A Cayley–Hamiltonian theorem for periodic finite band matrices

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I hope Pavel Exner will enjoy this birthday bouquet.

1 Introduction – The magic formula

Let \( J \) be a doubly infinite, self-adjoint, tridiagonal Jacobi matrix (i.e., \( J_{jk} = 0 \) if \( |j-k| > 1 \) and \( J_{j+1,j} > 0 \)) that is periodic, i.e., if

\[
(Su)_j = u_{j+1},
\]

then for some \( n \in \mathbb{Z}_+ \), \( S^n J = JS^n \). There is a huge literature on this subject – see Simon [7], Chapter 5.

\((J - E)u = 0\) is a second order difference equation, so there is a linear map \( T(E) : \mathbb{C}^2 \to \mathbb{C}^2 \) so that if \( u_0, u_1 \) are given, then \( T(E) \left( \begin{smallmatrix} u_0 \\ u_1 \end{smallmatrix} \right) \) for the solution of \((J - E)u = 0\). The discriminant of \( J \),

\( \Delta(E) = \text{Tr}(M(E)) \)

is called the discriminant of \( J \). We note that \( \det(T(E)) = 1 \) so \( T(E) \) has eigenvalues \( \lambda \) and \( \lambda^{-1} \) and \( \Delta(E) = \lambda + \lambda^{-1}. \) If \( \Delta(E) \in (-2, 2) \), then \( \lambda = e^{i\theta} \) for some \( \theta \) in \( \pm(0, \pi) \) and then \( J u = Eu \) has Floquet solutions, \( u^\pm \) obeying \( u_{j+nk}^\pm = e^{\pm ik\theta} u_j^\pm \). These are bounded and these are only bounded solutions if \( \Delta(E) \in [-2, 2] \). Thus \( \text{spec}(J) = \Delta^{-1}([-2, 2]) \). One often writes this relation as

\[
\Delta(E) = 2 \cos(\theta).
\]

In [2], Damanik, Killip and Simon emphasized and exploited the operator form of (2), namely

\[
\Delta(J) = S^n + S^{-n}.
\]

This follows from (2) and the view of \( J \) as a direct integral. More importantly, what they called the “magic formula”, [2] shows that a two sided, not \textit{a priori} periodic, Jacobi matrix, which obeys (3), is periodic and in the isospectral torus of \( J \).
A Laurent matrix is a finite band doubly infinite matrix that is constant along diagonals, so a polynomial in $S$ and $S^{-1}$, $S^n + S^{-n}$ is an example of such a matrix. The current paper had its genesis in a question asked me by Jonathan Breuer and Maurice Duits. They asked if $K$ is finite band and periodic but not triadiagonal if there is a polynomial $Q$ so that $Q(K)$ is a Laurent matrix. They guessed that $Q$ might be connected to the trace of a transfer matrix. While I don’t have a formal example where I can prove there is no such $Q$. I have found a related result which strongly suggests that, in general, the answer is no. I found an object which replaces $\Delta$ for more general $K$ which is width $2m + 1$ (i.e., $K_{jk} = 0$ if $|j - k| > m$), self-adjoint and non-degenerate in the sense that for all $j$, $K_{jk+m} \neq 0$. Namely we prove the existence of a polynomial, $p(x,y)$, in $x$ and $y$ of degree $2m$ in $y$, so that $p(K,S^n) = 0$. In the Jacobi case, 

$$p(x,y) = y^2 - y\Delta(x) + 1$$

so that $p(J,S^n) = 0$ is equivalent to (3).

We prove this theorem and begin the exploration of this object in Section 1. That a scalar polynomial vanishes when the variable is replaced by an operator is the essence of the Cayley–Hamiltonian theorem which says that a matrix obeys its secular equation. This was proven in 1853 by Hamilton [4] for the two special cases of three-dimensional rotations and for multiplication by quaternions and stated in general by Cayley [1] in 1858 who proved it only for $2 \times 2$ matrices although he said he’d done the calculation for $3 \times 3$ matrices. In 1878, Frobenius [3] proved the general result and attributed it to Cayley. We regard our main result, Theorem 2.1, as a form of the Cayley–Hamiltonian Theorem.

The magic formula had important precursors in two interesting papers of Naiman, namely [5] and [6]. These papers are also connected to our work here.

It is a pleasure to present this paper to Pavel Exner for his 70th birthday. I have long enjoyed his contributions to areas of common interest. I recall with fondness the visit he arranged for me in Prague. He was a model organizer of an ICMP. And he is an all around sweet guy. Happy birthday, Pavel.

2 Main result

By a width $2m + 1$ matrix, $m \in \{1, 2, \ldots \}$, we mean a doubly infinite matrix, $K$, with $K_{jk} = 0$ if $|j - k| > m$. If sup $|K_{jk}| < \infty$, $K$ defines a bounded operator on $\ell^2(\mathbb{Z})$ which we also denote by $K$. We know that $K$ is non-degenerate if $K_{jk+m} \neq 0$ for all $j$. $K$ is periodic (with period $n$) if $S^n K = KS^n$, where $S$ is the unitary operator given by (1).

We consider width $2m + 1$, non-degenerate, period-$n$ self-adjoint matrices. In that case, for any $E$, because $K$ is non-degenerate, $Ku = Eu$, as a finite difference equation, has a unique solution for each choice of $(u_j)_{j=0}^{2m+1}$. $T(E)$ will be defined as the map from $(u_j)_{j=0}^{2m+1}$ to $(u_j)_{j=0}^{2m+1}$ so it is $2m \times 2m$, degree $n$ matrix.

If $T(E)u = \lambda u$ for $\lambda \in \mathbb{C}$, $Ku = Eu$ has a Floquet solution with $u_{k+n} = \lambda^k u_j$. If $T(E)$ is diagonalizable, the set of Floquet solutions is a basis for all solutions of $Ku = Eu$. If $T(E)$ has Jordan anomalies (see [8] for background on linear algebra), there is a basis of modified Floquet solutions with some polynomial growth on top of the exponential $\lambda^j$.

The values of $\lambda$ are determined by $p(E, \lambda) = \det(\lambda I - T(E))$.

Since $\det(\lambda I - T(E)) = \lambda^{2m} \det(1 - \lambda^{-1} T(E))$

$$= \lambda^{2m} \left( \sum_{j=0}^{2m} (-\lambda)^j \text{Tr} \left( \frac{1}{2m-j} (T(E)) \right) \right)$$

$$= \sum_{j=0}^{2m} \lambda^j p_j(E),$$

where $\lambda^j$ is given by multilinear algebra (Section 1.3 of [8]) with $\lambda^n \lambda^{(n)}(T(E)) = 1$ on $\mathbb{C}$ so its trace is 1. Thus in (4),

$$p_{2m}(E) = 1, \quad p_j(E) = (-1)^j \text{Tr} \left( \frac{2m-j}{2m-j} (T(E)) \right)$$

and $p_j$ is of degree at most $(2m - j)n$ in $E$.

Since $S^n$ and $K$ are commuting bounded normal operators, they have a joint spectral resolution which is supported precisely on the solutions of $p(E, \lambda) = 0$ with $|\lambda| = 1$ because it is well known that the spectrum is precisely the set of energies with polynomially bounded solutions. By the spectral theorem (equivalently, a direct integral analysis), we thus have the main result of this note:

**Theorem 2.1.** Let $K$ be a self-adjoint, non-degenerate, width $2m + 1$, period $n$ matrix. Then for $p$ given by (4)/(5), we have that

$$p(K, S^n) = 0.$$
We used the self-adjointness of $K$ to be able to exploit the spectral theorem. But just as the Cayley–Hamilton Theorem for finite matrices holds in the non-self-adjoint case, it seems likely that Theorem 2.1 is valid for general non-degenerate, periodic $K$, even if not self-adjoint.

Since $K_{jj-w} \neq 0$, the transfer matrix, $T(E)$ is invertible and thus $\det(T(E))$ has no zeros. Since it is a polynomial, it must be constant, that is $\rho_0(E)$ is a constant. It is thus of much smaller degree than the bound, $2m_n$, obtained by counting powers of $E$.

In many cases of interest, $T(E)$ will be symplectic, i.e., there exists an antisymmetric $Q$ on $\mathbb{C}^{2m}$ with $Q^2 = -1$ so that $T(E)^{T} QT(E) = Q$. Such a $T(E)$ has $T(E)^{-1}$ and $T(E)^{T}$ similar, so the eigenvalues $\{\lambda_j\}_{j=1}^{2m_n}$ can be ordered so that $\lambda_{2m+1-j} = \lambda_j^{-1}$, $j = 1, \ldots, m$. It follows that $\det(T(E)) = 1$ but even more, we have that

$$\text{Tr} \left( \bigwedge^k (T(E)) \right) = \sum_{j_1 < \cdots < j_k} \lambda_{j_1} \cdots \lambda_{j_k}$$

$$= \sum_{j_1 < \cdots < j_{2m-k}} \lambda_{j_1}^{-1} \cdots \lambda_{j_{2m-k}}^{-1} \quad (7)$$

$$= \sum_{j_1 < \cdots < j_{2m-k}} \lambda_{j_1} \cdots \lambda_{j_{2m-k}} \quad (8)$$

$$= \text{Tr} \left( \bigwedge^{2m-k} (T(E)) \right)$$

and $p_{2m-k}(E) = p_k(E)$. In the above, (7) follows from the fact that the product of all the $\lambda$'s is 1, and we can sum over the complements of all $k$-sets. (8) then uses the fact that $\lambda_{2m+1-j} = \lambda_j^{-1}$, $j = 1, \ldots, m$.

One can ask whether there is a magic formula in this case, i.e., does $p(K, S^n) = 0$ imply that $K$ is periodic and isospectral to $K$. There is already one subtlety one faces at the start. If $K$ is not supposed a priori $n$-periodic, then $S^n p_j(K)$ may not equal $p_j(K) S^n$ so there is a question of what $p(K, S^n) = 0$ means. Even if one supposes that $K S^n = S^n K$, $p(K, S^n) = 0$ and the spectral theorem only implies that $\text{spec}(K) \subset \text{spec}(K)$, so there is more to be proven. Indeed, the isospectral set in this case remains to be explored.

It seems unlikely that there is another independent relation besides (6) between a polynomial in $K$ and Laurent polynomial in $S$. In general one cannot hope that $p(K, S^n) = 0$ yields a polynomial in one variable so that $Q(K)$ is a Laurent polynomial in $S^n$ but it remains to find an explicit example where one can prove that the Breuer–Duits question has a negative answer.

There are lots of interesting open questions connected to our main result, Theorem 2.1.

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References


