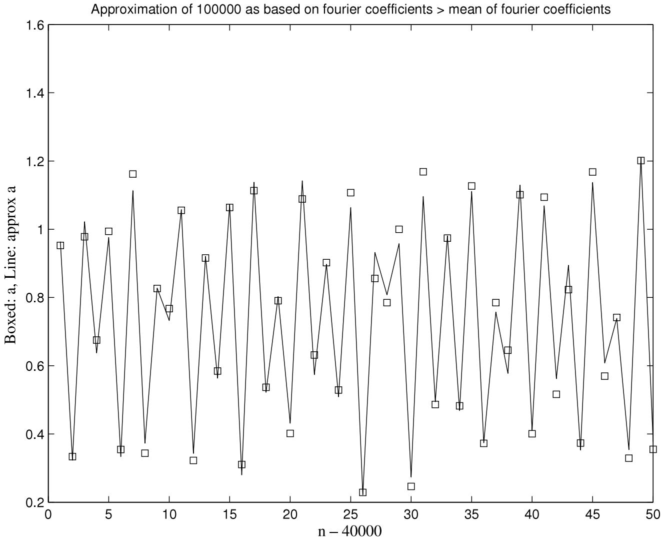

Fig. 3.3. An almost periodic fit (26 frequencies).

For the case of a finite number of arcs with a Szegő weight, Widom [32] proved an almost periodic behavior for $[a_1 \dots a_n / C(\mathfrak{e})^n]$ (in the non-real case, there are no a_n's but $a_1 \dots a_n$ is the inverse of the leading coefficient of p_n), a result studied further in [1,20,21,10]. We conjecture:

Conjecture 3.2. *For the Cantor polynomials, $W_n \equiv a_1 \dots a_n / C(\mathfrak{e})^n$ is asymptotic to an almost periodic function. In particular, it is bounded above and below (away from zero).*

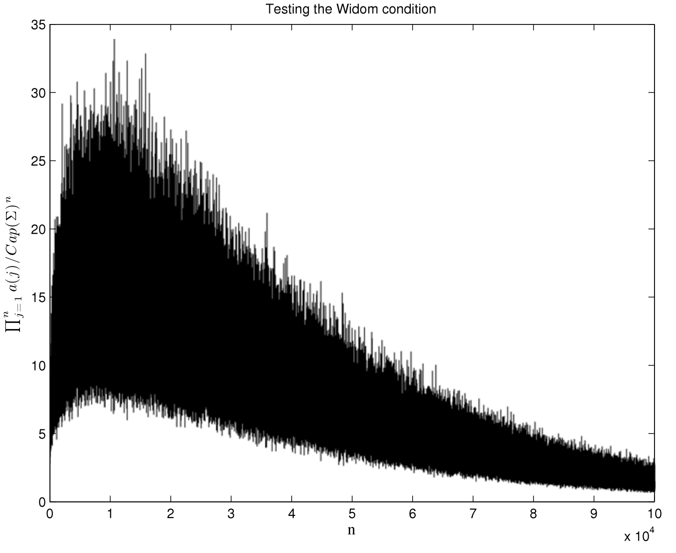

Fig. 3.4. Widom ratio for $1 \leq n \leq 100{,}000$.

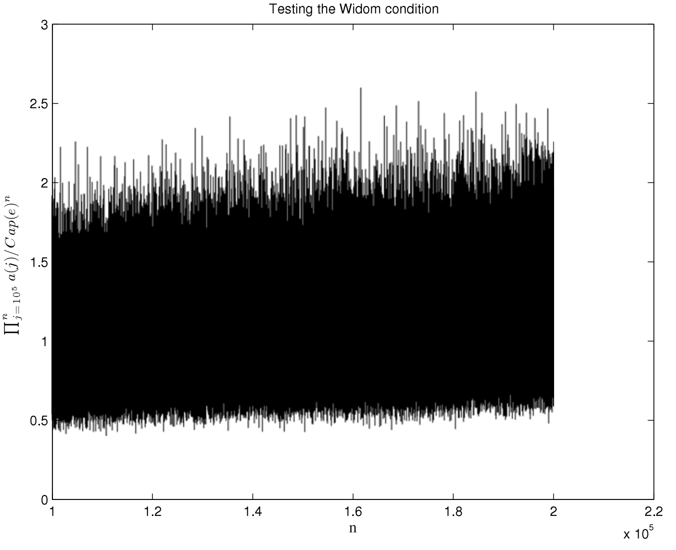

Fig. 3.5. Widom ratio for $100{,}000 \leq n \leq 200{,}000$.

For our numerics, we obtain $C(\mathfrak{e})$ as $(a_1 \dots a_N)^{1/N}$ (N being the largest n), so at $n = N$, W_n is 1. Thus, even if, say, $W_n \to 0$ for the true $C(\mathfrak{e})$, in our approximation it will not. Instead, as N increases, the small n values will be large. Fig. 3.4 shows the ratio for $N = 100{,}000$, that is, $1 \leq n \leq 100{,}000$ with $C(\mathfrak{e})$ defined so that $W_{100{,}000} = 1$.

It was not clear to us if this plot looked like that since it was indicating that $W_n \to 0$ or was an artifact of small n, so we plotted the ratio for $100{,}000 \leq n \leq 200{,}000$ as seen in Fig. 3.5 which we regard as support for Conjecture 3.2.

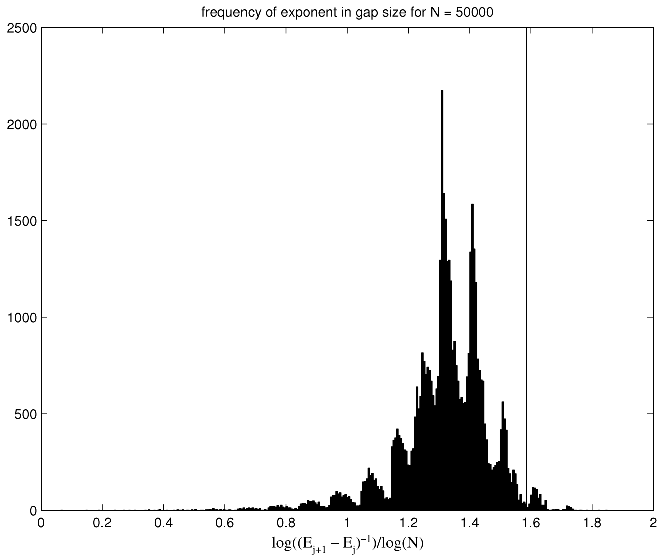

Fig. 3.6. Plot of zero gap exponent for $p_{50,000}$.

We should report on more results from our numeric explorations. For both random and a.c. cases, one knows the zeros of p_n scaled to scale $1/n$ have limiting distribution (see, e.g., [25] and references therein). For Cantor polynomials, one expects a small fraction of neighboring zero pairs to cross a gap of size much larger than $O(1/n)$. The others one might expect should be viewed on a scale $1/n^d$, where d is a suitable Hausdorff dimension, perhaps the dimension of the equilibrium measure. So we plotted for fixed n, $\log(E_{j+1}^{(n)} - E_j^{(n)})/\log(1/n)$ to see if they cluster near a single value. They did not (see Fig. 3.6), and were spread in a wide range (although bounded from below consistently with Theorem 2.5 and mainly bounded from above by Hausdorff dimension of the Cantor set). The issue of the distribution of zeros remains a mystery for now. The bar in Fig. 3.6 is $1/d_C$, the inverse of the Hausdorff dimension of C. $1/d_E$, the inverse of the Hausdorff dimension of the IDS, is even higher.

4. Isospectral tori

One possible dream one might have about the spectral theory of orthogonal polynomials on Cantor sets is that one might be able to generalize the theory of finite gap sets. Here

$$\mathfrak{e} = [\alpha_0, \beta_0] \cup \cdots \cup [\alpha_g, \beta_g] \tag{4.1}$$

consists of finitely many intervals separated by g gaps $\{(\beta_{j-1}, \alpha_j)\}_{j=1}^g$. A key fact in the theory of these operators is played by the isospectral torus

$$\mathcal{T}_\mathfrak{e}^{\text{refl}} = \{J : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}), \sigma(J) = \mathfrak{e} \text{ is reflectionless}\}. \tag{4.2}$$

We recall that J is reflectionless if the diagonal elements of its Green's function have vanishing real part on the spectrum

$$\lim_{\varepsilon \to 0^+} \operatorname{re}(\langle \delta_0, (J - E - \mathrm{i}\varepsilon)^{-1} \delta_0 \rangle) = 0, \quad \text{for almost every } E \in \mathfrak{e}. \tag{4.3}$$

We now list the key properties of this isospectral torus.

(1) $\mathcal{T}_{\mathfrak{e}}^{\mathrm{refl}}$ is homomorphic to a g dimensional torus $\mathbb{T}^g$.
(2) This homomorphism can be described as follows. Consider the restrictions $J^{\pm}$ of J to $\ell^2(\{\pm 1, \pm 2, \pm 3, \dots\})$. Then one has that either J^+ or J^- has an eigenvalue in each gap (β_{j-1}, α_j). Under appropriate identification of the endpoints of these two intervals, one obtains an explicit homomorphism.
(3) The elements J of $\mathcal{T}_{\mathfrak{e}}^{\mathrm{refl}}$ are almost periodic.
(4) The spectrum of $J \in \mathcal{T}_{\mathfrak{e}}^{\mathrm{refl}}$ is purely absolutely continuous.
(5) $\mathcal{T}_{\mathfrak{e}}^{\mathrm{refl}}$ is closed under the shift map.

Given this object, one can start to wonder what could replace it for the Cantor set. For this, one needs to replace the condition of being reflectionless by something as any Jacobi operator with spectrum of zero Lebesgue measure satisfies this. We recall that the forward upper Lyapunov exponent for a Jacobi operator J is given by

$$\overline{L}(E) = \limsup_{N \to \infty} \frac{1}{N} \log \|T_n(E)\| \tag{4.4}$$

where $T_n(E)$ was defined in (2.16).[1] Furthermore, we call a whole line Jacobi operator $J : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ regular if

$$\lim_{|N-M| \to \infty} \prod_{n=N}^{M} a(n) = \mathrm{Capacity}(\sigma(J)). \tag{4.5}$$

This in particular implies that the spectral measures of the restrictions $J^{\pm}$ to the right and the left half axis are regular.

A Jacobi operator J is almost-periodic if its orbit under the shift map, see (4.12), is pre-compact in $\mathcal{J}_C$ for an appropriate $C > 1$.

Proposition 4.1. *Let $\mathfrak{e}$ be a finite gap set and J a whole line Jacobi operator with $\sigma(J) = \mathfrak{e}$. Then the following conditions are equivalent.*

(1) *J is reflectionless on $\mathfrak{e}$.*
(2) *J is almost-periodic and $\overline{L}(E) = 0$ for $E \in \mathfrak{e}$.*
(3) *J is almost-periodic and regular.*

In order to prove this proposition, we recall a few things about almost-periodic Jacobi operators. Given an almost-periodic Jacobi operator J, we denote by $\Omega \subseteq \mathcal{J}_C$ the closure of its translates $\{J_n\}_{n \in \mathbb{Z}}$. We can define an additive structure on $\{J_n\}_{n \in \mathbb{Z}}$ by $J_n \oplus J_m = J_{n+m}$ with identity element $J_0 = J$. By continuity, this additive structure extends to Ω and thus makes Ω into a compact abelian group. We denote by μ the unique invariant Haar measure and note that (Ω, S, μ) is an uniquely ergodic dynamical system with S the shift defined in (4.12). For background, including Kotani and Remling's theorems, see Chapter 7 of Simon [27].

Proof of Proposition 4.1. A reflectionless Jacobi operator has purely absolutely continuous spectrum; in particular, its Lyapunov exponent vanishes. Thus (1) implies (2). In order to show that (2) implies (1), we first show that almost every Jacobi operator in the hull Ω of J is

[1] Note $\det(T_n(E))$ is not necessarily $= 1$ with our definition of $T_n(E)$.

reflectionless. For this observe that as (Ω, T) is uniquely ergodic, we also have that the ergodic Lyapunov exponent vanishes on $\mathfrak{e}$. Hence, the claim follows from Kotani's theorem. That this statement implies (1) follows from Remling's theorem.

The equivalence between (2) and (3) follows from Theorem B.1 in [15]. □

This proposition suggests the following definition of an isospectral torus, which we wish to nickname the almost-periodic isospectral torus

$$\mathcal{T}_{\mathfrak{e}}^{\mathrm{ap}} = \{J : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z}) : \ J \text{ is almost-periodic and regular } \sigma(J) = \mathfrak{e}\}. \tag{4.6}$$

The previous proposition shows that for $\mathfrak{e}$ a finite gap set, we have that

$$\mathcal{T}_{\mathfrak{e}}^{\mathrm{ap}} = \mathcal{T}_{\mathfrak{e}}^{\mathrm{refl}}. \tag{4.7}$$

For $\mathfrak{e}$ the Cantor set, we would have that Conjecture 3.1 would guarantee that $\mathcal{T}_{\mathfrak{e}}^{\mathrm{ap}}$ is non-empty. Unfortunately, we do not know how to achieve this.

We also wish to point out here that it is clear that if $J \in \mathcal{T}_{\mathfrak{e}}^{\mathrm{ap}}$ then its entire hull is already contained in $\mathcal{T}_{\mathfrak{e}}^{\mathrm{ap}}$. In particular, $\mathcal{T}_{\mathfrak{e}}^{\mathrm{ap}}$ is shift invariant.

In order to define an isospectral torus that is non-empty, we will need to discuss ergodic Jacobi operators, in a setting that is convenient for us. We first remark

Proposition 4.2. *Let J be a whole line Jacobi operator. Then*

$$|a(n)|, |b(n)| \leq \|J\| = \sup_{x \in \sigma(J)} |x|. \tag{4.8}$$

Proof. This follows from $a(n) = \langle \delta_n, J\delta_{n-1}\rangle$ and $b(n) = \langle \delta_n, J\delta_n\rangle$. □

This proposition implies that in order to define the isospectral torus, we may restrict ourself to Jacobi operators, where the coefficients satisfy

$$0 \leq a(n) \leq C, \quad -C \leq b(n) \leq C \tag{4.9}$$

for some $C > 0$. We now introduce

$$\mathcal{J}_C = \Big([0, C] \times [-C, C]\Big)^{\mathbb{Z}} \tag{4.10}$$

and identify $(a, b) \in \mathcal{J}_C$ with the Jacobi operator $J : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ with a, b as its recursion coefficients. We equip $\mathcal{J}_C$ with the usual topology of pointwise convergence, which is given by the metric

$$d((a, b), (\tilde{a}, \tilde{b})) = \sum_{n \in \mathbb{Z}} \frac{1}{2^{|n|}} \Big(|a(n) - \tilde{a}(n)| + |b(n) - \tilde{b}(n)|\Big), \tag{4.11}$$

and turns $\mathcal{J}_C$ into a compact topological space. Finally, we define the shift transformation $S : \mathcal{J}_C \to \mathcal{J}_C$ by

$$S(a, b)(n) = (a(n+1), b(n+1)). \tag{4.12}$$

This gives us a topological dynamical system $(\mathcal{J}_C, S)$. An inspection of the definition of $\mathcal{T}_{\mathfrak{e}}^{\mathrm{ap}}$ shows that we can extend this definition to subsets $A \subseteq \mathcal{J}_C$ such that (A, S) is minimal, i.e. for every $J \in A$ the orbit $\{S^n J\}_{n \in \mathbb{Z}}$ is dense in A.

We recall that μ a probability measure on $\mathcal{J}_C$ is ergodic if μ assigns weight 0 or 1 to an S invariant subset of $\mathcal{J}_C$.

Remark. One usually defines ergodic Jacobi operators by assuming that one is given a dynamical system (Ω, T, μ) and two bounded maps $f, g : \Omega \to \mathbb{R}$. Then one defines for $\omega \in \Omega$ the Jacobi operator J_ω as having recursion coefficients $a_\omega(n) = f(T^n\omega)$, $b_\omega(n) = g(T^n\omega)$. This definition is equivalent to the one given above as on one hand one can take the dynamical system $(\mathcal{J}_C, S, \mu)$ with maps $f(a,b) = a(0)$, $g(a,b) = b(0)$. On the other hand, one can take as the probability measure the pushforward of μ under the map $\omega \mapsto J_\omega$.

Let μ be a probability measure on $\mathcal{J}_C$. We denote by $\text{supp}(\mu)$ the support of μ, which is the smallest closed subset of $\mathcal{J}_C$ such that μ assigns 0 weight to its complement. We define

$$\log A(\mu) = \int \log(a(J;0))d\mu(J) \tag{4.13}$$

which is equal to $\lim_{N\to\infty} \frac{1}{N}\sum_{n=0}^{N-1} \log(a(J;n))$ for μ almost-every J. For $\log A(\mu) > -\infty$, we define the Lyapunov exponent

$$L_\mu(E) = \lim_{N\to\infty} \frac{1}{N} \int \log \|T(J;E,n)\| d\mu(J). \tag{4.14}$$

We know that $L_\mu(E)$ is a subharmonic function of E.

We denote by $\mathcal{M}^1_{e,+}$ the set of all ergodic probability measures on $\mathcal{J}_C$ such that $A(\mu) > 0$.

Theorem 4.3. *Let* $\mu \in \mathcal{M}^1_{e,+}$. *Then*

(1) *There exists a compact, perfect set* $\Sigma \subseteq \mathbb{R}$ *such that for* μ *almost every* J, *we have*

$$\sigma(J) = \Sigma. \tag{4.15}$$

Similar claims hold for the absolutely continuous, singular continuous, and pure point spectrum.

(2) *Let* $E \in \mathbb{R}$, *then for* μ *almost every* J, *we have*

$$L_\mu(E) = \overline{L}(J;E). \tag{4.16}$$

(3) *In particular, either* μ *almost every* J *is regular, or not.*

(4) *We have*

$$\Sigma_{\text{ac}} = \overline{\{E : L_\mu(E)\}}^{\text{ess}} \tag{4.17}$$

where the essential closure of a set A *is given by*

$$\overline{A}^{\text{ess}} = \{E : \forall\varepsilon > 0\ |A \cap [E-\varepsilon, E+\varepsilon]| > 0\}. \tag{4.18}$$

Proof. (1), (2), and (4) are standard results in Kotani theory. For (3) we first note that $L(E)$ being upper semi-continuous implies that $L(E)$ is continuous at the points where $L(E) = 0$. Thus almost every J being regular is equivalent to

$$L(E) = 0, \quad \text{for every } E \in \Sigma.$$

This statement clearly either holds or not. □

Given this theorem it makes sense to define

$$\mathcal{T}^{\text{erg}}_{\mathfrak{e}} = \bigcup_{\substack{\mu\in\mathcal{M}^1_{e,+},\ \Sigma=\mathfrak{e} \\ \mu \text{ a.e. } J \text{ is regular}}} \text{supp}(\mu). \tag{4.19}$$

One has that $\mathcal{T}_{\mathfrak{e}}^{\text{erg}}$ is a shift-invariant subset. Using standard facts about almost-periodic functions, one can show that

$$\mathcal{T}_{\mathfrak{e}}^{\text{ap}} \subseteq \mathcal{T}_{\mathfrak{e}}^{\text{erg}}. \tag{4.20}$$

Furthermore, Proposition 4.1 implies that for $\mathfrak{e}$ a finite gap set, we have that $\mathcal{T}_{\mathfrak{e}}^{\text{erg}} = \mathcal{T}_{\mathfrak{e}}^{\text{refl}}$.

Unfortunately, we still have no way to ensure that this set is non-empty. In order to do this, we need to define the following variant of $\mathcal{J}_C$ by

$$\mathcal{J}_{C,-} = \left(\left[\frac{1}{C}, C\right] \times [-C, C]\right)^{\mathbb{Z}}. \tag{4.21}$$

It is clear how to make it into a topological space and define the shift map and the notion of ergodicity of it.

Given a compact set $\mathfrak{e} \subseteq \mathbb{R}$, there is an unique measure $\rho_{\mathfrak{e}}$ called the equilibrium measure minimizing $\mathcal{E}(\eta) = \int \log(|z-w|^{-1}) d\eta(z) d\eta(w)$ over all probability measures η supported on $\mathfrak{e}$. The set $\mathfrak{e}$ is called potentially perfect if $\mathfrak{e} = \operatorname{supp}(\rho_{\mathfrak{e}})$. By Corollary 5.5.13 in [27], we have that these are exactly the measures for which $\overline{L}(E) = 0$ for quasi-every E is meaningful.

There exists now a half-line operator H such that its spectral measure is equal to $\rho_{\mathfrak{e}}$. If we knew for this operator that $a(n) \geq \frac{1}{C}$ for some $C > 1$, we could use the following theorem to show that the set $\mathcal{T}_{\mathfrak{e}}^{\text{erg}}$ is non-empty.

Theorem 4.4. *Let H be a half-line Jacobi operator with $\sigma(H) = \mathfrak{e}$ that is regular and satisfies $C^{-1} \leq a(n) \leq C$, $|b(n)| \leq C$. Then there exists an ergodic measure μ on $\mathcal{J}_{C,-}$ such that μ almost every J is regular and satisfies $\sigma(J) = \mathfrak{e}$.*

Proof. This is a consequence of Theorem 2.1 in [15]. Let μ be as in (2.13). Then we have for μ almost-every J that it is reflectionless on $\mathfrak{e}$. As its spectrum must be contained in $\mathfrak{e}$, the claim follows. □

In particular, this implies

Corollary 4.5. *Let $\mathfrak{e}$ be a potentially perfect set and assume that* $\inf_{n\geq 1} a(\rho_{\mathfrak{e}}; n) > 0$. *Then*

$$\mathcal{T}_{\mathfrak{e}}^{\text{erg}} \neq \emptyset. \tag{4.22}$$

Unfortunately, showing that $\inf_n a(n) > 0$ seems quite challenging if one does not assume the existence of the absolutely continuous spectrum.

For the sake of completeness, we wish to point out here that regularity implies that $\frac{1}{N}\sum_{n=1}^{N} \log a(n) \geq -C$ for some $C > 0$ and all $N \geq 1$. Thus one obtains the weak lower bound $a(n) \geq \mathrm{e}^{-Cn}$ or in other words that the $a(n)$ are at most exponentially decaying (on a subsequence). The average behavior is of course much better.

Finally, one can use (2.8) to show that for the Cantor polynomials, one has a polynomial lower bound on $a(n)$.

A final problem with the definition of $\mathcal{T}_{\mathfrak{e}}^{\text{erg}}$ is that, we cannot guarantee that it is a closed set. Fortunately, we have that

Proposition 4.6. *The closure of $\mathcal{T}_{\mathfrak{e}}^{\text{erg}}$ is the set*

$$\mathcal{T}_{\mathfrak{e}}^{\text{shift}} = \bigcup_{\substack{\mu \text{ shift invariant} \\ \mu \text{ a.e. } J, \sigma(J) = \mathfrak{E}, \text{ and regular}}} \operatorname{supp}(\mu). \tag{4.23}$$

Proof. As every ergodic measure is shift invariant, we clearly have that

$$\mathcal{T}_{\mathfrak{e}}^{\text{erg}} \subseteq \mathcal{T}_{\mathfrak{e}}^{\text{shift}}$$

and thus $\overline{\mathcal{T}_{\mathfrak{e}}^{\text{erg}}} \subseteq \mathcal{T}_{\mathfrak{e}}^{\text{shift}}$. The other conclusion follows from Choquet's theory, i.e. that given a shift-invariant measure μ, there exists a measure α on the ergodic measures such that $\mu = \int \beta d\alpha(\beta)$. In particular that

$$\text{supp}(\mu) = \overline{\bigcup_{\beta \in \text{supp}(\alpha)} \text{supp}(\beta)}.$$

By Theorem 2.1 in [15] and $L_\beta(E) \geq 0$, we have that for α almost every β, we have that $L_\beta(E) = 0$ for $E \in \mathfrak{e}$. Thus for α almost every β, we have that $\text{supp}(\beta) \subseteq \mathcal{T}_{\mathfrak{e}}^{\text{erg}}$. Now, the claim follows. □

At the end of this section, we wish to discuss some further problems. We have already noted that for almost-every J the Lyapunov exponent vanishes. In light of our results of the growth on the transfer matrices, one might conjecture that for almost every $J \in \mathcal{T}_{\mathfrak{e}}$ one has that the transfer matrices grow subexponentially, i.e.

$$\|T(J; E, n)\| \leq C n^{\alpha} \tag{4.24}$$

for some $C, \alpha > 0$. However, this might be too strong of a conjecture as such a result might well only hold almost everywhere with respect to the spectral measure of J.

5. Gap labeling and Borg's theorem

So far we have discussed how to define the isospectral torus. Now we wish to discuss some properties of it. As $\mathfrak{e} \subseteq \mathbb{R}$ is a compact set, we have that

$$\mathfrak{e} = [a, b] \setminus \bigcup_{j \in J} I_j \tag{5.1}$$

for disjoint open intervals I_j and $[a, b]$ the convex hull of $\mathfrak{e}$. The set J is always countable. In the cases we consider, we usually have that J is infinite. Given the interval $I_j = (a_j, b_j)$, we define a torus $\mathbb{T}_j$ as the disjoint union of two copies $(\overline{I_j}, +)$ and $(\overline{I_j}, -)$ of the closure of I_j, where we identify the endpoints $(a_j, +)$ and $(a_j, -)$ and similarly $(b_j, +)$ with $(b_j, -)$.

Theorem 5.1. *There exists a map*

$$D : \mathcal{T}_{\mathfrak{e}} \to \{\mathbb{T}_j\}_{j \in J} \tag{5.2}$$

such that if $D(J)_j = (E, \pm)$ *with* $E \in (a_j, b_j)$*, then*

$$E \in \sigma(J_\pm). \tag{5.3}$$

Proof. We recall that for J, we have the diagonal Green's function

$$g(z) = \langle \delta_0, (J - z)^{-1} \delta_0 \rangle \tag{5.4}$$

and we have defined the two $m_\pm$ functions given by

$$m_\pm(z) = \langle \delta_{\pm 1}, (J_\pm - z)^{-1} \delta_{\pm 1} \rangle. \tag{5.5}$$

A computation shows that

$$g(z) = \frac{1}{b_0 - z - (a_0)^2 m_+(z) - (a_{-1})^2 m_-(z)}. \tag{5.6}$$

As g is a Herglotz function, we have $g(z) = \int \frac{d\gamma(t)}{t-z}$. In particular, $g'(z) = \int \frac{d\gamma(t)}{(t-z)^2}$ and thus g is a strictly increasing function in the gaps I_j. In particular, g can have at most one zero $E \in I_j$. The only way $g(E) = 0$ can happen is that either $m_+(E)$, $m_-(E)$, or both are infinite. As $J = J_+ + J_+ +$ rank one, we have that $m_+(E), m_-(E)$ cannot be both equal to 0. Hence, we can define

$$D(J)_j = (E, \pm) \quad \text{if } E \in I_j,\ g(E) = 0,\ E \in \sigma(J_\pm). \tag{5.7}$$

Finally, if $g(E) \neq 0$ for all $E \in I_j$, we have by continuity of g that either $g(E) > 0$ or $g(E) < 0$ for all $E \in I_j$. If $g(E) > 0$, we set $D(J) = a_j$ and if $g(E) < 0$, we set $D(J) = b_j$. □

It is clear that $D_j : \mathcal{T}_{\mathfrak{e}} \to \mathbb{T}_j$ is continuous whenever $D_j(J) = (E, \pm)$ for $E \in I_j$. One can finally show that this map is continuous.

The map D plays an essential role in the inverse spectral theory for finite gap matrices, as it gives the explicit identification of the isospectral problem with a finite dimensional torus. It would be nice to be able to verify this is the case here. Unfortunately, it is neither clear that the map D is surjective or injective.

As the equilibrium measure $\rho_{\mathfrak{e}}$ is supported on $\mathfrak{e}$. We may now define a label function by

$$\ell(I_j) = \rho_{\mathfrak{e}}((-\infty, E)), \quad E \in I_j. \tag{5.8}$$

This definition makes sense since the right hand side is constant in $E \in I_j$. A fundamental theorem, see for example Section 4.1 in [5], now states

Theorem 5.2 (*Gap Labeling*)**.** *Denote by A the C^* algebra obtained from μ. Then*

$$\{\ell(I_j)\}_j \subseteq \tau(K_0(A)). \tag{5.9}$$

We remark on several variations on what we wish to dub Borg's theorem.

Theorem 5.3. *Let $\mu \in \mathcal{M}^1_{e,+}$ such that μ almost every J is regular and assume that $\Sigma = \sigma(J)$ is an interval. Then μ assigns weight to a single element $J = (a, b)$ with a and b constant sequences.*

Proof. As we have demonstrated the isospectral torus agrees for finite gap sets with the reflectionless one. Now, this follows from the standard result. □

It would be nice to have some generalization of this to the situation where the spectrum contains gaps. We first recall the notion of the frequency module $\mathcal{M}(f)$ of an almost-periodic function f. We define

$$\mathcal{M}(f) = \mathbb{Z} - \text{module generated by} \left\{ \omega : \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} f(n) e^{2\pi i n\omega} \neq 0 \right\}. \tag{5.10}$$

We note that for almost periodic functions, this frequency module is exactly the set of allowed labels. We conjecture

Conjecture 5.4. *Let J be an almost-periodic Jacobi operator. Then its frequency module equals the module generated by the labels.*

A theorem of Borg asserts that this is true when the frequency module is finite, i.e. that J is a periodic Jacobi operator.

6. Perturbations of the isospectral torus

In the previous section, we discussed how to define an isospectral torus $\mathcal{T}_{\mathfrak{e}}$ for fairly general sets $\mathfrak{e}$. In this section, we wish to discuss some consequences of this definition. We have already used the following result to compute the capacity of the Cantor set. We say that a statement holds for almost every $J \in \mathcal{T}_{\mathfrak{e}}$ if it holds outside a set A such that every ergodic measure assigns zero weight to it.

Theorem 6.1. *For almost every $J \in \mathcal{T}_{\mathfrak{e}}$, we have that*

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n=0}^{N-1} \log(a(n)) = \log(\text{Capacity}(\mathfrak{e})). \tag{6.1}$$

Proof. For each ergodic measure $\int \log(a(J;0))d\mu(J) = \log(\text{Capacity}(\mathfrak{e}))$. Hence, the claim follows from the ergodic theorem. □

We note that in order to compute the capacity of the Cantor set from this statement, one would need to quantify the rate of convergence. For this, we note

Proposition 6.2. *Let $f : \Omega \to \mathbb{R}$ be an integrable function, and (Ω, T, μ) ergodic. Then*

$$\sum_{n=0}^{N-1} f(T^n\omega) - N\int f d\mu = O(1) \tag{6.2}$$

for almost every ω if and only if there exists an integrable function $g : \Omega \to \mathbb{R}$ such that

$$f(\omega) - \int f d\mu = g(T\omega) - g(\omega) \tag{6.3}$$

that is f is a coboundary.

Proof. That the second statement implies the first is straightforward. For the other direction, define g as a limit point of $\sum_{n=0}^{N-1} f(T^n\omega) - N\int f d\mu$ as $N \to \infty$. □

We thus see that the strong sense of convergence claimed above relies on our conjecture that the quantity a_N is almost-periodic. We should also note here that

Proposition 6.3. *Let $J \in \mathcal{T}_{\mathfrak{e}}$ and $H = J^+ + A$, where A decays to zero. Then if* $\inf_{n\geq 0} a(J;n) > 0$

$$\frac{1}{N}\sum_{n=1}^{N} \log(a(J;n)) - \frac{1}{N}\sum_{n=1}^{N} \log(a(H;n)) = o(1). \tag{6.4}$$

Proof. Follows as $a(J;n) - a(H;n) \to 0$. □

A second topic, we wish to discuss in this section is if one could define the isospectral torus $\mathcal{T}_{\mathfrak{e}}$ in terms of half-line operators. The first thing to note is that the splitting

$$J = J^+ \oplus J^- + \text{ rank one} \tag{6.5}$$

has minor effects on the spectral properties of J if J has absolutely continuous spectrum. This follows from the invariance of the absolutely continuous spectrum under trace class perturbations. The same is true for the essential spectrum and regularity. In particular, we have that our definition of $\mathcal{T}_{\mathfrak{e}}$ could be written the same way for half-line operators.

However, the following theorem shows that this must not be the case for the spectral measures.

Theorem 6.4. *Let μ be the Cantor measure and H the corresponding Jacobi operator. Then for $H_\beta = H + \beta\langle\delta_1, .\rangle\delta_1$, we have that the spectral measure is pure point with the pure points lying outside the Cantor set.*

Proof. For $m_\beta(z) = \langle\delta_1, (H_\beta - z)^{-1}\delta_1\rangle$, we have that

$$m_\beta(z) = \frac{m_0(\beta)}{1 - \beta m_0(z)}.$$

As $\text{Im}(m_0(z)) \to \infty$ on the Cantor set, the claim follows. □

Other properties are however stable under this type of perturbations. One example is the growth of transfer matrices.

Another interesting question is how to extend the Szegő class and similar objects to the Cantor set and more general sets. For example it is completely, unclear how to characterize the set of spectral measures μ_{J^+} for $J \in \mathcal{T}_{\mathfrak{e}}$, where we recall that $\langle\delta_1, (J^+ - z)^{-1}\delta_1\rangle = \int \frac{d\mu_{J^+}(t)}{t-z}$. As we discuss in the next section it is not even clear what dimension the measures μ_{J^+} have.

Finally, we remark that it might be useful to relax this problem a little bit and instead of trying to characterize all J such that $J \in \mathcal{T}_{\mathfrak{e}}$ to characterize the J such that

$$\sum_{m=1}^{\infty} d(S^m J, \mathcal{T}_{\mathfrak{e}})^2 < \infty. \tag{6.6}$$

For comparison, in the case of periodic Jacobi operators one has that this condition leads to the conditions

(1) $\text{supp}_{\text{ess}}(\mu_{J^+}) = \mathfrak{e}$.
(2) $\sum_{E \in \text{supp}(\mu_{J^+}) \setminus \mathfrak{e}} \text{dist}(E, \mathfrak{e})^{\frac{3}{2}} < \infty$.
(3) $\int d(x, \mathbb{E} \setminus \mathfrak{e})^{\frac{1}{2}} \log(\frac{d\mu_{J^+}}{dx}) dx > -\infty$.

See Theorem 8.6.1 in [27], whereas the corresponding question for periodic Jacobi operators leads to an implicit equation for the m function $m(z) = \int \frac{d\mu_{J^+}(t)}{t-z}$ given by

$$\alpha(z)m(z)^2 + \beta(z)m(z) + \gamma(z) = 0 \tag{6.7}$$

where α, β, and γ are polynomials, see Theorem 5.2.1 in [27].

7. Dimension of the spectrum

Recall that one defines the dimension of a measure μ by

$$\dim(\mu) = \inf_{A \text{ Borel}, \ \mu(\mathbb{R} \setminus A) = 0} \dim_{\text{Hausdorff}}(A). \tag{7.1}$$

One can show that for $\mathfrak{e}$ a finite-gap set, one has that the Hausdorff dimension of $\mathfrak{e}$ agrees with the dimension of the equilibrium measure $\rho_\mathfrak{e}$ and the dimension of the spectral measure of the Jacobi operators $J \in \mathcal{T}_\mathfrak{e}$.

For $\mathfrak{e}$ the usual one third Cantor set a theorem of Makarov [17] shows that

$$\dim(\rho_\mathfrak{e}) < \dim(\mathfrak{e}). \tag{7.2}$$

In particular, one might conjecture from this that also the other inequality fails, i.e. that for $J \in \mathcal{T}_\mathfrak{e}$, we have that the dimension of the spectral measure is strictly smaller than $\dim(\rho_\mathfrak{e})$.

This might seem contradictory to our claim that the Cantor polynomials should be asymptotic to an object in $\mathcal{T}_\mathfrak{e}$ as we have that the dimension of the Cantor measure agrees with the dimension of the Cantor set. This is not a problem as a decaying perturbation might well be increasing the dimension.

The definition given here does not allow us to compute the Hausdorff dimension of a measure. In order to do this, we note that for measures μ of exact dimension α, one has that

$$\lim_{r\to\infty} \frac{\log(\mu([x-r, x+r]))}{\log(2r)} = \alpha, \quad \text{for } \mu \text{ almost every } x. \tag{7.3}$$

By standard convergence theorems, this implies that

$$\lim_{r\to\infty} \int \frac{\log(\mu([x-r, x+r]))}{\log(2r)} d\mu(x) = \dim_{\text{Hausdorff}}(\mu). \tag{7.4}$$

In the case of $\mathfrak{e}$ being the middle third Cantor set, we can write $\mathfrak{e} = \bigcap_{\ell=1}^\infty \mathfrak{e}_\ell$ with $\mathfrak{e}_\ell$ consisting of 2^ℓ disjoint intervals of length $3^{-\ell}$. Furthermore, these bands are at least $3^{-\ell}$ apart. Letting $\mathfrak{e}_\ell = \bigcup_{j=1}^{2^\ell} [a_j, b_j]$, we thus see that

$$\int \frac{\log(\rho_\mathfrak{e}([x-r, x+r]))}{\log(2r)} d\rho_\mathfrak{e}(x) = \sum_{j=1}^{2^\ell} \frac{\log(\alpha_j)}{\log(2\cdot 3^{-\ell})}\alpha_j \tag{7.5}$$

where $\alpha_j = \rho_\mathfrak{e}([a_j, b_j])$. In our computations, we approximate $\rho_\mathfrak{e}(a, b)$ by

$$\tilde{\rho}([a, b]) = \frac{1}{N}\mathrm{Tr}\left(\#\{\text{eigenvalues of } H^{[1,N]} \text{ in } [a, b]\}\right). \tag{7.6}$$

In order to ensure that $\sum_j \alpha_j = 1$, we thus take $\alpha_j = \tilde{\rho}([\frac{b_{j-1}+a_j}{2}, \frac{b_j+a_{j+1}}{2}))$.

We note the following result.

Theorem 7.1. *Let $\mathfrak{e}$ be potentially perfect. Then we have for almost every J that*

$$\dim_{\text{Hausdorff}}(\mu_J) \le \dim_{\text{Hausdorff}}(\rho_\mathfrak{e}) \le \dim_{\text{Hausdorff}}(\mathfrak{e}). \tag{7.7}$$

Here $2\int \frac{d\mu_J(t)}{t-z} = \langle \delta_0, (J-z)^{-1}\delta_0\rangle + \langle \delta_1, (J-z)^{-1}\delta_1\rangle$.

Proof. The second inequality is trivial as $\operatorname{supp}(\rho_\mathfrak{e}) = \mathfrak{e}$. For the first inequality observe that for β an involved ergodic measure, we have that

$$\int \mu_J(A) d\beta(J) = \rho_\mathfrak{e}(A)$$

for any Borel set A. Thus if $\rho_\mathfrak{e}(\mathbb{R} \setminus A) = 0$, we have for μ almost every J that $\mu_J(\mathbb{R} \setminus A) = 0$. Thus the claim follows. □

Table A.1
The capacity of the Cantor measure.

N	$\prod_{n=1}^{N} a(n)$
100	0.677875
1 000	0.664837
10 000	0.663075
100 000	0.662875

Table A.2
IDS on gaps.

N	$\rho_{\mathfrak{e}}(\frac{1}{27}, \frac{2}{27})$	$\rho_{\mathfrak{e}}(\frac{1}{9}, \frac{2}{9})$	$\rho_{\mathfrak{e}}(\frac{7}{27}, \frac{8}{27})$
100	0.171717	0.282828	0.383838
1 000	0.170170	0.287287	0.387387
10 000	0.170217	0.287229	0.387239
100 000	0.170242	0.287263	0.387264

Thus, we see that Makarov's theorem shows that in most cases the second inequality is strict and we conjectured the first one to be strict.

We note the following case, where one shows something resembling the first inequality.

Theorem 7.2. *Assume that $L_\mu(E) > 0$ for every E. Then for μ almost every J, there exists a Borel set A such that*

$$\mu_J(\mathbb{R} \setminus A) = 0, \qquad \rho_{\mathfrak{e}}(A) = 0 \tag{7.8}$$

that is the two measures are singular.

Proof. By Theorem 7.3 in [26], we have that μ_J is supported on a set of capacity zero. Now, we know that $\rho_{\mathfrak{e}}$ assigns zero measure to such sets. □

It should be possible to weaken this result to fast enough polynomial growth of the transfer matrices. However, even weakening this to $L(E) > 0$ for almost every E seems beyond reach.

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Appendix. Some numerical results

The main goal of this section is to provide tables of numerical data for various computations. We refer to Section 3 for some details on how these computations were done. (see Tables A.1–A.3)

We briefly discuss how to compute the Fourier coefficients. Consider the sequence $a(n) = a \cdot e^{2\pi i k n}$ and define

$$\hat{a}_N(\ell) = \frac{1}{N} \sum_{n=0}^{N-1} a(n) e^{-2\pi i \frac{n\ell}{N}}.$$

Table A.3
A few Fourier coefficients.

k	$\lvert\hat{a}_{1000}(k)\rvert$	$\lvert\hat{a}_{10000}(k)\rvert$	$\lvert\hat{a}_{100000}(k)\rvert$
0.5	0.263657	0.265101	0.265856
0.28726	0.0827705	0.0828834	0.0829444
0.71274	0.0827705	0.0828834	0.0829444
0.17025	0.0307355	0.0305749	0.0306035
0.38729	0.0310281	0.0310683	0.0307494
0.61271	0.0310281	0.0310683	0.0307494
0.82975	0.0307355	0.0305749	0.0306035

Take $k = \frac{\ell+\varepsilon}{N}$ for $\ell \in \mathbb{Z}$ and $\epsilon \in [0, 1)$. Then a quick computation yields

$$|\hat{a}_N(\ell)| \approx \frac{|a \sin(\pi\epsilon)|}{\pi\epsilon}, \qquad |\hat{a}_N(\ell+1)| \approx \frac{|a \sin(\pi\epsilon)|}{\pi(1-\epsilon)}.$$

From these one computes

$$\epsilon = \frac{1}{1 + \frac{|\hat{a}_N(\ell)|}{|\hat{a}_N(\ell+1)|}}, \qquad |a| = |\hat{a}_N(\ell)| \cdot \frac{\pi\epsilon}{|\sin(\pi\epsilon)|}.$$

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