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A Cayley–Hamiltonian theorem for periodic finite band matrices

Barry Simon

I hope Pavel Exner will enjoy this birthday bouquet.

1 Introduction – The magic formula

Let J be a doubly infinite, self-adjoint, tridiagonal Jacobi matrix (i.e., $J_{jk} = 0$ if $|j-k| > 1$ and $J_{jj+1} > 0$) that is periodic, i.e., if

$$(Su)_j = u_{j+1}, \tag{1}$$

then for some $n \in \mathbb{Z}_+$, $S^n J = JS^n$. There is a huge literature on this subject – see Simon [7], Chapter 5.

$(J-E)u = 0$ is a second order difference equation, so there is a linear map $T(E) : \mathbb{C}^2 \to \mathbb{C}^2$ so that if u_0, u_1 are given, then $T(E)\binom{u_0}{u_1} = \binom{u_n}{u_{n+1}}$ for the solution of $(J-E)u = 0$. $\Delta(E) = \operatorname{Tr}(M(E))$ is called the *discriminant* of J. We note that $\det(T(E)) = 1$ so $T(E)$ has eigenvalues λ and λ^{-1} and $\Delta(E) = \lambda + \lambda^{-1}$. If $\Delta(E) \in (-2,2)$, then $\lambda = e^{i\theta}$ for some θ in $\pm(0,\pi)$ and then $Ju = Eu$ has Floquet solutions, $u^{\pm}$ obeying $u^{\pm}_{j+nk} = e^{\pm ik\theta}u^{\pm}_j$. These are bounded and these are only bounded solutions if $\Delta(E) \in [-2,2]$. Thus $\operatorname{spec}(J) = \Delta^{-1}([-2,2])$. One often writes this relation as

$$\Delta(E) = 2\cos(\theta). \tag{2}$$

In [2], Damanik, Killip and Simon emphasized and exploited the operator form of (2), namely

$$\Delta(J) = S^n + S^{-n}. \tag{3}$$

This follows from (2) and the view of J as a direct integral. More importantly, what they called the "magic formula", [2] shows that a two sided, not *a priori* periodic, Jacobi matrix, which obeys (3), is periodic and in the isospectral torus of J.

A Laurent matrix is a finite band doubly infinite matrix that is constant along diagonals, so a polynomial in S and S^{-1}. $S^n + S^{-n}$ is an example of such a matrix. The current paper had its genesis in a question asked me by Jonathan Breuer and Maurice Duits. They asked if K is finite band and periodic but not tridiagonal if there is a polynomial Q so that $Q(K)$ is a Laurent matrix. They guessed that Q might be connected to the trace of a transfer matrix.

While I don't have a formal example where I can prove there is no such Q, I have found a related result which strongly suggests that, in general, the answer is no. I found an object which replaces Δ for more general K which is width $2m+1$ (i.e., $K_{jk} = 0$ if $|j-k| > m$), self-adjoint and non-degenerate in the sense that for all j, $K_{jj+m} \neq 0$. Namely we prove the existence of a polynomial, $p(x, y)$, in x and y of degree $2m$ in y, so that $p(K, S^n) = 0$. In the Jacobi case,

$$p(x, y) = y^2 - y\Delta(x) + 1$$

so that $p(J, S^n) = 0$ is equivalent to (3).

We prove this theorem and begin the exploration of this object in Section 1. That a scalar polynomial vanishes when the variable is replaced by an operator is the essence of the Cayley–Hamiltonian theorem which says that a matrix obeys its secular equation. This was proven in 1853 by Hamilton [4] for the two special cases of three-dimensional rotations and for multiplication by quaternions and stated in general by Cayley [1] in 1858 who proved it only for 2×2 matrices although he said he'd done the calculation for 3×3 matrices. In 1878, Frobenius [3] proved the general result and attributed it to Cayley. We regard our main result, Theorem 2.1, as a form of the Cayley–Hamiltonian Theorem.

The magic formula had important precursors in two interesting papers of Naĭman, namely [5] and [6]. These papers are also connected to our work here.

It is a pleasure to present this paper to Pavel Exner for his 70$^{\text{th}}$ birthday. I have long enjoyed his contributions to areas of common interest. I recall with fondness the visit he arranged for me in Prague. He was a model organizer of an ICMP. And he is an all around sweet guy. Happy birthday, Pavel.

2 Main result

By a width $2m+1$ matrix, $m \in \{1, 2, \dots\}$, we mean a doubly infinite matrix, K, with $K_{jk} = 0$ if $|j-k| > m$. If $\sup |K_{jk}| < \infty$, K defines a bounded operator on $\ell^2(\mathbb{Z})$ which we also denote by K. We say that K is *non-degenerate* if $K_{jj\pm m} \neq 0$ for all j. K is periodic (with period n) if $S^n K = K S^n$, where S is the unitary operator given by (1).

We consider width $2m+1$, non-degenerate, period-n self-adjoint matrices. In that case, for any E, because K is non-degenerate, $Ku = Eu$, as a finite difference equation, has a unique solution for each choice of $\{u_j\}_{j=0}^{2m-1}$. $T(E)$ will be defined as the map from $\{u_j\}_{j=0}^{2m-1}$ to $\{u_j\}_{j=n}^{n+2m-1}$ – it is a $2m \times 2m$, degree n matrix. If $T(E)u = \lambda u$ for $\lambda \in \mathbb{C}$, $Ku = Eu$ has a Floquet solution with $u_{kn+j} = \lambda^k u_j$. If $T(E)$ is diagonalizable, the set of Floquet solutions is a basis for all solutions of $Ku = Eu$. If $T(E)$ has Jordan anomalies (see [8] for background on linear algebra), there is a basis of modified Floquet solutions with some polynomial growth on top of the exponential λ^k.

The values of λ are determined by

$$p(E, \lambda) = \det(\lambda \mathbf{1} - T(E)).$$

Since

$$\begin{aligned} \det(\lambda \mathbf{1} - T(E)) &= \lambda^{2m} \det(\mathbf{1} - \lambda^{-1} T(E)) \\ &= \lambda^{2m} \Big(\sum_{j=0}^{2m} (-\lambda)^j \operatorname{Tr} \Big(\bigwedge^j (T(E)) \Big) \Big) \\ &= \sum_{j=0}^{2m} \lambda^j p_j(E), \end{aligned} \tag{4}$$

where $\bigwedge^j$ is given by multilinear algebra (Section 1.3 of [8]) with $\bigwedge^0(T(E)) = \mathbf{1}$ on $\mathbb{C}$ so its trace is 1. Thus in (4),

$$p_{2m}(E) = 1, \quad p_j(E) = (-1)^j \operatorname{Tr} \Big(\bigwedge^{2m-j} (T(E)) \Big) \tag{5}$$

and p_j is of degree at most $(2m-j)n$ in E.

Since S^n and K are commuting bounded normal operators, they have a joint spectral resolution which is supported precisely on the solutions of $p(E, \lambda) = 0$ with $|\lambda| = 1$ because it is well known that the spectrum is precisely the set of energies with polynomially bounded solutions. By the spectral theorem (equivalently, a direct integral analysis), we thus have the main result of this note:

Theorem 2.1. *Let K be a self-adjoint, non-degenerate, width $2m+1$, period n matrix. Then for p given by* (4)/(5), *we have that*

$$p(K, S^n) = 0. \tag{6}$$

We end with a number of comments.

(1) We used the self-adjointness of K to be able to exploit the spectral theorem. But just as the Cayley–Hamilton Theorem for finite matrices holds in the non-self-adjoint case, it seems likely that Theorem 2.1 is valid for general non-degenerate, periodic K, even if not self-adjoint.

(2) Since $K_{jj-m} \neq 0$, the transfer matrix, $T(E)$ is invertible and thus $\det(T(E))$ has no zeros. Since it is a polynomial, it must be constant, that is $p_0(E)$ is a constant. It is thus of much smaller degree than the bound, $2mn$, obtained by counting powers of E.

(3) In many cases of interest, $T(E)$ will be symplectic, i.e., there exists an antisymmetric Q on $\mathbb{C}^{2m}$ with $Q^2 = -\mathbf{1}$ so that $T(E)^t QT(E) = Q$. Such a $T(E)$ has $T(E)^{-1}$ and $T(E)^t$ similar, so the eigenvalues $\{\lambda_j\}_{j=1}^{2m}$ can be ordered so that $\lambda_{2m+1-j} = \lambda_j^{-1}$, $j = 1,\dots,m$. It follows that $\det(T(E)) = 1$ but even more, we have that

$$\begin{aligned}\operatorname{Tr}\Big(\bigwedge^k(T(E))\Big) &= \sum_{j_1<\dots<j_k} \lambda_{j_1}\dots\lambda_{j_k} \\ &= \sum_{j_1<\dots<j_{2m-k}} \lambda_{j_1}^{-1}\dots\lambda_{j_{2m-k}}^{-1} \qquad (7)\\ &= \sum_{j_1<\dots<j_{2m-k}} \lambda_{j_1}\dots\lambda_{j_{2m-k}} \qquad (8)\\ &= \operatorname{Tr}\Big(\bigwedge^{2m-k}(T(E))\Big)\end{aligned}$$

and $p_{2m-k}(E) = p_k(E)$. In the above, (7) follows from the fact that the product of all the λ's is 1, and we can sum over the complements of all k-sets. (8) then uses the fact that $\lambda_{2m+1-j} = \lambda_j^{-1}$, $j = 1,\dots,m$.

(4) One can ask whether there is a magic formula in this case, i.e., does $p(\widetilde{K}, S^n) = 0$ imply that $\widetilde{K}$ is periodic and isospectral to K. There is already one subtlety one faces at the start. If $\widetilde{K}$ is not supposed *a priori* n-periodic, then $S^{nj} p_j(\widetilde{K})$ may not equal $p_j(\widetilde{K})S^{nj}$ so there is a question of what $p(\widetilde{K}, S^n) = 0$ means. Even if one supposes that $\widetilde{K}S^n = S^n\widetilde{K}$, $p(\widetilde{K}, S^n) = 0$ and the spectral theorem only implies that $\operatorname{spec}(\widetilde{K}) \subset \operatorname{spec}(K)$, so there is more to be proven. Indeed, the isospectral set in this case remains to be explored.

(5) It seems unlikely that there is another independent relation besides (6) between a polynomial in K and Laurent polynomial in S. In general one cannot hope that $p(K, S^n) = 0$ yields a polynomial in one variable so that $Q(K)$ is a Laurent polynomial in S^n but it remains to find an explicit example where one can prove that the Breuer–Duits question has a negative answer.

There are lots of interesting open questions connected to our main result, Theorem 2.1.

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