The $P(\phi)_2$ Euclidean quantum field theory as classical statistical mechanics

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Part I of this paper including the Bibliography appeared at the end of the preceding issue of this journal.

Part II

IV. Lattice Markov fields

To date in constructive field theory, two types of ultraviolet cutoffs have been used [32]:

1) A smeared field, i.e., $\phi(x)$ is replaced by $\phi_h(x) = \int h(x-y)\phi(y)dy$, where h is some smooth positive approximation to the delta function;

2) Box and sharp momentum cutoff, i.e., $\phi(x)$ is replaced by the periodic field $\phi_{K,V}(x)$, which is obtained by approximating the Fourier integral defining $\phi(x)$ by a finite Fourier sum.

In this section we wish to introduce a new cutoff method in which $\mathbf{R}^d$ (space-imaginary time) is replaced by a lattice and the Laplacian Δ in the Euclidean propagator is approximated by a finite difference operator. The effect of this approximation is to replace the measure (II.25) by a perturbed Gaussian measure in a finite number of variables, $q_1, \cdots, q_N$, each associated with a lattice point:

$$\text{(IV.1)} \qquad e^{-U(g)}d\mu_0 \longrightarrow \text{const.}\ e^{-\Sigma_n P_n(q_n)} \exp\left(-\frac{1}{2}\vec{q}\cdot B\vec{q}\right)d^N q\ .$$

where the P_n are semibounded polynomials and B is a positive-definite $N \times N$ matrix with nonpositive off-diagonal entries. There are two main advantages to this approximation which play a key role in our proof of correlation inequalities (§ V): First, it is *locality preserving* in the sense that $U(g)$ is approximated by a sum $\sum P_n(q_n)$ in which each term involves the field q_n at only a single lattice point. Secondly, the Gaussian exponential $e^{-1/2\vec{q}\cdot B\vec{q}}$ provides a direct analogy with the *ferromagnetic* Ising model. Moreover the field theory on the lattice satisfies the Markov property.

Lattice cutoff fields have been previously discussed (but for time-zero Hamiltonian theories) by Wentzel [115] and Schiff [87].

IV.1. *The Lattice Approximation.* Although the lattice approximation for the free theory makes sense for an arbitrary number of dimensions, we shall take $d = 2$. Let $\delta > 0$ be the spacing parameter for the lattice $L_\delta = \{n\delta \mid n = (n_1, n_2) \in \mathbf{Z}^2\}$ in $\mathbf{R}^2$. Our definition of the lattice cutoff field $\phi_\delta(n)$ is motivated by the following observation of Nelson (private communication): If we view $\phi(x)$ as a continuum labelled (formal) family of Gaussian random variables, the joint covariance "matrix", $\int \phi(x)\phi(y)d\mu_0$, is just the integral kernel of $(-\Delta + m^2)^{-1}$ (cf. (II.8)). Recall that in the formula for the joint density of a family of Gaussian random variables,

$$f(q_1, \cdots, q_N) = (2\pi)^{-N/2} \,|\, C \,|^{-1/2} \exp\left[-\frac{1}{2}(\vec{q} - \vec{m})\cdot C^{-1}(\vec{q} - \vec{m})\right], \tag{IV.2}$$

it is the inverse of the covariance matrix, $C_{jk} = \operatorname{Cov}(q_j, q_k)$, which appears in the exponent; here, $m_j = E(q_j)$ and $|\, C \,| = \det C$. Thus, formally, the free field measure is $e^{-1/2(\phi, A\phi)}d\phi$ where A is the operator $(-\Delta + m^2)$. From a formal point of view, $-\Delta$ is positive on-diagonal and negative infinitesimally off-diagonal, as is evident from the finite difference approximation to $-\Delta$:

$$(-\Delta_\delta f)(n\delta) = \delta^{-2}[4f(n\delta) - \textstyle\sum_{|n'-n|=1} f(n'\delta)] \tag{IV.3}$$

where we norm $\mathbf{Z}^2$ by $|(n_1, n_2)| = |n_1| + |n_2|$ so that the sum over $n' \in \mathbf{Z}^2$ takes place over the 4 nearest neighbours of n.

We now convert the above heuristic discussion to a precise one. Consider the Fourier transform from $l^2(\mathbf{Z}^2)$ to $L^2([-\pi/\delta, \pi/\delta]^2)$ defined by

$$\hat{h}(k) = \frac{\delta}{2\pi} \textstyle\sum_{n \in \mathbf{Z}^2} h(n)e^{-ik\cdot n\delta} . \tag{IV.4}$$

If we regard $A = (-\Delta_\delta + m^2)$ as an operator on $l^2(\mathbf{Z}^2)$ by means of (IV.3), then we see it is a convolution operator $(Ah)(n) = \sum a(n - n')h(n')$, where

$$a(n) = \begin{cases} m^2 + 4\delta^{-2} & n = (0, 0) \\ -\delta^{-2} & n = (\pm 1, 0), (0, \pm 1) \\ 0 & \text{otherwise} . \end{cases} \tag{IV.5}$$

Therefore, its image $\hat{A}$ on L^2 is multiplication by $(2\pi/\delta)\hat{a}(k)$; by a simple computation,

$$(2\pi/\delta)\hat{a}(k) = \delta^{-2}(4 - 2\cos(\delta k_1) - 2\cos(\delta k_2)) + m^2 \equiv \mu_\delta(k)^2 . \tag{IV.6}$$

We wish the cutoff field $\phi_\delta(n)$ to have covariance matrix $\delta^{-2}A^{-1}$; i.e.,

$$\int \phi_\delta(n)\phi_\delta(n')d\mu_0 = (2\pi)^{-2}\int_\delta e^{ik\cdot(n-n')\delta}\mu_\delta(k)^{-2}d^2k , \tag{IV.7}$$

where $\int_\delta$ means that the range of integration is $|k_i| \leqq \pi/\delta$. Accordingly we define the lattice cutoff field $\phi_\delta(n)$ in terms of the field $\phi(x)$ of (II.22) by $\phi_\delta(n) = \phi(f_{\delta,n})$, where

$$f_{\delta,n}(x) = (2\pi)^{-2}\int_\delta e^{ik\cdot(x-n\delta)}\mu(k)/\mu_\delta(k)d^2k .$$

Put differently, we have the first of the following definitions:

Definitions.

(i)

$$\phi_\delta(n) = (2\pi)^{-1}\int_\delta e^{-ik\cdot n\delta}[a^*(k) + a(-k)]\mu_\delta(k)^{-1}d^2k. \tag{IV.8}$$

(ii) $:\phi_\delta^r(n): = (2\pi)^{-r}\int_\delta e^{-i\delta n\cdot\Sigma k^{(j)}}\sum_{j=0}^r \binom{r}{j} a^*(k^{(1)})\cdots a^*(k^{(j)})$
$\cdot a(-k^{(j+1)})\cdots a(-k^{(r)})\prod_i \mu_\delta(k_i)^{-1}dk_i.$

(iii) If $g \in C_0^\infty(\mathbf{R}^2)$, then $\phi_\delta(g) = \sum_{n\in\mathbf{Z}^2}\delta^2\phi_\delta(n)g(n\delta)$.

(iv) If P is a polynomial and $g \in C_0^\infty$ we set $U(g) = \int g(x) :P(\phi(x)): dx\omega_0$, as in § II, and

$$U_\delta(g) = \sum_n \delta^2 :P(\phi_\delta(n)): g(n\delta)\omega_0 . \tag{IV.9}$$

The basic convergence result is:

THEOREM IV.1. *Let P be a semibounded polynomial and let $g \in C_0^\infty(\mathbf{R}^2)$ be nonnegative. Then, as $\delta \to 0$,*

$$\phi_\delta(g) \longrightarrow \phi(g) , \tag{IV.10a}$$

$$U_\delta(g) \longrightarrow U(g) , \tag{IV.10b}$$

$$\exp(-U_\delta(g)) \longrightarrow \exp(-U(g)) , \tag{IV.10c}$$

where each of the above limits takes place in $L^p(Q, d\mu_0)$ for any $p < \infty$.

The proof of the theorem depends on

LEMMA IV.2.

(i) *For each $k \in \mathbf{R}^2$, $\mu_\delta(k) \to \mu(k)$ as $\delta \to 0$.*

(ii) *If $|k_i| \leqq \pi/\delta$, $\mu_\delta(k)^{-1} \leqq (\pi/2)\mu(k)^{-1}$.*

Proof. From the definition (IV.6) we see that (i) is trivial. Clearly, (ii) is a consequence of the estimate

$$1 - \cos y \geqq \frac{2}{\pi^2}y^2 \quad \text{if} \quad y \in [-\pi, \pi] \tag{IV.11}$$

which we prove as follows. Consider the C^∞ function $F(y) = 1 - \cos y - 2y^2/\pi^2$ in the interval $[0, \pi]$. Note that $F(0) = F(\pi) = 0$, $F'(0) = 0$, and $F''(0) > 0$. Moreover $F''(y) = \cos y - 4/\pi^2$ has exactly one zero in $[0, \pi]$. Since F'' must vanish between zeros of F' and since $F'(0) = 0$, F' can vanish at most once in $(0, \pi)$. Therefore F does not vanish in $(0, \pi)$. But $F(y) > 0$ for small y, and so $F(y) \geqq 0$. □

By Lemma IV.2(ii), Theorem IV.1 is a consequence of an extended conditioning convergence theorem (see the remark after Theorem II.23). We prefer a direct proof:

Proof of Theorem IV.1.

(a), (b) Since, for varying δ, the vectors $U_\delta(g)$ have a fixed finite number of particles, it is sufficient by (II.23) to prove L^2-convergence. The L^2-convergence of the relevant kernels follows from Lemma IV.2 and the dominated convergence theorem.

(c) By mimicking the proof of Theorem II.10 that $e^{-U(g)} \in L^p$ and by using the uniform bound of Lemma IV.2(ii), we see that $\exp(-U_\delta(g))$ is uniformly bounded in L^p.

The convergence thus follows from (b), the estimate (II.24), and Hölder's inequality. □

The lattice cutoff (smeared) Schwinger functions are defined by

$$S_{g,\delta}(h_1, \cdots, h_r) = \frac{\int \phi_\delta(h_1) \cdots \phi_\delta(h_r) e^{-U_\delta(g)} d\mu_0}{\int e^{-U_\delta(g)} d\mu_0} \tag{IV.12}$$

where $h_j \in C_0^\infty$. As an immediate consequence of Theorem IV.1, we deduce:

COROLLARY IV.3. *Let* $h_1, \cdots, h_r \in C_0^\infty(\mathbf{R}^2)$. *As* $\delta \to 0$,

$$S_{g,\delta}(h_1, \cdots, h_r) \longrightarrow S_g(h_1, \cdots, h_r) .$$

IV.2. *Properties of the Lattice Theory.* In this section we investigate the properties of the lattice approximation, and in particular show that the measure reduces to the form (IV.1). To this end, note that in the expression (IV.12) for the lattice cutoff Schwinger function, only a finite number of Gaussian random variables are involved, namely the fields $q_n \equiv \phi_\delta(n) \equiv \phi(f_{\delta,n})$ for $n\delta \in \Lambda_\delta \equiv L_\delta \cap \Lambda$, where $\Lambda = \operatorname{supp} g \cup \operatorname{supp} h_1 \cdots \cup \operatorname{supp} h_r$. Thus by (IV.2) and the definition of the measure $d\mu_0$, the numerator in (IV.12) reduces to a sum of terms of the form

$$\begin{aligned}(2\pi)^{-N/2} |C|^{-1/2} \int q_{n_1} \cdots q_{n_r} \exp\left[-\textstyle\sum_n \delta^2 :P(q_n): g(n\delta)\right] \\ \cdot \exp\left(-\frac{1}{2}\vec{q}\cdot C^{-1}\vec{q}\right) d^N q ,\end{aligned} \tag{IV.13}$$

where $N = |\Lambda_\delta|$, the number of points in Λ_δ, and C is the $N \times N$ covariance matrix of the q's. The same conclusion holds for the denominator of (IV.12), and, more generally, for any expectation

$$\frac{\int F(\phi_\delta(n_1), \cdots, \phi_\delta(n_r)) e^{-U_\delta(g)} d\mu_0}{\int e^{-U_\delta(g)} d\mu_0} \tag{IV.14}$$

where F is a function on $\mathbf{R}^r$.

Now we constructed the lattice cutoff field so that its covariance matrix

$$C_{nn'} = \int \phi_\delta(n)\phi_\delta(n')d\mu_0 \tag{IV.15}$$

gives rise to an operator C on l^2 whose inverse $C^{-1} = \delta^2 A$ has particularly simple matrix elements (see (IV.5)). In fact, by (IV.5) and (IV.6), $A \in M_\infty$ and $C \in K_\infty$ where we define:

Definitions. Let M_∞ denote the class of positive invertible operators on l^2 with nonpositive off-diagonal entries. Let $K_\infty = \{C \mid C^{-1} \in M_\infty\}$.

These classes of infinite matrices have rather interesting properties some of which are established in Appendix A. In particular, Theorem A.2 shows that any *finite submatrix* of $C \in K_\infty$ retains the property that its inverse is a positive definite matrix with nonpositive off-diagonal entries. We summarize these facts about C:

THEOREM IV.4.

(i) *The covariance matrix* (IV.15) *defines a bounded, positive, invertible operator C on $l^2(\mathbf{Z}^2)$.*

(ii) *$C^{-1} = \delta^2 A$ has nonpositive off-diagonal matrix elements* (see (IV.5)); i.e., $C \in K_\infty$.

(iii) *Let S be a finite subset of $\mathbf{Z}^2$ and let D be the matrix $(C_{nn'})_{n,n' \in S}$. Then D is strictly positive-definite and D^{-1} is a positive-definite matrix with nonpositive off-diagonal entries.*

If for a fixed $\delta > 0$ we are dealing only with the lattice fields in a bounded region Λ, let us denote the (finite) covariance matrix of these fields by C^Λ (and similarly for submatrices of the operator A). According to the above discussion, Theorem IV.4(iii) shows that in expectations of the form (IV.14) the measure reduces to the form (IV.1) where $B = (C^\Lambda)^{-1}$ is a positive-definite matrix with *nonpositive off-diagonal entries.* Although this conclusion is sufficient for the purposes of the next section, we can go further in determining $(C^\Lambda)^{-1}$. Of course, $(C^\Lambda)^{-1} \neq \delta^2 A^\Lambda$; but, as we show in Theorem IV.7 below,

$$\delta^{-2}(C^\Lambda)^{-1} = A^\Lambda - B_{\partial\Lambda} \tag{IV.16}$$

where the matrix $B_{\partial\Lambda}$ is "concentrated on the boundary of Λ_δ". By this we mean the following: Given a set $\Lambda \subset \mathbf{R}^2$ with its enclosed lattice points Λ_δ, we make these definitions:

Definitions.

(i) $\Lambda_\delta^{\text{ext}} = L_\delta \backslash \Lambda_\delta$.

(ii) $\Lambda_\delta^{\text{int}} = \{n\delta \in \Lambda_\delta \mid m\delta \in \Lambda_\delta \text{ if } |m - n| = 1\}$.

(iii) The boundary $\partial\Lambda_\delta = \Lambda_\delta \backslash \Lambda_\delta^{\text{int}}$.

(iv) P_Λ is the projection in $l^2(\mathbf{Z}^2)$ onto sequences a such that $a_n = 0$ if $n\delta \notin \Lambda$.

We say that an operator B is *concentrated on the set* Λ if $B = P_\Lambda B P_\Lambda$, and similarly for a matrix B, where we identify P_Λ with its restriction to the finite-dimensional subspace of l^2 on which the matrix acts. In addition we identify C^Λ with $P_\Lambda C P_\Lambda$.

The relation (IV.16) turns out to be connected with the lattice Markov property (see Theorem IV.8 below) and will be useful in understanding the results of the following sections. The basic idea in the proof of (IV.16) is that $(C^\Lambda)^{-1}$ can be calculated from

(IV.17) $$\delta^2 C^\Lambda = \lim_{R\to\infty} P_\Lambda (A^R)^{-1} P_\Lambda \ .$$

To prove (IV.17) we first note:

LEMMA IV.5. *For some* $\alpha > 0$, $C_{nn'} = 0(e^{-\alpha|n-n'|})$.

Proof. It is sufficient to show that C_{n0} decreases exponentially in $n_1 > 0$, uniformly in n_2. From (IV.7) and the definition (IV.15),

$$C_{n0} = (2\pi)^{-2} \int_\delta dk_2 e^{ik_2 n_2 \delta} \int_{-\pi/\delta}^{\pi/\delta} dk_1 e^{ik_1 n_1 \delta} \mu_\delta(k)^{-2} \ .$$

For fixed k_2, $\mu_\delta(k)^{-2}$ is an analytic function of k_1 in the strip $-2\pi/\delta < \operatorname{Re} k_1 < 2\pi/\delta$ except at the two zeros of $\mu_\delta(k) = 0$, i.e., at $k_1 = \pm i\kappa(k_2)$, where by an elementary computation, $\kappa \geqq \delta^{-1} \log(1 + m^2\delta^2)$. Therefore by the Cauchy integral theorem, the integral in k_1 along the line segment $[-\pi/\delta, \pi/\delta]$ is the same as the integral along $[(-\pi + ia)/\delta, (\pi + ia)/\delta]$ for any $0 < a < \kappa$, since the integrals over the other two sides of the rectangle cancel by periodicity. Thus the integral over k_1 is $0(e^{-an_1})$, uniformly in k_2. □

LEMMA IV.6. *If* $\Lambda \subset \mathbf{R}^2$ *is a fixed bounded region then, as the regions* R *go to infinity,* (IV.17) *holds.*

Proof. Define the operator $B = P_R A P_{R'} + P_{R'} A P_R$, where R' is the complement of R. Then we have $A - B = A^R \oplus A^{R'}$. From (IV.6) we note the bounds $m^2 \leqq A \leqq 8\delta^{-2} + m^2$, so that A, A^R, $A^{R'}$ and B are bounded and the first three have bounded inverses (all bounds uniform in R). Therefore

$$(A - B)^{-1} = (A^R)^{-1} \oplus (A^{R'})^{-1} \ .$$

Assuming that $\Lambda \subset R$ we have $P_\Lambda (A^R)^{-1} = 0$ and so

$$\begin{aligned} P_\Lambda (A^R)^{-1} P_\Lambda &= P_\Lambda (A - B)^{-1} P_\Lambda = P_\Lambda \big(A^{-1} + (A - B)^{-1} B A^{-1}\big) P_\Lambda \\ &= \delta^2 C^\Lambda + \delta^2 P_\Lambda (A^R)^{-1} B C P_\Lambda \ . \end{aligned}$$

But since A has matrix elements only between nearest neighbours, B is concentrated on the set $R_0 = \partial R_\delta \cup \partial R'_\delta$. From the previous lemma and the uniform bounds on $(A^R)^{-1}$ and B we thus conclude that

$$P_\Lambda(A^R)^{-1}P_\Lambda = \delta^2 C^\Lambda + 0(\exp[-\alpha \operatorname{dist}(R_0, \Lambda)])$$

and (IV.17) follows. □

We then obtain this improvement of Theorem IV.4 (iii):

THEOREM IV.7. *If* $\Lambda \subset \mathbf{R}^2$ *is a bounded region, then* $(C^\Lambda)^{-1} = \delta^2(A^\Lambda - B_{\partial\Lambda})$, *where* $B_{\partial\Lambda}$ *is a positive semi-definite matrix with nonnegative elements that is concentrated on the boundary* $\partial\Lambda_\delta$.

Proof. Let $M = |\Lambda_\delta|$ and $N = |R_\delta|$. We determine $(C^\Lambda)^{-1}$ by formula (IV.2), i.e., as the matrix in the exponential of the density function for the Gaussian random variables on the lattice Λ_δ:

$$C^\Lambda_{nn'} = (2\pi)^{-M/2}|C^\Lambda|^{-1/2}\int q_n q_{n'} e^{-1/2q\cdot(C^\Lambda)^{-1}q} d^M q \ , \tag{IV.18}$$

where $n\delta$, $n'\delta \in \Lambda_\delta$. On the other hand by Lemma IV.6 we also have

$$\delta^2 C^\Lambda_{nn'} = \lim_{R\to\infty}(2\pi)^{-N/2}|A^R|^{1/2}\int q_n q_{n'} e^{-1/2q\cdot A^R q} d^N q \ , \tag{IV.19}$$

where we have again used (IV.2). Now suppose that for a fixed $R \supset \Lambda$ we integrate out the variables in $R\backslash\Lambda$ in the integral in (IV.19). Denote the resulting matrix in the Gaussian by A^Λ_R, i.e.,

$$(2\pi)^{-N/2}|A^R|^{1/2}\int e^{-1/2q^R\cdot A^R q^R} dq^{R\backslash\Lambda} = (2\pi)^{-M/2}|A^\Lambda_R|^{1/2} e^{-1/2q^\Lambda\cdot A^\Lambda_R q^\Lambda} \tag{IV.20}$$

where q^R stands for the variables on R_δ. It is obvious from the definition (IV.20) that A^Λ_R is a positive definite matrix with

$$\inf \sigma(A^\Lambda_R) \geqq \inf \sigma(A^R) \ . \tag{IV.21}$$

We claim that the matrix $B^R_{\partial\Lambda} = A^\Lambda - A^\Lambda_R$ is a positive semi-definite matrix with nonnegative elements that is concentrated on the boundary $\partial\Lambda_\delta$. Since we are assured by Lemma IV.6 that the limit in (IV.19) exists for all n, n' in $\delta^{-1}\Lambda_\delta$, we see from (IV.18)-(IV.20) that these properties of $B^R_{\partial\Lambda}$ yield the theorem. The claim is proved by induction on the number of variables in $R\backslash\Lambda$. What is the effect of integrating out a single variable q in a Gaussian integral? If the terms involving q are of the form $\exp(-(c/2)q^2 - q\sum a_n q_n)$, then integrating out q leaves a factor const. $\exp[1/2c(\sum a_n q_n)^2]$. Thus as each variable in $R\backslash\Lambda$ is integrated out, more couplings are introduced for the remaining variables. However, since A^R couples only *nearest neighbours*, the variables in $\Lambda^{\text{int}}_\delta$ are not coupled to the variables in $\Lambda^{\text{ext}}_\delta$, and integrating out variables in $\Lambda_\delta^{\text{ext}}$ cannot affect the terms involving variables in $\Lambda_\delta^{\text{int}}$. Clearly, then, $B_{\partial\Lambda}^R$ is always concentrated on $\partial\Lambda_\delta$. As for the sign of $B_{\partial\Lambda}^R$, note that each integration modifies the positive-definite quadratic form in the remaining variables by subtracting a term of the form $1/2c(\sum a_n q_n)^2$ in which, we claim, all the coefficients are nonnegative. The reason is that c is always positive, since by (IV.21) the quadratic form remains positive-definite after each integration, and the a_n's always have the same sign, namely, negative. For, from the definition (IV.5) of A^R, the couplings start out nonpositive and each integration only decreases them. □

Example. To illustrate the theorem, we can explicitly calculate $(C^\Lambda)^{-1}$ and $B_{\partial\Lambda}$ when $d = 1$. In one dimension, $A = (-\Delta_\delta + m^2)$ has matrix elements $A_{nn'} = m^2 + 2\delta^{-2} \equiv a_0$ when $n = n'$, $A_{nn'} = -\delta^{-2} \equiv -a_1$ when $n = n' \pm 1$, and $A_{nn'} = 0$ otherwise. In momentum space A is multiplication by $\mu_\delta(k)^2 = m^2 + 2\delta^{-2}(1 - \cos k\delta)$. The matrix elements of the covariance operator, $C_{nn'} = (2\pi)^{-1}\int_\delta e^{ik\delta(n-n')}\mu_\delta(k)^{-2}dk$, can be evaluated by contour integration and we find

$$C_{nn'} = \delta x^{|n-n'|+1}/(1 - x^2) \tag{IV.22}$$

where x is the solution of $a_1x^2 - a_0x + a_1 = 0$ in $0 < x < 1$, i.e., $x = [(1 + (m\delta/2)^2)^{1/2} - m\delta/2]^2$. Note that $C_{nn'}$ decreases exponentially with $|n - n'|$.

Let $\Lambda = (0, M\delta]$ so that $\Lambda_\delta = \{\delta, 2\delta, \cdots, M\delta\}$ and $\partial\Lambda_\delta = \{\delta, M\delta\}$. From the definition (IV.22) we can readily compute $(C^\Lambda)^{-1}$, and we obtain

$$(C^\Lambda)^{-1} = \frac{1}{x\delta}\begin{pmatrix} 1 & -x & 0 & 0 & \cdots \\ -x & 1+x^2 & -x & 0 & \cdots \\ 0 & -x & 1+x^2 & -x & \cdots \\ & & & \ddots & -x \\ & \cdots & 0 & -x & 1 \end{pmatrix}.$$

Thus we see that $\delta^{-1}(C^\Lambda)^{-1} = A^\Lambda - B_{\partial\Lambda}$, where

$$B_{\partial\Lambda} = \begin{pmatrix} a_1x & 0 & \cdots & \\ 0 & 0 & & \vdots \\ \vdots & & 0 & 0 \\ \vdots & \cdots & 0 & a_1x \end{pmatrix}.$$

Remarks 1. The main point of Theorem IV.7 is that even in a bounded region, the measure still couples only nearest neighbours, except for the boundary variables.

2. The role of the boundary matrix $B_{\partial\Lambda}$ is to adjust the boundary condition associated with the (finite difference) Laplacian. Thus, if to the $M = |\Lambda_\delta|$ Gaussian random variables on the lattice Λ_δ we associate the measure const. $e^{-1/2q\cdot(C^\Lambda)^{-1}q}d^Mq$, we are taking "free" boundary conditions; on the other hand, it is heuristically reasonable that the measure const. $e^{(-1/2)\delta^2 q\cdot A^\Lambda q}d^Mq$ gives Dirichlet B.C. since the effect of the A^Λ is to ignore the variables just outside Λ_δ in the definition of the finite difference approximation to $-\Delta$. We return to this question and the related convergence theorem in § IV.3. Similarly we expect that arbitrary B.C. may be obtained by an appropriate choice of boundary matrix.

3. The assertions about the sign and semi-definiteness of $B_{\partial\Lambda}$ also follow from the method of Appendix A (see the remark after Theorem A.2). The semi-definiteness, which can be expressed as $\delta^{-2}(A^\Lambda)^{-1} \leqq C^\Lambda$, is the lattice version of the positive-definiteness relationship between Dirichlet and free B.C. noted in § II. The significance of the sign of $B_{\partial\Lambda}$ will become clear in § V when we show by correlation inequalities that the free Schwinger functions are greater than the Dirichlet Schwinger functions.

4. The fact that the influence of the variables in Λ^{ext} can be felt by the variables in Λ only on $\partial\Lambda$ is of course an expression of the Markov property of the free field theory. In § VII we return to this idea of boundary terms in our discussion of equilibrium equations.

Finally, we discuss the Markov property for the lattice theory. In the continuous case the Markov property amounts to a statement of the fact that $(-\Delta + m^2)$ is a local operator (see the proof of Proposition II.3). In the lattice case the corresponding fact is that the finite difference operator $A = (-\Delta_\delta + m^2)$ couples nearest neighbours only. An abstract discussion of the lattice theory and its Markov property proceeds just as in § II.1. In fact the lattice theory is just a *sub-theory* of the continuous one: The single-particle Hilbert space N_δ consists of sequences a on the lattice L_δ with inner product

(IV.23) $$\langle a, b\rangle_\delta = \sum_{n,n'} \bar{a}_n C_{nn'} b_{n'} \,.$$

If we identify a with the function $f(x) = \sum_n a_n f_{\delta,n}(x)$ we have an isometric imbedding $N_\delta \subset N$. On the associated Q-space, $Q_\delta \equiv Q_{N_\delta}$, the Markov property takes the usual form: We define e_{R_δ} to be the projection in N_δ onto sequences supported in R_δ, and E_{R_δ} to be $\Gamma(e_{R_\delta})$. Then

THEOREM IV.8. *If R and S are closed disjoint regions in $\mathbf{R}^2$, then*

(IV.24) $$E_{R_\delta} E_{S_\delta} = E_{\partial R_\delta} E_{S_\delta} \,.$$

Proof 1. The proof of (IV.24) reduces to the corresponding single particle relation which we verify as in Proposition II.3: For any $a \in \operatorname{Ran} e_{S_\delta}$, we must show that $e_{R_\delta}a$ is supported on ∂R_δ, or equivalently that $\sum_n \overline{(e_{R_\delta}a)}_n b_n = 0$ for all finite sequences b supported in R_δ^{int}. But

$$\begin{aligned}\textstyle\sum_n \overline{(e_{R_\delta}a)}_n b_n &= \langle e_{R_\delta}a,\ Ab\rangle_\delta \\ &= \langle a,\ e_{R_\delta}Ab\rangle_\delta \\ &= \langle a,\ Ab\rangle_\delta \\ &= \textstyle\sum \overline{a_n} b_n = 0\end{aligned}$$

where $e_{R_\delta}Ab = Ab$ since A couples only nearest neighbours. □

Proof 2. But a more concrete version of (IV.24) is at hand (at least for bounded regions) since the lattice approximation has provided a concrete representation of Q_δ. Thus if F is a function of the random variables q^S on S_δ, the projection $E_{R_\delta}F$ is defined by the identity

$$\begin{aligned}&(2\pi)^{-|R_\delta|/2}\,|C^R|^{-1/2}\int G(q^R)(E_{R_\delta}F)(q^R)e^{(-1/2)q^R\cdot(C^R)^{-1}q^R}dq^R \\ &\quad= (2\pi)^{-|R_\delta\cup S_\delta|/2}\,|C^{R\cup S}|^{-1/2}\int G(q^R)F(q^S)e^{(-1/2)q^{R\cup S}\cdot(C^{R\cup S})^{-1}q^{R\cup S}}dq^{R\cup S}\ ,\end{aligned}$$

for all $G \in C_0^\infty(\mathbf{R}^{|R_\delta|})$. But by Theorem IV.7, $(C^{R\cup S})^{-1}$ does not couple any of the variables in R_δ^{int} to variables in S_δ. Therefore, upon integrating over dq^S on the right hand side, we see that $E_{R_\delta}F$ is a function only of the variables in ∂R_δ. □

As for the interacting theory, it is clear from (IV.13) that the interaction does not destroy the Markov property since it does not couple different q's; i.e., the interaction is local:

(IV.25)
$$U_{\Lambda_\delta} = \textstyle\sum_{\Lambda_\delta} {:}P_n(q_n){:}\ ,$$

where the sum takes place over sites $n\delta \in \Lambda_\delta$ ($\Lambda = \operatorname{supp} g$).

In Theorem IV.8 we were of course considering the lattice theory with free boundary conditions, i.e., the covariance in the region Λ is given by C^Λ where $(C^\Lambda)^{-1} = \delta^2(A^\Lambda - B_{\partial\Lambda})$, as in Theorem IV.7. However, it is obvious from the second proof of Theorem IV.8 that the Markov property depends only on the nearest neighbour form of the Gaussian density; hence the Markov property will hold for arbitrary boundary conditions; i.e., if the inverse of the covariance matrix has the form $\delta^2(A^\Lambda - B_{\partial\Lambda})$ where $B_{\partial\Lambda}$ is an arbitrary matrix concentrated on $\partial\Lambda_\delta$. Given the lattice L_δ, these observations lead us to:

Definition. In two Euclidean space dimensions, the *polynomially inter-*

acting lattice Markov field theory with boundary condition $\{B_{\partial\Lambda}\}$ is defined in the bounded region $\Lambda \subset \mathbf{R}^2$ by the following measure for the fields q_n on the lattice Λ_δ:

$$d\nu_\Lambda(q) = \frac{\exp\left(-U_{\Lambda_\delta} - \frac{1}{2}q\cdot Bq\right)d^N q}{\int \exp\left(-U_{\Lambda_\delta} - \frac{1}{2}q\cdot Bq\right)d^N q}\ ; \tag{IV.26}$$

here U_{Λ_δ} is defined by (IV.25) in terms of the given semi-bounded polynomials P_n, $N = |\Lambda_\delta|$, and B is the $N \times N$ positive definite matrix, $B = \delta^2(A^\Lambda - B_{\partial\Lambda})$ where A^Λ is the matrix of the operator (IV.5) restricted to Λ_δ and $B_{\partial\Lambda}$ is concentrated on $\partial\Lambda_\delta$.

In closing this section we remark that one can prove L^p-estimates (e.g., hypercontractivity, checkerboard = pegboard) for the lattice theory, just as in § III. The hypercontractivity of projections is based on a calculation similar to (IV.22).

IV.3. *Dirichlet Boundary Conditions.* We now consider the lattice theory with Dirichlet B.C. in more detail. Our main goal is to verify the claim made in Remark 2 following Theorem IV.7 that the lattice measure const. $e^{(-1/2)\delta^2 q\cdot A^\Lambda q}dq$ corresponds to Dirichlet B.C. on the boundary of Λ. Throughout this discussion the region Λ will be fixed, and we shall generally neglect attaching a subscript Λ to Λ-dependent objects.

As in the continuum case the Dirichlet field ϕ_δ^D is obtained by "conditioning on $\Lambda_\delta^{\text{ext}}$". Thus let P_δ be the projection orthogonal in N_δ (cf. (IV.23)) onto sequences supported in $\Lambda_\delta^{\text{ext}}$. We write $\phi_\delta(a) = \sum a_n\phi_\delta(n)$ if $a \in l^2(\mathbf{Z}^2)$ and we let $e^{(n)}$ be the vector in l^2 with components $e_m^{(n)} = \delta_{nm}$. Then we have:

Definition. On the lattice L_δ, the Dirichlet field in the region Λ is given by

$$\phi_\delta^D(n) = \phi_\delta((I - P_\delta)e^{(n)})\ . \tag{IV.27}$$

Note that if $n\delta \notin \Lambda$, then $\phi_\delta^D(n) \equiv 0$, and that if $n\delta \in \Lambda$,

$$\phi_\delta^D(n) = \phi_\delta(n) - \sum\nolimits_m \alpha_m^{(n)}\phi_\delta(m)\ , \tag{IV.28}$$

where by the Markov property the vector $\alpha^{(n)} = P_\delta e^{(n)}$ has nonzero components $\alpha_m^{(n)}$ only if $m\delta \in \partial(\Lambda')_\delta$ where $\Lambda' = \mathbf{R}^2\backslash\Lambda$. The interaction $U_\delta^D(g)$ is defined as in (IV.9) with the replacement of $\phi_\delta(n)$ by $\phi_\delta^D(n)$. Notice that since ϕ_δ^D vanishes outside Λ, only the values of g in Λ matter.

First we verify that the covariance matrix of the Dirichlet fields

$$C_{jk}^{\Lambda,D} = \int \phi_\delta^D(j)\phi_\delta^D(k)d\mu_0\ , \tag{IV.29}$$

has boundary matrix $B_{\partial\Lambda} = 0$:

THEOREM IV.9. *Fix $\delta > 0$ and Λ a bounded region in $\mathbf{R}^2$. Then as $|\Lambda_\delta| \times |\Lambda_\delta|$ matrices,*

$$(C^{\Lambda,D})^{-1} = \delta^2 A^\Lambda \,. \tag{IV.30}$$

Proof. From the definitions (IV.29), (IV.27) and (IV.23)

$$\begin{aligned} C_{jk}^{\Lambda,D} &= \langle e^{(j)}, (I - P_\delta)e^{(k)}\rangle_\delta \\ &= C_{jk} - \sum_m C_{jm}\alpha_m^{(k)} \,. \end{aligned}$$

This equality holds true even if $j\delta \notin \Lambda$ in which case the last expression vanishes; thus if k, $i \in \Lambda/\delta$,

$$\begin{aligned} \delta^2 \sum_{j\in\Lambda/\delta} A_{ij}C_{jk}^{\Lambda,D} &= \delta^2 \sum_{j\in \mathbf{Z}^2} A_{ij}\Big(C_{jk} - \sum_m C_{jm}\alpha_m^{(k)}\Big) \\ &= \delta_{ik} - \sum \delta_{im}\alpha_m^{(k)} \\ &= \delta_{ik} \end{aligned}$$

since the sum over m is over $m\delta \in \partial(\Lambda')_\delta$ which is disjoint from Λ_δ. □

Secondly, we establish the convergence of the lattice Dirichlet theories (both "full" and "half") to the continuum Dirichlet theories in the same region. However, it is necessary to impose a regularity condition on Λ, as is shown by the following.

Example. Let Λ_1 be the unit disk and let $\Lambda_2 = \Lambda_1 \backslash S$ where S is the line segment $\{(x, \pi x) \,|\, x > 0\}$. Obviously $(\Lambda_1)_\delta = (\Lambda_2)_\delta$ if δ is rational, in which case the lattice Dirichlet theories agree for the two regions. On the other hand, there are elements in the one-particle space N of (II.6) which are supported in $\Lambda_1 \backslash \Lambda_2$ so that the continuum Dirichlet theories differ for the two regions.

Definition. Let $\Lambda \subset \mathbf{R}^2$ be open and bounded and let Λ' be its complement. We say Λ is *regular* if and only if any distribution $f \in N$ with support in Λ' is a limit (in N) of elements in $C_0^\infty(\Lambda')$.

Remark. By scaling and convolution it is easy to show that Λ is regular if it is convex.

We now state the convergence theorem for the *full* Dirichlet theory:

THEOREM IV.10. *Let Λ be an open, bounded, regular region in $\mathbf{R}^2$. Let $g \in C_0^\infty(\mathbf{R}^2)$ be nonnegative. Let $\phi_\delta^D(\phi^D)$ denote the Dirichlet field on the lattice L_δ (respectively, continuum Dirichlet field), both in the region Λ, and let $U_\delta^D(g)$* (resp. $U^D(g)$) *be the corresponding interaction defined above* (resp. in (II.63)) *in terms of a fixed semibounded polynomial. Then, as $\delta \to 0$,* (IV.31a) $$\phi_\delta^D(g) \longrightarrow \phi^D(g) ,$$

(IV.31b) $$U_\delta^D(g) \longrightarrow U^D(g) ,$$

(IV.31c) $$\exp(-U_\delta^D(g)) \longrightarrow \exp(-U^D(g))$$

where each of the above limits takes place in $L^p(Q, d\mu_0)$ for any $p < \infty$.

Remark. g need not have support in Λ; moreover only the values of g in Λ matter. Thus, for example, we can replace g with χ_Λ, the characteristic function of Λ.

The proof of the theorem depends on

LEMMA IV.11. *Let Λ be an open, bounded regular region in $\mathbf{R}^2$, and let P be the orthogonal projection in N onto the distributions with support in Λ'. Then*

(IV.32) $$s-\lim_{\delta\to 0} P_\delta = P .$$

Remark. By the isometric imbedding of N_δ in N, we regard P_δ as an operator on N; i.e., P_δ is the projection onto the subspace of N spanned by $\{f_{n,\delta} \mid n\delta \notin \Lambda\}$.

Proof. Given $g \in C_0^\infty(\mathbf{R}^2)$, define $g_\delta(x) = \delta^2 \sum_n g(n\delta) f_{\delta,n}(x)$. Then it is easy to see by Lemma IV.2 and the dominated convergence theorem that $g_\delta \to g$ and

(IV.33) $$(-\Delta_\delta + m^2)g_\delta \longrightarrow (-\Delta + m^2)g$$

in N. Suppose that $g \in C_0^\infty(\Lambda')$. Then $g_\delta \in \operatorname{Ran} P_\delta$, so that by Bessel's inequality $\|P_\delta g - g\| \leqq \|g_\delta - g\| \to 0$ as $\delta \to 0$. Since Λ is regular, we conclude that

(IV.34) $$P_\delta g \longrightarrow g \quad \text{if} \quad g \in \operatorname{Ran} P .$$

Now suppose that $g \in N$. Since the ball of radius $\|g\|$ in N is weakly compact we know that $\{P_\delta g\}$ has a weakly convergent subsequence. Let f be any weak limit point of $\{P_\delta g\}$. We claim that to prove (IV.32) it is sufficient to show that $f = Pg$, for then $P_\delta g \to Pg$ weakly and hence strongly since P_δ and P are projections. To this end, let $h \in C_0^\infty(\Lambda)$ be arbitrary. Since $\operatorname{supp} h \cap \partial(\Lambda')_\delta = \phi$, $((-\Delta_\delta + m^2)h_\delta, P_\delta g)_N = 0$. Thus by (IV.33), $((-\Delta + m^2)h, f)_N = \int \overline{h(x)} f(x) dx = 0$ so that $f \in \operatorname{Ran} P$. Now let h be arbitrary in N. Then $(h, f) = (Ph, f) = \lim_n (Ph, P_{\delta_n} g) = (Ph, g)$ by (IV.34). We conclude that $f = Pg$. □

Proof of Theorem IV.10.

(a), (b) By the general theory of conditioning (cf. §II.5), $U^D(g) = \Gamma(I - P)U(g)$ and $U_\delta^D(g) = \Gamma(I - P_\delta)U_\delta(g)$. Clearly L^2-convergence follows from Theorem IV.1 and Lemma IV.11, and L^p-convergence from (II.23).

(c) By (II.24) we need only show that $\sup \| \exp(-U_\delta^D(g)) \|_p$ is finite for each fixed $p < \infty$. But by Corollary II.21,

$$\| \exp(-U_\delta^D(g)) \|_p \leqq \| \exp(-U_\delta(g)) \|_p$$

so that the uniform L^p bound follows from the proof of Theorem IV.1. □

Theorems IV.9 and IV.10 together justify our conclusion that the choice of boundary matrix $B_{\partial\Lambda} = 0$ gives Dirichlet B.C.

The lattice cutoff (smeared) Schwinger functions with Dirichlet B.C. in the region Λ are defined by

$$S_{\Lambda,\delta}^D(h_1, \cdots, h_r) = \frac{\int \phi_\delta^D(h_1) \cdots \phi_\delta^D(h_r) e^{-U_\delta^D(\chi_\Lambda)} d\mu_0}{\int e^{-U_\delta^D(\chi_\Lambda)} d\mu_0}, \tag{IV.35}$$

where $h_j \in C_0^\infty(\mathbf{R}^2)$. As an immediate consequence of Theorem IV.10 we deduce:

COROLLARY IV.12. *Let Λ be an open, bounded, regular region in $\mathbf{R}^2$. Let $h_1, \cdots, h_r \in C_0^\infty(\mathbf{R}^2)$. As $\delta \to 0$,*

$$S_{\Lambda,\delta}^D(h_1, \cdots, h_r) \longrightarrow S_\Lambda^D(h_1, \cdots, h_r) .$$

Finally, we note that the half-Dirichlet lattice theory converges as $\delta \to 0$. We adopt the second point of view as explained in § II.6; i.e., we realize the noninteracting lattice Dirichlet theory in terms of the Dirichlet measure

$$d\mu_{\Lambda,\delta}^D = (2\pi)^{-N/2} | A^\Lambda |^{1/2} \delta^N e^{(-1/2)\delta^2 q \cdot A^\Lambda q} dq$$

where $q_n = \phi_\delta(n)$, $n = 1, 2, \cdots, N$, are the field variables in Λ. The (smeared) lattice cutoff Schwinger functions with half-Dirichlet B.C. are (cf.(II.91))

$$S_{\Lambda,\delta}^{HD}(h_1, \cdots, h_r) = \frac{\int \phi_\delta(h_1) \cdots \phi_\delta(h_r) e^{-U_\delta(\chi_\Lambda)} d\mu_{\Lambda,\delta}^D}{\int e^{-U_\delta(\chi_\Lambda)} d\mu_{\Lambda,\delta}^D} . \tag{IV.36}$$

Here $U_\delta(\chi_\Lambda)$ is defined by (IV.9), that is, in terms of Wick powers $:\phi_\delta(n)^r:$ rather than $:\phi_\delta(n)^r:_D$. Let $P(X) = \sum_{r=0}^{2m} a_r X^r$. By Lemma V.27

$$:\phi_\delta(n)^r: = \sum_{j=0}^{[r/2]} \begin{Bmatrix} r \\ j \end{Bmatrix} (-B_\delta(n))^j :\phi_\delta(n)^{r-2j}:_D \tag{IV.37}$$

where $\begin{Bmatrix} r \\ j \end{Bmatrix} = r!/j!\,(r-2j)!\,2^j$, and $B_\delta(n) = C_{n,n} - C_{n,n}^{\Lambda;D}$ (cf. (IV.15) and (IV.29)). It is convenient to regard the lattice cutoffs in $B_\delta(n)$ and $\phi_\delta^D(n)$ as being independent of one another and so we write by (IV.37)

$$U_{\delta,\delta'}(\chi_\Lambda) = U_{\delta'}^D(\chi_\Lambda) + \sum_{r=2}^{2m} a_r \sum_{j=1}^{[r/2]} \begin{Bmatrix} r \\ j \end{Bmatrix} \sum_{n\delta' \in \Lambda} \delta'^2 (-g_\delta(n\delta'))^j :\phi_{\delta'}(n)^{r-2j}:_D$$

where $U_{\delta,\delta}(\chi_\Lambda) = U_\delta(\chi_\Lambda)$ and $g_\delta(x) = B_\delta(n)$ if $x \in \Lambda$ is in the lattice square centered at $n\delta$. By Theorem IV.10, when $\delta' \to 0$

$$U_{\delta,\delta'}(\chi_\Lambda) \longrightarrow V_\delta = U^D(\chi_\Lambda) + \sum_{j=2}^{2m} a_r \sum_{j=1}^{[r/2]} \begin{Bmatrix} r \\ j \end{Bmatrix} \int (-g_\delta(x))^j :\phi(x)^{r-2j}:_D \, dx$$

in each $L^p(Q, d\mu_\Lambda^D)$, $p < \infty$. Moreover, $\exp(-U_{\delta,\delta'}(\chi_\Lambda)) \to \exp(-V_\delta)$ in L^p, $p < \infty$. In addition, one can show by using the ideas of Lemma II.37 and Theorem II.38 that this convergence is uniform in δ. Now let $\delta \to 0$. For $x \in \Lambda$, $g_\delta(x) \to \delta G(x)$ as defined in Lemma II.37(i), where the convergence takes place in $L^p(\Lambda)$, $p < \infty$. Thus by Theorem II.38(iv), we have in each $L^p(Q, d\mu_\Lambda^D)$ for $p < \infty$,

$$V_\delta \longrightarrow U^D(\chi_\Lambda) + \sum_{r=2}^{2m} a_r \sum_{j=1}^{[r/2]} \begin{Bmatrix} r \\ j \end{Bmatrix} \int (-\delta G(x))^j :\phi(x)^{r-2j}:_D \, dx = U_\Lambda$$

by (II.88). Moreover, $e^{-V_\delta} \to e^{-U_\Lambda}$ in each L^p, $p < \infty$. We have thus verified:

THEOREM IV.13. *Let Λ be an open, regular, log-normal region in $\mathbf{R}^2$, and let $h_1, \cdots, h_r \in C_0^\infty(\mathbf{R}^2)$. As $\delta \to 0$*

$$S_{\Lambda,\delta}^{HD}(h_1, \cdots, h_r) \longrightarrow S_\Lambda^{HD}(h_1, \cdots, h_r) .$$

IV.4. *The lattice Theory as an Ising Ferromagnet.* E. Nelson [69] has pointed out a useful interpretation of the lattice model which helps explain why we have been able to establish the correlation inequalities of the next section. Assume that the boundary matrix $B_{\partial\Lambda}$ is nonnegative, as we have proved it is in the case of free or Dirichlet B.C. Then the free part of the measure (IV.26) can be written

$$\exp\Big(\frac{1}{2} \sum_{n\neq n'} |B_{nn'}| \, q_n q_{n'}\Big) \prod_n e^{(-1/2) B_{nn} q_n^2} dq_n \, ,$$

where we have made use of the fact that $B_{nn'} \leqq 0$ if $n \neq n'$. Thus we see that the free measure gives a Gibbs state for a set of Gaussian spins with a *ferromagnetic* interaction Hamiltonian $H = (1/2) \sum_{n\neq n'} B_{nn'} q_n q_{n'}$. In fact by considering the entire array of spins, we conclude that the *free lattice theory is an infinite array of Gaussian spins with nearest neighbour Ising interactions of ferromagnetic type.* The extent to which the Gaussian model simulates the actual Ising model was discussed in 1952 by Berlin and Kac [4].

With the interaction turned on ("perturbed Gaussian model") we have already noted that the local interaction does not change the coupling between spin sites but only the distribution of each uncoupled spin. *An in-*

teracting lattice Markov field theory is thus an array of continuous spins with ferromagnetic pair interactions between spins.

We warn the reader of a clash between the natural terminologies of field theory and statistical mechanics: What we have called Dirichlet B.C. ($B_{\partial\Lambda} = 0$) corresponds in the Ising model picture of the lattice theory to what would be called "free B.C." in statistical mechanics. Our "free B.C." ($B_{\partial\Lambda}$ given by Theorem IV.7) are very different. It is useful to bear this difference in mind when trying to understand the monotonicity of the Dirichlet states ([69] and § V.4).

V. Correlation inequalities

In this section we wish to prove and apply correlation inequalities of the type that have been so useful in statistical mechanics [38], [85]. These inequalities have the general form of a positivity statement:

(G-I) $$\langle A \rangle \geqq 0 \; ;$$

or a statement of positive correlation:

(G-II and FKG) $$\langle AB \rangle \geqq \langle A \rangle \langle B \rangle \; .$$

Here $\langle \cdot \rangle$ denotes expectation on some space such that $\langle I \rangle = 1$, A and B are suitably chosen observables on this space, G-I and G-II refer to Griffiths' first and second inequalities respectively, and FKG refers to the inequalities of Fortuin-Kasteleyn-Ginibre [20]. The proof of G-I and G-II in our case (Theorems V.1–2, 7–9 below) uses ideas developed in a beautiful analysis of general Griffiths' inequalities by Ginibre [27].

In the first subsection, motivated by the form of the lattice approximation of § IV, we prove inequalities for a class of measures on $\mathbf{R}^n$ which we call "normally perturbed Gaussian measures of ferromagnetic type". It is then easy (§ V.2) to obtain correlation inequalities for the $P(\phi)_2$ field theory. After discussing the possibilities for correlation inequalities involving Wick powers (§ V.3), we turn to applications in § V.4.

Perhaps the most significant consequence of the $P(\phi)_2$ correlation inequalities to date is Nelson's proof ([69] and see Theorem V.20 below) that if P is even and the coupling constant $\lambda > 0$ arbitrary, then the non-coincident Schwinger functions with half-Dirichlet B.C. converge monotonically as the volume cutoff is removed. By "non-coincident Schwinger functions" we mean the smeared Schwinger functions whose smearing functions have pairwise disjoint supports.

V.1. *Gaussian Measures of Ferromagnetic Type.* The form (IV.1) of the measure in the lattice approximation leads us to a consideration of these measures on $\mathbf{R}^n$:

Definitions: A measure $d\mu_0 = e^{-\vec{x}\cdot A\vec{x}}d^n x$ where A is a positive definite matrix with nonpositive off-diagonal elements is called a *Gaussian measure of ferromagnetic type*. If $F_1, \cdots, F_n$ are bounded, continuous, positive functions on $\mathbf{R}$, the measure $d\mu(x) = F_1(x_1) \cdots F_1(x_n)d\mu_0(x)$ is called a *normally perturbed Gaussian measure of ferromagnetic type*, or, for short, a *ferromagnetic measure*. If each F_i is an even function of x_i, we say that μ is an *even* ferromagnetic measure.

If ν is any of the above measures, we define the normalized expectation value of a function f on $\mathbf{R}^n$ as

$$\langle f \rangle_\nu = \int f(x)d\nu(x) \Big/ \int d\nu(x) . \tag{V.1}$$

Finally, if f satisfies $f(x) \geqq f(y)$ whenever $x_i \geqq y_i$ for $i = 1, \cdots, n$, we say that f is an increasing function on $\mathbf{R}^n$ and we write $f\uparrow$.

With these definitions, we are able to state the main results of this subsection:

THEOREM V.1. *If μ is an even ferromagnetic measure and $j_1, \cdots, j_n$ are nonnegative integers, then*

$$\langle x_1^{j_1} \cdots x_n^{j_n} \rangle_\mu \geqq 0 .$$

THEOREM V.2. *If μ is an even ferromagnetic measure and $j_1, \cdots, k_n$ are nonnegative integers, then*

$$\langle x_1^{j_1+k_1} \cdots x_n^{j_n+k_n} \rangle_\mu \geqq \langle x_1^{j_1} \cdots x_n^{j_n} \rangle_\mu \langle x_1^{k_1} \cdots x_n^{k_n} \rangle_\mu .$$

THEOREM V.3. *Let μ be any ferromagnetic measure. Then*

$$\langle fg \rangle_\mu \geqq \langle f \rangle_\mu \langle g \rangle_\mu$$

if f and g are continuous, polynomially bounded functions with $f, g\uparrow$.

Remarks 1. H. Leff [59] has proved Theorems 1 and 2 by explicit computation under the special hypothesis that $F_1 = \cdots = F_n = 1$, and the weaker condition on A that A^{-1} be positive definite with nonnegative elements.

2. F. Spitzer (private communication) has also proved Theorem 3.

3. These three theorems are of type G-I, G-II, and FKG respectively.

4. The assumptions on the perturbing functions F_j can be weakened somewhat. For instance, F_j need not be bounded but must merely be of order 2 and type less than any eigenvalue of A; i.e., $F_j(x) = O(e^{ax^2})$ where $a < \min \sigma(A)$. At the end of this subsection we consider the extent to which we can relax the assumption that μ be even in the first two theorems.

Typically the functions F_i are of the form $e^{-\beta P_i}$ where $P_i(x)$ is a polynomial. Since the earliest simplifications of Griffiths' work [36], [53], [26], the standard method of proving correlation inequalities has been to expand the exponential. In our case this does not work: first, the expansion generally does not converge and, secondly, the inequalities are not true order by order in β. Instead we shall expand the *off-diagonal* part of the Gaussian measure. Thus we take as our base measure

$$d\nu = F_1(x_1) \cdots F_n(x_n) e^{-x \cdot A_0 x} d^n x \ , \tag{V.2}$$

where A_0 is the diagonal part of A, the matrix whose off-diagonal elements vanish and whose diagonal elements are the same as A's We also define, as the interpolation between $d\nu$ and $d\mu$,

$$d\mu^{(\lambda)} = F_1(x_1) \cdots F_n(x_n) e^{-x \cdot A_\lambda x} d^n x \tag{V.3}$$

where $\lambda \in [0, 1]$ and $A_\lambda = \lambda A + (1 - \lambda) A_0$. An important role is played by:

LEMMA V.4. *If μ is an even ferromagnetic measure, then the function $g(\lambda) = \int x_1^{k_1} \cdots x_n^{k_n} d\mu^{(\lambda)}$ is analytic in a circle with center at the origin and radius greater than* 1.

Proof. Since A is invertible and positive definite, so are A_0 and thus A_λ if $0 \leqq \lambda \leqq 1$. It follows that $g(\lambda)$ is analytic in a complex neighbourhood of $[0, 1]$. In particular g is analytic about $\lambda = 0$. Now the m^{th} derivative

$$g^{(m)}(0) = \int [\textstyle\sum_{i \neq j} (-a_{ij}) x_i x_j]^m x_1^{k_1} \cdots x_n^{k_n} d\nu \geqq 0$$

since each $a_{ij} \leqq 0$ and each F_1 is even and positive. But an analytic function with nonnegative Taylor series at the origin must have its nearest singularity on the positive real axis. Thus there is an $\varepsilon > 0$ such that the Taylor series for g at 0 converges if $|\lambda| < 1 + \varepsilon$. □

Proof of Theorem V.1. By the lemma, $g(1) = \sum_0^\infty g^{(m)}(0)/m! \geqq 0$. □

Remark. We owe to E. Lieb (private communication) the observation that Theorem V.1 is actually equivalent to the usual statement of Griffiths' first inequality: For ferromagnetic spin systems, G-I asserts that the (unnormalized) expectation

$$\langle \sigma_1^{k_1} \cdots \sigma_n^{k_n} \rangle_\beta \equiv \textstyle\sum_{\sigma_j = \pm 1} \sigma_1^{k_1} \cdots \sigma_n^{k_n} \exp\left(\sum_{i \neq j} J_{ij}(\beta) \sigma_i \sigma_j\right) \geqq 0 \tag{V.4}$$

provided the couplings are ferromagnetic; i.e., $J_{ij} = J_{ji} \leqq 0$ for $i \neq j$. If we set $\sigma_i = \operatorname{sgn} x_i$, then (V.4) implies

$$\int x_1^{k_1} \cdots x_n^{k_n} d\mu \equiv \int_{\mathbf{R}_+^n} x_1^{k_1} \cdots x_n^{k_n} F_1(x_1) \cdots F_n(x_n) \langle \sigma_1^{k_1} \cdots \sigma_n^{k_n} \rangle_x d^n x \geqq 0$$

where $\mathbf{R}_+^n = \{x \mid x_i > 0\}$ and $J_{ij}(x) = A_{ij}|x_i||x_j|$.

Conversely, given the ferromagnetic couplings J_{ij} $(i \neq j)$ we can find a matrix A with $A_{ij} = J_{ij}$ if $i \neq j$ and $A_{ii} = a$ such that A is positive definite. Taking $F_i(x_i) = \delta(x_i + 1) + \delta(x_i - 1)$ (or a limit of smooth approximations to this function) we recover the spin inequality from Theorem V.1.

In his general analysis of Griffiths' inequalities, Ginibre [27] isolates a condition (Q3) that the measure and observables should satisfy to yield correlation inequalities. In our case (Q3) is just:

LEMMA V.5. *Let $d\nu$ be the even measure* (V.2) *and let $P_1, \cdots, P_m$ be polynomials in $x \in \mathbf{R}^n$ with nonnegative coefficients. Then for any choice of the plus or minus signs,*

$$\int \prod_{i=1}^{m} (P_i(x) \pm P_i(y)) d\nu(x) d\nu(y) \geqq 0 . \tag{V.5}$$

Proof. Clearly it is sufficient to prove the lemma if each P_i is a monomial of the form $x_1^{i_1} \cdots x_n^{i_n}$. First note that if $n = 1$,

$$\int (x - y)^i (x + y)^j d\nu(x) d\nu(y) \geqq 0 \tag{V.6}$$

for all nonnegative integers i, j. For if $i + j$ is odd the integral vanishes by $(x, y) \to (-x, -y)$ symmetry. If i is odd, it vanishes by $(x, y) \to (y, x)$ symmetry. Thus the integral can be non-zero only if i and j are both even, in which case the integrand is positive. Since

$$(x^i \pm y^i) = 2^{-i+1} \sum_{j \left\{ \substack{\text{even} \\ \text{odd}} \right.} \binom{i}{j} (x + y)^{i-j} (x - y)^j ,$$

we conclude from (V.6) that

$$\int (x^{i_1} \pm y^{i_1}) \cdots (x^{i_m} \pm y^{i_m}) d\nu(x) d\nu(y) \geqq 0 \tag{V.7}$$

for any choice of $\pm$ from each pair and any nonnegative integers $i_1, \cdots, i_m$.

Finally, if $n > 1$ we appeal to the product nature of the measure $d\nu$ and deduce (V.5) from (V.7) by repeated use of

$$\begin{aligned} h_1(x_1)h_2(x_2) &\pm h_1(y_1)h_2(y_2) \\ &= \frac{1}{2}[h_1(x_1) + h_1(y_1)][h_2(x_2) \pm h_2(y_2)] \\ &\quad + \frac{1}{2}[h_1(x_1) - h_1(y_1)][h_2(x_2) \mp h_2(y_2)] . \end{aligned}$$

□

Proof of Theorem V.2. We must show that

$$\int (x^{(j)} - y^{(j)})(x^{(k)} - y^{(k)}) d\mu(x) d\mu(y) \geqq 0 \tag{V.8}$$

where $x^{(j)} = x_1^{j_1} \cdots x_n^{j_n}$. By Lemma V.4, the integral (V.8) can be evaluated by a power series expansion in the off-diagonal part of the Gaussian measure. But each term in this expansion

$$\frac{1}{m!}\int (x^{(j)} - y^{(j)})(x^{(k)} - y^{(k)})[\textstyle\sum_{i\neq j}(-a_{ij})(x_i x_j + y_i y_j)]^m d\nu(x) d\nu(y)$$

is nonnegative by Lemma V.5. □

Finally, to prove Theorem V.3 we observe that a ferromagnetic measure satisfies the FKG condition:

LEMMA V.6. *Let $d\mu(x) = G(x)d^n x$ be a ferromagnetic measure on $\mathbf{R}^n$. Then*

$$G(x \wedge y)G(x \vee y) \geqq G(x)G(y)\,. \tag{V.9}$$

Notation. Under the ordering $x \succ y$ if $x_i \geqq y_i$, $i = 1, \cdots, n$, $\mathbf{R}^n$ is a distributive lattice with the operations $\vee$ and $\wedge$ given by $(x \vee y)_i = \max(x_i, y_i)$ and $(x \wedge y)_i = \min(x_i, y_i)$.

Proof. By definition $G(x) = F(x)e^{-x\cdot Ax}$. Since $F(x) = F_1(x_1) \cdots F_n(x_n)$ $F(x \vee y)F(x \wedge y) = F(x)F(y)$. Moreover,

$$\begin{aligned}(x \wedge y)\cdot A(x \wedge y) &+ (x \vee y)\cdot A(x \vee y) - x\cdot Ax - y\cdot Ay \\ &= 2 \sum_{\substack{i,j \\ x_i<y_i,\, x_j>y_j}} a_{ij}(x_i y_j + y_i x_j - x_i x_j - y_i y_j) \\ &= 2 \sum_{\substack{i,j \\ x_i<y_i,\, x_j>y_j}} a_{ij}(x_i - y_i)(y_j - x_j) \leqq 0\end{aligned}$$

since $a_{ij} \leqq 0$ if $i \neq j$. □

Proof of Theorem V.3. By Lemma V.6 we see that the theorem is just a continuum version of the usual FKG inequality [20], [49]. To reduce the theorem to the case of a *finite* distributive lattice, we simply approximate $\langle f\rangle_\mu$ by

$$\langle f\rangle_N = \frac{\sum_{|j_i|\leqq N^2} f(j_1/N, \cdots, j_n/N)G(j_1/N, \cdots, j_n/N)N^{-n}}{\sum_{|j_i|\leqq N^2} G(j_1/N, \cdots, j_n/N)N^{-n}}\,.$$

Then by FKG, $\langle fg\rangle_N \geqq \langle f\rangle_N\langle g\rangle_N$, and letting $N \to \infty$ yields the theorem. □

We conclude this subsection with a discussion of the necessity of the condition that μ be even in Theorems V.1 and V.2. This condition cannot be dropped completely as the following example shows:

Example. Consider the ferromagnetic measure $d\mu(x) = e^{(-1/2)\vec{x}\cdot A\vec{x} - \vec{\lambda}\cdot\vec{x}}$, i.e., $F_i(x_i) = e^{-\lambda_i x_i}$, $i = 1, \cdots, n$. Setting $m = -A^{-1}\lambda$, we see that

$$\frac{1}{2}x\cdot Ax + \lambda\cdot x = \frac{1}{2}(x - m)\cdot A(x - m) + \frac{1}{2}m\cdot\lambda ,$$

so that the effect of the linear perturbation is to change the means but not the covariance of x_i's. From the joint characteristic function of the x_i's,

$$\chi(q) = \int e^{iq\cdot x}d\mu(x)\Big/\int d\mu(x) = \exp\left[im\cdot q - \frac{1}{2}q\cdot A^{-1}q\right] ,$$

we can read off the various expectations of products of x_i's. For example,

$$\langle x_1x_2\rangle_\mu = \frac{1}{i^2}\frac{\partial^2\chi}{\partial q_1\partial q_2}\bigg|_{q=0} = (A^{-1})_{12} + m_1m_2 ,$$

$$\langle x_1x_2x_3\rangle_\mu = m_1m_2m_3 + m_1(A^{-1})_{23} + m_2(A^{-1})_{13} + m_3(A^{-1})_{12} ,$$

and so on. Therefore,

$$\langle x_1x_2x_3\rangle_\mu - \langle x_1x_2\rangle_\mu\langle x_3\rangle_\mu = m_1(A^{-1})_{23} + m_2(A^{-1})_{13} . \tag{V.10}$$

Now we know by Theorem A.1 that A^{-1} always has nonnegative entries; certainly by a specific choice of A we can arrange that A^{-1} have strictly positive entries. Consequently the right side of (V.10) is given by an expression of the form $\sum c_i\lambda_i$ where $c_i < 0$. We conclude that G-II holds if all the $\lambda_i \leqq 0$, but fails if $\lambda_i > 0$ (see Remark 3 after Theorem V.10).

On the other hand, it is formally clear that Theorem V.1 will still hold if F even is replaced by the assumption that

$$F(x) = G(x)\sum_0^\infty c_ix^i \tag{V.11}$$

where $G(x)$ is even and $c_i \geqq 0$. From the role of Lemma V.5, we see that Theorem V.2 is formally true if in addition products of F's satisfy

$$\prod_{j=1}^n F_j(x_j)F_j(y_j) = \prod_{j=1}^n G_j(x_j)G_j(y_j)\sum_k \prod_{i=1}^{m_k}(P_{ki}(x) \pm P_{ki}(y)) , \tag{V.12}$$

where $G_j(x_j)$ is an even function of x_j and the P_{ki} are polynomials in $x \in \mathbf{R}^n$ with nonnegative coefficients. Rather than attempting to characterize such F_j (not to mention proving the convergence of the series!), we are content to notice that one important class of F_j's satisfying (V.11) and (V.12) is the class

$$F_j(x_j) = e^{-P_j(x_j)+Q_j(x_j)} \tag{V.13}$$

where P_j is an even semibounded polynomial and Q_j is an odd polynomial with nonnegative coefficients. Obviously (V.11) and (V.12) are satisfied if we set $G_j = e^{-P_j}$ and expand the exponentials e^{Q_j}. If in addition we require that

(V.14) $$\deg Q_j < \deg P_j$$

then we can prove the convergence of the series involved. For, as in the proof of Lemma V.4, define

$$f(\lambda) = \int x^{(k)} \prod_j e^{-P_j(x_j)+\lambda Q_j(x_j)} e^{-x\cdot Ax} d^n x$$

for $\lambda \in [0, 1]$. By (V.14), $f(\lambda)$ is analytic in a neighbourhood of $[0, 1]$, and since its derivatives at the origin are nonnegative by Theorem V.1, $f(1)$ can be evaluated by a convergent power series about $\lambda = 0$ and is thus nonnegative. The argument for G-II is similar and we obtain:

THEOREM V.7. *Let μ be a ferromagnetic measure on $\mathbf{R}^n$ whose perturbing functions are exponentials of polynomials as described in* (V.13) *and* (V.14). *Then we have the inequalities $\langle x^{(j)}\rangle_\mu \geqq 0$ and $\langle x^{(j+k)}\rangle_\mu \geqq \langle x^{(j)}\rangle_\mu \langle x^{(k)}\rangle_\mu$, where j and k are n-tuples of nonnegative integers.*

V.2. *Correlation Inequalities for Markov Fields.* We now wish to prove correlation inequalities for two-dimensional interacting Markov fields. The reader will recognize that in § V.1 we have already established the inequalities for interacting *lattice* Markov fields with general boundary conditions, and so in this section we simply invoke the approximation theorem of § IV.1. The same methods yield corresponding correlation inequalities in one dimension; see Appendix B for alternate proofs of these results for the anharmonic oscillator which do *not* use the lattice approximation.

For P a given semibounded polynomial and $g \in C_0^\infty(\mathbf{R}^2)$ we denote the expectation value with respect to the measure (II.25) by $\langle \cdot \rangle_g$, and we define the Schwinger functions

$$S_g(x_1, \cdots, x_n) = \langle \phi(x_1) \cdots \phi(x_n)\rangle_g$$

as distributions. A distributional inequality such as $S_g(x_1, x_2) \geqq 0$ means that $\int S_g(x_1, x_2) f_1(x_1) f_2(x_2) dx \geqq 0$ for all nonnegative $f_1, f_2 \in C_0^\infty(\mathbf{R}^2)$.

THEOREM V.8. *Let the* (semibounded) *polynomial $P(X) = P_e(X) + \lambda X$ where P_e is even and $\lambda \leqq 0$. Then if $x_1, \cdots, x_n \in \mathbf{R}^2$,*

$$S_g(x_1, \cdots, x_n) \geqq 0 .$$

THEOREM V.9. *Let $P = P_e + \lambda X$, P_e even, $\lambda \leqq 0$. If $x_1, \cdots, x_{n+m} \in \mathbf{R}^2$,*

$$S_g(x_1, \cdots, x_{n+m}) \geqq S_g(x_1, \cdots, x_n) S_g(x_{n+1}, \cdots, x_{n+m}) .$$

THEOREM V.10. *Let P be an arbitrary* (semibounded) *polynomial. Let F and G be continuous, polynomially bounded functions on $\mathbf{R}^r$ such that $F, G \uparrow$, as in Theorem* V.3. *If $h_1, \cdots, h_r \in C_0^\infty(\mathbf{R}^2)$ are nonnegative, then*

$$\langle F(\phi(h_1), \cdots, \phi(h_r))G(\phi(h_1) \cdots)\rangle_g \geqq \langle F(\phi(h_1) \cdots)\rangle_g \langle G(\phi(h_1) \cdots)\rangle_g .$$

Remarks 1. We have taken the functions g and h_j in C_0^∞ only for convenience. By a limiting argument Theorems 8-10 extend to more general g and h_j (e.g., g the characteristic function of a bounded region) and *to any limit point of the* $\langle\cdot\rangle_g$ as $g \to 1$.

2. On the basis of Theorem V.7 one might expect that in Theorems V.8 and V.9 the odd part of P could be arbitrary as long as the coefficients were nonpositive. This is so for the *one-dimensional* case *without Wick ordering.* The difficulty is that Wick ordering introduces lower degree powers of opposite sign (e.g., $:\phi^3: = \phi^3 - \infty\phi$) so that it is not possible to prove Griffiths' inequalities for $P(\phi)_2$ theories with cubic (or higher odd) terms.

3. If $P = P_e + \lambda X$ we see that the odd (even) Schwinger functions are odd (even) in λ by $X \to -X$ covariance. Thus for $\lambda > 0$,

$$(-1)^n S_g(x_1, \cdots, x_n) \geqq 0 , \tag{V.15a}$$

$$(-1)^{n+m}[S_g(x_1, \cdots, x_{n+m}) - S_g(x_1, \cdots, x_n)S_g(x_{n+1}, \cdots, x_{n+m})] \geqq 0 . \tag{V.15b}$$

4. Although we have stated our theorems for the case of "free" boundary conditions, correlation inequalities hold for general boundary conditions according to the discussion of § IV.2.

Proof of Theorems V.8 *and* V.9. By the convergence theorem (Corollary IV.3) it is sufficient to prove the inequalities in the lattice approximation when ϕ is replaced by ϕ_δ. But then each expectation involves only finitely many Gaussian variables (see IV.13) and the corresponding measure is ferromagnetic by Theorem IV.4. Thus Theorems V.8–9 follow from Theorem V.7. □

Proof of Theorem V.10. We may suppose that F and G are bounded. For let H_M be the function $H_M(x) = x$ if $|x| \leqq M$, $H_M(x) = 0$ otherwise. Then $F^{(M)} \equiv H_M \circ F$ is increasing ($\uparrow$) and if we can prove that $\langle F^{(M)}G^{(M)}\rangle \geqq \langle F^{(M)}\rangle\langle G^{(M)}\rangle$, then the inequality holds for F and G by limits. Now if F is bounded, then $F_\delta \equiv F(\phi_\delta(h_1), \cdots, \phi_\delta(h_r)) \to F$ in each L^p as $\delta \to 0$. Thus we need only show that $\langle F_\delta G_\delta\rangle \geqq \langle F_\delta\rangle\langle G_\delta\rangle$ where the expectations are in terms of the lattice variables $q_1, \cdots, q_N$ as in (IV.13). This latter inequality follows from Theorem V.3 and the observation that $F_\delta(q_1, \cdots, q_N)\uparrow$ whenever $F(X_1, \cdots, X_r)\uparrow$ since the smearing functions h_j are nonnegative. □

V.3. *Correlation Inequalities for Wick Powers*? Correlation inequalities involving Wick powers would be extremely useful. Our discussion of such inequalities is divided into three parts:

a) Inequalities that hold;

b) inequalities that fail (but seem tempting);

c) inequalities that may hold (some conjectures).

We regret that part a) is considerably shorter than parts b) and c).

a) *Inequalities that hold*: We have but one result to report here:

THEOREM V.11. *Consider the spatially cutoff $P(\phi)_2$ theory with $P = P_e + \lambda X$, $\lambda \leqq 0$. Then*

$$\langle \phi(x_1) \cdots \phi(x_n) : \phi^2(y) : \rangle_g \geqq \langle \phi(x_1) \cdots \phi(x_n) \rangle_g \langle : \phi^2(y) : \rangle_g , \tag{V.16}$$

where $x_1, \cdots, x_n, y \in \mathbf{R}^2$.

Proof. Let h be a nonnegative approximation to the δ function in $C_0^\infty(\mathbf{R}^2)$, and let $\phi_h(x) = \int \phi(y)h(x-y)dy$. By Theorem V.9,

$$\langle \phi_h(x_1) \cdots \phi_h(x_n)\phi_h(y_1)\phi_h(y_2) \rangle \geqq \langle \phi_h(x_1) \cdots \phi_h(x_n) \rangle \langle \phi_h(y_1)\phi_h(y_2) \rangle .$$

Both sides remain non-singular for coincident arguments, in particular for $y_1, y_2 \to y$. Since $\langle A(B - \langle B \rangle) \rangle \geqq \langle A \rangle \langle B - \langle B \rangle \rangle$ if $\langle AB \rangle \geqq \langle A \rangle \langle B \rangle$, we have

$$\langle \phi_h(x_1) \cdots \phi_h(x_n) : \phi_h^2(y) : \rangle \geqq \langle \phi_h(x_1) \cdots \phi_h(x_n) \rangle \langle : \phi_h^2(y) : \rangle .$$

Letting $h \to \delta$ we have convergence in the sense of distributions so that (V.16) holds as a distributional inequality. □

b) *Inequalities that fail*:

(i) *Singly Wick-ordered inequalities.* We want to show that the inequality

$$\langle \phi(x_1)\phi(x_2) : \phi^4(x) : \rangle_g \geqq \langle \phi(x_1)\phi(x_2) \rangle_g \langle : \phi^4(x) : \rangle_g \tag{V.17a}$$

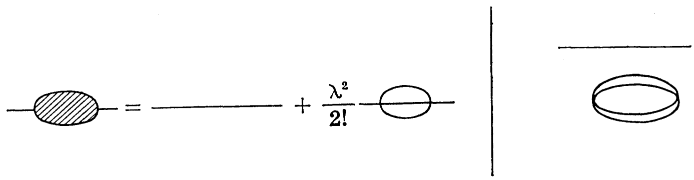
is false for a cutoff $(\phi^4)_2$ Markov theory. Rather than disprove the inequality directly, we shall show that one of its consequences is false. The following theorem answers a question raised in [101] and shows that ϕ^2 [82] can be a misleading guide to ϕ^4:

THEOREM V.12. *Fix the bare mass $m > 0$ and the cutoff $l > 0$. Let $E(\lambda)$ be the vacuum energy for the Hamiltonian $H_l = H_0 + \lambda \int_{-l/2}^{l/2} : \phi^4(x) : dx$, and Ω_λ the vacuum vector. Then for $\lambda > 0$ sufficiently small, there is an eigenvalue in $(E(\lambda), E(\lambda) + m)$ with an eigenvector ψ satisfying $(\psi, \phi(x)\Omega_\lambda) \neq 0$.*

Remarks 1. One reason for expecting such a result is that in second order Rayleigh-Schrödinger theory the gap for a one-dimensional $\lambda : q^4 :$ oscillator is decreasing in λ.

2. A second source of intuition comes from the Feynman perturbation series for the two-point function in the Euclidean region. There is no first

order term and the second order term is positive (Fig. V.1(a)):



(a) (b)

FIGURE V.1. The propagator in second order.

Thus in perturbation theory, $S_2^{\text{int}}(p = 0) \geqq S_2^{\text{non-int}}(p = 0)$. From the Kallen-Lehmann representation we conclude that

$$\int_0^\infty \frac{d\rho(\mu^2)}{\mu^2} \geqq \frac{1}{m^2}\,.$$

Since $\int_0^\infty d\rho = 1$ (the CCR hold!), $m_{\text{phys}} \leqq m$ with equality possible only if $d\rho = \delta(\mu^2 - m^2)d\mu^2$. Of course there is a cancellation of a second order vacuum bubble (Fig. V.1(b)) in second order. Our calculation below follows this intuitive idea including the vacuum cancellation.

3. It is easy to extend this result to an arbitrary space cutoff $g \in L^1 \cap L^{1+\epsilon}(\mathbf{R})$, $g \geqq 0$, $g \not\equiv 0$.

4. Our proof illustrates the usefulness of Dirichlet states even for the study of H_l. For it is hard to work directly with H_l because there is no excited eigenvalue about which to perturb: An approximate eigenpacket is not appropriate for a small λ calculation. Of course the Dirichlet Hamiltonian $H_{0,l}^D$ of § II.6 has purely discrete spectrum. Thus we first prove the theorem for the Dirichlet Hamiltonian H_l^D of (II.93) from which it follows for H_l.

5. This result also shows that for small coupling $(\phi^4)_2$, $m_{\text{phys}} < m$ (see (V.22b)).

Proof. We claim that it is sufficient to prove that for small λ, H_l^D has an eigenvalue in the "gap" $(E^D(\lambda), E^D(\lambda) + m)$, i.e., an eigenvalue $E^D(\lambda) + \Delta E^D(\lambda)$ where $\Delta E^D(\lambda) < m$. For by Theorem V.24 below it follows that H_l has spectrum in $(E(\lambda), E(\lambda) + m)$. Since we know that H_l has only point spectrum in the "gap" $(E(\lambda), E(\lambda) + m)$ [32], we conclude that there is an eigenvalue in the "gap" and that, by Theorem V.14 below, a corresponding eigenvector is coupled by the field to the vacuum.

It remains then to find a vector η_λ orthogonal to Ω_λ^D such that

$$\Delta(\lambda) \equiv E^D(\lambda) + m - \frac{(\eta_\lambda, H_l^D \eta_\lambda)}{(\eta_\lambda, \eta_\lambda)}$$

is positive. Define

$$\eta_\lambda = a^*(0)\Omega_0 - \lambda(H_{0,l}^D - m)^{-1}H_I^D a^*(0)\Omega_0$$

where $a^*(0)$ is the zero momentum creation operator and $H_I^D = \int_{-l/2}^{l/2} :\phi^D(x)^4: dx$. Since Ω_λ^D is even in ϕ^D, η_λ is orthogonal to Ω_λ^D. Moreover, since the perturbation series for $E^D(\lambda)$ is asymptotic in λ [106], $\Delta(\lambda)$ is given, up to second order, by the λ^2 diagram of Fig. V.1 and is thus positive. □

COROLLARY V.13. *For* $(\phi^4)_2$, *the inequality* (V.17a) *does not hold for all* $x_1, x_2, x \in \mathbf{R}^2$ *and all cutoffs* $g \in C_0^\infty(\mathbf{R}^2)$.

Proof. If it did, then by the arguments of § V.4 below we would have

$$\langle\phi(x_1)\phi(x_2)\rangle_g \leqq \langle\phi(x_1)\phi(x_2)\rangle_0 . \tag{V.17b}$$

But this would imply by Theorem II.17 that

$$\left(\Omega_h, \phi(f)e^{-t(H(h)-E(h))}\phi(f)\Omega_h\right) \leqq \left(\Omega_0, \phi(f)e^{-tH_0}\phi(f)\Omega_0\right)$$

for any $f \geqq 0$ in $C_0^\infty(\mathbf{R})$ and any $h \geqq 0$ in $L^1 \cap L^{1+\epsilon}(\mathbf{R})$. Consequently for any ψ with $(\psi, \phi(f)\Omega_h) \neq 0$,

$$\left(\psi, (H(h) - E(h))\psi\right) \geqq m \,\|\psi\|^2 ,$$

contradicting Theorem V.12. □

Since we expect that the mass gap for $(\phi^4)_2$ will be monotone decreasing in the cutoff (cf. Theorem V.23) it is possible that the *reverse* of (V.17) holds. Nevertheless

Example 1. For the $(\phi^4)_2$ theory, neither

$$\langle\phi(x_1)\cdots\phi(x_4):\phi^4(x):\rangle \leqq \langle\phi(x_1)\cdots\phi(x_4)\rangle\langle:\phi^4(x):\rangle \tag{V.17c}$$

nor its reverse holds for all $x_1, \cdots, x_4, x \in \mathbf{R}^2$ and all cutoffs g. For when $g = 0$, $\langle\phi(x_1)\cdots\phi(x_4):\phi^4(x):\rangle > 0$ by explicit calculation so that (V.17c) fails. On the other hand, if the reverse of (V.17c) held then we would conclude that

$$\langle\phi(x_1)\cdots\phi(x_4)\rangle_g \leqq \langle\phi(x_1)\cdots\phi(x_4)\rangle_0 .$$

Taking the distance between x_1, x_2 and x_3, x_4 to infinity, we deduce (V.17b) by clustering, thereby contradicting Corollary V.17.

(ii) *Doubly Wick-ordered inequalities.* For simplicity of notation we discuss the one-dimensional theory. As we shall describe below we believe that in a $:q^4:$ theory,

$$\langle:q^4(t)::q^4(s):\rangle \geqq \langle:q^4(t):\rangle\langle:q^4(s):\rangle$$

may hold. However

Example 2. For all $:q^4:$ theories, neither

$$\langle:q(t_1)\cdots q(t_6)::q^4(s):\rangle \geqq \langle:q(t_1)\cdots q(t_6):\rangle\langle:q^4(s):\rangle$$

nor its reverse holds. For in lowest order perturbation theory (first order) the left side is negative and the right size is zero. On the other hand if $t_1 = t_2 \to \infty$ and $t_3 = \cdots = t_6 = s$, the left side goes to $\langle : q^2 : \rangle \langle (: q^4 :)^2 \rangle$ which strictly exceeds $\langle : q^2 : \rangle \langle : q^4 : \rangle^2$ (the limit of the right side) in lowest order non-zero perturbation theory (second!).

(iii) *Triply Wick-ordered inequalities.* One might hope that

$$\langle : q^4(s) : : q^4(t) : : q^4(u) : \rangle \leqq \langle : q^4(s) : \rangle \langle : q^4(t) : : q^4(u) : \rangle .$$

Unfortunately this is false for the free field, and, if our conjectures (i) and (ii) below are true, the reverse also fails, as can be seen by clustering.

c) *Inequalities that may hold* (*some conjectures*).

(i) *Monotonicity of the energy.* Since we know that the vacuum energy $E_l = E(\chi_l)$ is monotone in l [42], it is natural to conjecture that $E(g)$ is monotone in g. Such a result would follow easily from:

Conjecture. In a $P(\phi)_2$ theory with P even and $P(0)=0$, $\langle : P(\phi(x)) : \rangle_g \leqq 0$; i.e., for any f, $g \geqq 0$ in $C_0^\infty(\mathbf{R}^2)$,

$$\int d\mu_0 \int : P(\phi(x)) : f(x) dx \exp\Big(-\int g(x) : P(\phi(x)) : dx\Big) \leqq 0 .$$

(ii) *Strong correlations on $P(\phi)$.* The fact that the energy per unit volume is monotone suggests that the regions in Q-space where $: P(\phi(x)) :$ is very negative are correlated for different x's; i.e.,

Conjecture. In a $P(\phi)_2$ theory with P even,

$$\langle : P(\phi(x_1)) : : P(\phi(x_2)) : \rangle_g \geqq \langle : P(\phi(x_1)) : \rangle_g \langle : P(\phi(x_2)) : \rangle_g .$$

(iii) *Correlations on the correlation functions.* The preceding conjecture suggests that

Conjecture. In a $P(\phi)_2$ theory with P even let $U(f) = \int f(x) : P(\phi(x)) : dx$. For any f, g, $h \geqq 0$ in $C_0^\infty(\mathbf{R}^2)$,

$$\langle e^{-U(f)} e^{-U(h)} \rangle_g \geqq \langle e^{-U(f)} \rangle_g \langle e^{-U(h)} \rangle_g .$$

This is our most significant conjecture since it is equivalent to

$$\langle e^{-U(f)-U(h)-U(g)} \rangle_0 \langle e^{-U(g)} \rangle_0 \geqq \langle e^{-U(f)-U(g)} \rangle_0 \langle e^{-U(h)-U(g)} \rangle_0$$

or

(V.17d) $$\langle e^{-U(h)} \rangle_{f+g} \geqq \langle e^{-U(h)} \rangle_g .$$

(V.17d) together with a local L^p-estimate would imply convergence of $\langle e^{-U(h)} \rangle_g$ as $g \to 1$. We expect that this in turn would lead to local L^p-convergence of the states.

V.4. *Applications.* The traditional applications of Griffiths' inequalities for ferromagnets in statistical mechanics are of three types [38], [85]:

(A) monotonic convergence of states in the infinite volume limit;

(B) monotonic behaviour of correlation lengths as the interaction is made more ferromagnetic;

(C) persistence of phase transitions if an interaction is made more ferromagnetic.

This section is divided into three parts (infinite volume limit of states, monotonicity of the mass gap, and broken symmetry) as we discuss the corresponding applications of Griffiths' inequalities to $P(\phi)$ field theories. For $P(\phi)_2$, "more ferromagnetic" means that the coefficient of ϕ^2 has been decreased, which amounts to a *decrease in the bare mass.* Nelson's important application of Griffiths' inequalities to the monotone convergence of half-Dirichlet states is discussed briefly and we explain how monotonicity under a change in "*local* bare mass" is involved; for details we refer the reader to Nelson's paper [69].

The FKG inequalities have also proved useful for $P(\phi)_2$, having been exploited by Simon in the proof of the following results:

THEOREM V.14 (Simon [103]). *Let P be an arbitrary semibounded polynomial.*

(i) *For nonnegative $g \in L^1 \cap L^{1+\epsilon}(\mathbf{R})$, let Ω_g be the vacuum for $H(g)$. Then $\{\phi(f)\Omega_g \mid f \geqq 0, f \in \mathcal{S}(\mathbf{R})\}$ is coupled to the first excited state of $H(g)$.*

(ii) *Consider an infinite volume $P(\phi)_2$ theory satisfying Nelson's Axioms* (A)–(F) *of* § II.2, *and assume that the* FKG *inequalities hold* (they will if, in particular, the theory is a limit of cutoff theories with free, half-Dirichlet, or Dirichlet B.C.). *Let H be the corresponding Hamiltonian and Ω its* (unique) *vacuum in the relativistic Hilbert space $\mathcal{H} = E_0 L^2(Q)$* (cf. Theorem II.7). *Then $\{E_0(\phi(f)) \mid f \geqq 0, f \in C_0^\infty(\mathbf{R}^2)\}$ is coupled to the first excited state of H.*

The statement that a set of vectors S is "coupled to the first excited state" of a Hamiltonian H with ground state Ω means that the infimum of the supports of the spectral measures of H associated with the vectors in S is equal to $E_1 = \inf \sigma(H \restriction \{\Omega\}^\perp)$ [103]. In addition, if E_1 is an eigenvalue then there is a $\psi \in S$ such that $(\Omega_1, \psi) \neq 0$, where Ω_1 is an eigenvector corresponding to E_1. It thus follows from Theorem V.14 that if the polynomial P is *even,* then (a) there is an eigenvector Ω_1 which is odd under the symmetry $\phi \to -\phi$, and (b) the rate of fall-off of the two-point Schwinger function precisely determines the gap ΔE between E_1 and the vacuum energy E_0:

COROLLARY V.15 [103]. *Consider either the case* (i) *or* (ii) *of Theorem* V.14 *and assume further that P is even. Then the gap*

$$\Delta E = -\sup_{\substack{f \geq 0 \\ f \in C_0^\infty(\mathbf{R}^2)}} \lim_{t\to\infty} \frac{1}{t} \ln \int S(x, s; y, u) f(x, s) f(y, u - t) dx ds\, dy du\ ,$$

where in case (i) *S stands for the Schwinger function defined in* (II.38), i.e., *with cutoff function* $h(x, s) = g(x)$.

(A) *Infinite volume convergence of states*

By an analytic continuation argument using the uniform bounds on the fields of [33], [42], Glimm and Spencer have shown [35] that the convergence of the non-coincident Schwinger functions in the infinite volume limit implies the convergence of the Wightman functions or of the C^*-states of the relativistic cutoff theory. We thus focus our attention on the limit $\lim_{\Lambda\to\infty} S_\Lambda(x_1, \cdots, x_n)$ where S_Λ is the Schwinger function with spatial cutoff χ_Λ.

In the one-dimensional case, Griffiths' inequalities give more detailed information about the approach to infinite volume than the transfer matrix techniques of § II.4:

THEOREM V.16. *Let* $d\mu_0$ *be the free Markov measure for* $d = 1$ *as defined in* § II, *and let* $P(q) = \sum_{i=1}^n a_{2i} q^{2i}$ *where* $a_{2i} \geqq 0$. *As* $a, b \longrightarrow \infty$ *the Schwinger functions*

$$S_{a,b}(t_1, \cdots, t_r) = \langle q(t_1) \cdots q(t_r)\rangle_{a,b}$$
$$= \frac{\int d\mu_0 q(t_1) \cdots q(t_r) \exp\left(-\int_{-a}^b P(q(s))ds\right)}{\int d\mu_0 \exp\left(-\int_{-a}^b P(q(s))ds\right)}$$

decrease monotonically to a (nonnegative) *infinite volume limit.*

Proof. By the first Griffiths' inequality (§ V.2 or Appendix B), $S_{a,b} \geqq 0$, so that convergence is established by the monotonicity statement. Consider the function $f(b) = \exp\left(-\int_{-a}^b P(q(s))ds\right)$ as a function in $L^p(Q, d\mu_0)$ for $p < \infty$. It is easy to see by the Duhamel formula that $f(b)$ is differentiable in b and that $f'(b) = -P(q(b))f(b)$. Therefore

$$\frac{\partial}{\partial b} S_{a,b} = -\langle q(t_1) \cdots q(t_r) P(q(b))\rangle_{a,b} + \langle q(t_1) \cdots q(t_r)\rangle_{a,b} \langle P(q(b))\rangle_{a,b} \leqq 0$$

by the second Griffiths' inequality. Similarly $S_{a,b}$ is decreasing in a. □

In the two-dimensional case we are only in a position to control Wick powers of degree 2 or less:

LEMMA V.17. *Consider the spatially cutoff Markov theory with interaction* $U = :Q(\phi)(g): + \mu :\phi^2(h): + \lambda\phi(f)$, *where Q is an even polynomial,*

f, g, h are nonnegative functions in $C_0^\infty(\mathbf{R}^2)$, and μ, λ are real. If $\lambda \leqq 0$, the Schwinger functions are monotonically decreasing functions of μ and λ. If $\lambda > 0$, the even Schwinger functions are decreasing in μ and increasing in λ, whereas the odd Schwinger functions are increasing in μ and decreasing in λ.

Remark. The conclusion of the lemma can be summarized more succinctly in this way: By Remark 3 following Theorem V.10 we see that if $\lambda > 0$, the odd (even) Schwinger functions are negative (positive). If we define the absolute value of the Schwinger function by,

$$|S_g(x_1, \cdots, x_n)| = \begin{cases} S_g(x_1, \cdots, x_n) & \text{if } \lambda \leqq 0 \quad \text{or } n \text{ even} \\ -S_g(x_1, \cdots, x_n) & \text{if } \lambda > 0 \quad \text{and } n \text{ odd} , \end{cases}$$

then the lemma asserts that $|S_g|$ is a decreasing function of μ and an increasing function of $|\lambda|$.

Proof. Take $\lambda \leqq 0$. The proof of monotonicity in μ is analogous to that of Theorem V.16. One uses the formula for derivatives of $\langle A\rangle = \int Ae^{-U}d\mu_0 \Big/ \int e^{-U}d\mu_0$:

$$\frac{\partial}{\partial\mu}\langle A\rangle = -\langle A : \phi^2(h) :\rangle + \langle A\rangle\langle : \phi^2(h) :\rangle$$

which is valid for all $A \in L^p$, $p < \infty$. Thus $\partial S/\partial\mu$ is nonpositive by Theorem V.11. The case $\lambda > 0$ and the monotonicity in λ follow similarly from (V.15). □

As an immediate consequence of the lemma, we obtain these two results:

THEOREM V.18. *For the $(\phi^2)_2$ theory, the Schwinger functions converge monotonically downward to an infinite volume limit.*

THEOREM V.19. *Let $E(\cdot)$ be the expectation associated with any Euclidean field theory arising as an infinite volume limit of spatially cutoff $P(\phi)_2$ theories, where P is even. Suppose that for all $g \in C_0^\infty(\mathbf{R}^2)$ with $0 \leqq g \leqq 1$, $e^{-:\phi^2(g):} \in L^1$;* i.e., *$E(e^{-:\phi^2(g):}) < \infty$. Then the Schwinger functions for the theory with expectation $E_g(\cdot) = E(\cdot\, e^{-:\phi^2(g):})/E(e^{-:\phi^2(g):})$ converge as $g \to 1$.*

Remark. Presumably Theorem V.18 could be proved by direct calculation as in the ϕ^2 Hamiltonian theory [82].

The above results are rather disappointing inasmuch as they apply only to quadratic interactions when $d = 2$. In fact, as the discussion of § V.3 indicates, it is unlikely that any of the *free* B.C. Schwinger functions converge monotonically for $:\phi_2^{2n}:$, $n \geqq 2$. At this stage of knowledge, our problem was that we did not understand the significance of boundary conditions, but E. Nelson clarified the situation in [69] by considering the Schwinger functions for *half-Dirichlet* B.C. His result is:

THEOREM V.20A. (Nelson [69]). *Consider the $P(\phi)_2$ Markov field theory where P is even. Then the Schwinger functions S_Λ^{HD} with half-Dirichlet* B.C. *on $\partial\Lambda$ are monotonically increasing in Λ and the non-coincident Schwinger functions converge to an infinite volume limit as $\Lambda \to \infty$.*

Remarks 1. As in (V.15) the convergence also holds if there is a linear term in P: $P = P_e + \lambda X$, P_e even, λ real. In this case it is the absolute value $|S_\Lambda^{HD}|$ which increases with Λ.

2. For small coupling constant but arbitrary semibounded P, Glimm and Spencer [35] have proved that the "free" Schwinger functions, S_Λ, converge as $\Lambda \to \infty$.

Note that Nelson's convergence result is valid for arbitrary coupling constant.

3. As in the case with the free B.C. result of Glimm and Spencer, the convergence of the half-Dirichlet Schwinger functions should imply convergence of the corresponding Wightman functions. See [127]

4. The assumption of non-coincidence is used to bound the sequence $\{S_\Lambda^{HD}\}$ uniformly in Λ; the proof relies on (V.18) and the linear estimates of [33], [42].

5. Since the convergence of the theorem holds for arbitrary regions (under the ordering of inclusion) it is easy to prove the Euclidean invariance of the limiting Schwinger functions.

6. Given Theorem V.20A it is not hard to show that the limiting Schwinger functions are those associated with a Minkowski field theory satisfying all of the Wightman axioms except possibly uniqueness of the vacuum. This follows by verifying a suitable set of Osterwalder-Schrader axioms [74], [119]. The following alternate method should also be possible:

Prove the convergence of suitable Wightman functions which establishes all of the axioms except Lorentz invariance and the spectral condition. Lorentz invariance follows from Euclidean invariance as in Nelson [67] and then the spectral condition follows from the positivity of the Hamiltonian.

For details we refer the reader to Nelson's paper, but we comment briefly here on the significance of half-Dirichlet B.C. It is critical that half-Dirichlet B.C. rather than Dirichlet B.C. are used; for then Wick ordering is defined with respect to the *free measure* and as a result the definition does not change as Λ is changed. The role of half-Dirichlet (as opposed to free) B.C. is to decouple the variables in Λ and its complement so that increasing Λ has the effect of introducing more ferromagnetic bonds. To understand this statement consider the lattice approximation of § IV. If $\Lambda \subset \Lambda'$, a half-Dirichlet expectation of fields in Λ is the same as the expectation of these fields in Λ' but with the matrix $A^{\Lambda'}$ in the Gaussian replaced by $A^{\Lambda} \oplus A^{\Lambda'\backslash\Lambda}$. The correct expectation in Λ' is then obtained by turning on the neglected couplings in the matrix $B = A^{\Lambda'} - A^{\Lambda} \oplus A^{\Lambda'\backslash\Lambda}$. But B is nonpositive and so G-II implies that the expectation in Λ is less than the expectation in Λ'. A similar argument fails for the free B.C. expectations because in addition to turning on the neglected couplings B we must replace $B_{\partial\Lambda}$ by $B_{\partial\Lambda'}$ and thus there is no consistent increase of ferromagnetic bonds. Incidentally, the fact that half-Dirichlet expectations are obtained from free B.C. expectations by adding the quadratic from $\delta^2\vec{q}\cdot B_{\partial\Lambda}\vec{q}$ in the exponent, where $B_{\partial\Lambda}$ is nonnegative (Theorem IV.7), shows by G-II that

$$S_\Lambda^{HD}(x_1, \cdots, x_n) \leqq S_\Lambda(x_1, \cdots, x_n) . \tag{V.18}$$

The significance of (V.18) for Theorem V.20A is that it provides the bound needed for convergence.

This inequality, as well as the monotonicity of $\{S_\Lambda^{HD}\}$, can also be understood on the basis of a change of *local* bare mass. We sketch the ideas: In the classical variational problem [8, p. 398] the boundary condition

$$\frac{\partial\phi}{\partial n} + \sigma\phi = 0 \quad \text{for} \quad -\Delta + m^2$$

is arranged through the surface term in the form to be minimized:

$$D(\phi) = \int_\Lambda [(\Delta\phi)^2 + m^2\phi^2] + \sigma\int_{\partial\Lambda}\phi^2 ;$$

Dirichlet B.C. correspond to taking $\sigma \to \infty$. Analogously we expect that S_Λ^{HD} can be obtained from S_Λ by

$$S_\Lambda^{HD} = \lim_{\sigma\to\infty} \frac{\int d\mu_0\phi(x_1)\cdots\phi(x_n)\exp\left(-\sigma\int_{\partial\Lambda}:\phi^2: - U_\Lambda\right)}{\int d\mu_0 \exp\left(-\sigma\int_{\partial\Lambda}:\phi^2: - U_\Lambda\right)} \tag{V.19a}$$

or by

$$S_\Lambda^{HD} = \lim_{\substack{\sigma\to\infty \\ R\to\infty}} \frac{\int d\mu_0\phi(x_1)\cdots\phi(x_n)\exp\left(-\sigma\int_{R\backslash\Lambda}:\phi^2: - U_R\right)}{\int d\mu_0 \exp\left(-\sigma\int_{R\backslash\Lambda}:\phi^2: - U_R\right)} . \tag{V.19b}$$

From (V.19a) and Theorem V.11 we again deduce the inequality (V.18), while from (V.19b) we see that $S_\Lambda^{HD} \leqq S_{\Lambda'}^{HD}$ if $\Lambda \subset \Lambda'$ since less "infinite mass is turned on" when we consider the larger region.

For general even polynomials P we can say nothing about Dirichlet as opposed to half-Dirichlet states since a change in Λ changes the definition of the Wick ordering in the interaction. However, when deg $P = 4$, we can control this change:

THEOREM V.20B. *Consider the $P(\phi)_2$ Markov field theory with $P(X) = aX^4 + bX^2 + cX$ ($a > 0$, $c \leqq 0$). Then the Schwinger functions S_Λ^D with Dirichlet boundary conditions on $\partial\Lambda$ are monotonically increasing in Λ and the non-coincident Schwinger functions converge to an infinite volume limit as $\Lambda \to \infty$.*

Remarks 1. As usual, $c > 0$ is allowed if we consider $|S_\Lambda^D|$.

2. As in the case of Theorem V.20A, the limit Schwinger functions are the Schwinger functions for a quantum field theory obeying all the Wightman axioms except possibly uniqueness of the vacuum.

3. At the current stage of our knowledge, the $(\phi^4)_2$ Dirichlet theory is of special interest, because it is only for this theory that we have controlled both the convergence of the Schwinger functions and the pressure (see § VI.1).

Proof. As in Nelson's proof of Theorem V.20A, we need only prove that

(V.20a) $$S_\Lambda^D(x_1, \cdots, x_n) \leqq S_{\Lambda'}^D(x_1, \cdots, x_n) \qquad \text{if } \Lambda \subset \Lambda'$$

and for boundedness that

(V.20b) $$S_\Lambda^D(x_1, \cdots, x_n) \leqq S_\Lambda(x_1, \cdots, x_n) .$$

We prove (V.20a); the proof of (V.20b) is similar. Let $S_\Lambda^{D,\Lambda'}(x_1, \cdots, x_n)$ denote the Schwinger functions for a theory with a "free field" which is the Dirichlet field in region Λ but with an interaction which is Wick ordered relative to the Dirichlet theory in Λ'. Obviously (V.20a) follows from

(V.20c) $$S_\Lambda^{D,\Lambda'} \leqq S_{\Lambda'}^D$$

and

(V.20d) $$S_\Lambda^D \leqq S_\Lambda^{D,\Lambda'} .$$

(V.20c) follows as in Nelson's proof; for example, in the lattice approximation, we note that to go from $S_\Lambda^{D,\Lambda'}$ to $S_{\Lambda'}^D$, we add extra (nearest neighbour) ferromagnetic couplings. To prove (V.20d), we need to compute

$$:P(\phi(x)):_{D,\Lambda'} - :P(\phi(x)):_{D,\Lambda} \equiv Q(\phi(x)) .$$

By the special nature of P, Q has only a quadratic and a constant term. The quadratic term is, by (V.24) below,

$$6[\delta G_{\Lambda'}(x) - \delta G_\Lambda(x)] : \phi(x)^2 :_{D,\Lambda}$$

where $\delta G_\Lambda(x)$ is given by (II.87). Thus, by Lemma II.37(V), Q has a quadratic term which is negative. Since

$$S_\Lambda^{D,\Lambda'}(x_1, \cdots, x_n) = \frac{\left\langle \phi(x_1) \cdots \phi(x_n) \exp\left(-\int Q\right)\right\rangle_{D,A}}{\left\langle \exp\left(-\int Q\right)\right\rangle_{D,\Lambda}},$$

the constant term in Q does not affect $S_\Lambda^{D,\Lambda'}$. By the Griffiths inequalities (Theorem V.11), the quantity

$$\left\langle \phi(x_1) \cdots \phi(x_n) \exp\left(-\alpha \int Q\right)\right\rangle_{D,A} \Big/ \left\langle \exp\left(-\alpha \int Q\right)\right\rangle_{D,A}$$

is monotone increasing in α, and we obtain (V.20d). □

(B) *Monotonicity of the mass gap*

The field theory analogue of the correlation length for ferromagnets is the mass gap. For $d = 1$ we have:

THEOREM V.21. *Consider the $P(\phi)_1$ Markov field theory for some fixed bare mass with interaction $P(q) = \sum_1^n a_{2i} q^{2i}$ $(a_{2n} > 0)$, and with expectation and Schwinger functions as defined in Theorem* V.16. *Let $\Delta E(a_2, \cdots, a_{2n})$ be the difference between the first two eigenvalues for the anharmonic oscillator Hamiltonian $H = H_0 + P(q)$.* Then

(i) *$S_{a,b}(t_1, \cdots, t_r)$ is a monotone decreasing function of each a_i.*

(ii) *This monotonicity remains true for $S = \lim_{a,b\to\infty} S_{a,b}$.*

(iii) *$\Delta E(a_2, \cdots, a_{2n})$ is a monotone increasing function of each a_i.*

Proof. (i) The proof is similar to that of Theorem V.16 since

$$\frac{\partial}{\partial a_{2i}} S_{a,b}(t_1, \cdots, t_r) = -\int_{-a}^{b} dt[\langle q(t_1) \cdots q(t_r) q^{2i}(t)\rangle - \langle q(t_1) \cdots q(t_r)\rangle\langle q^{2i}(t)\rangle] .$$

(ii) Follows directly from (i).

(iii) Let $\Omega^{(0)}$ and $\Omega^{(1)}$ be the first two eigenfunctions of H. By the proof of Theorem II.18,

$$\langle q(t)q(0)\rangle_{\infty,\infty} = (q\Omega^{(0)}, e^{-t(H-E)} q\Omega^{(0)}) \tag{V.21}$$

where E is the lowest eigenvalue of H. Since $\Omega^{(0)}(q) > 0$ and $\Omega^{(1)}$ has a node only at the origin, $(\Omega^{(1)}, q\Omega^{(0)}) > 0$ by a suitable normalization of $\Omega^{(1)}$; by symmetry $(\Omega^{(0)}, q\Omega^{(0)}) = 0$. Therefore from (V.21)

$$\Delta E(a_2, \cdots, a_{2n}) = -\lim_{t\to\infty} t^{-1} \log \langle q(t)q(0)\rangle_{\infty,\infty} ,$$

and (iii) follows from (ii). □

Remark. While it is intuitively obvious that the gap is monotone in λ

for a $(p^2 + q^2 + \lambda q^4)$ oscillator, we know of no proof of this fact in the literature.

In the two dimensional case we can handle changes only in the quadratic term:

THEOREM V.22. *Consider the $P(\phi)_2$ theory with fixed spatial cutoff g, where P is even. Let $\Delta E_\lambda(g)$ be the gap in the spectrum of $H_\lambda(g) = H_0 + :P(\phi):(g) + \lambda :\phi^2:(g)$ above $E_\lambda^{(0)}(g) = \inf \sigma(H_\lambda(g))$. Then $\Delta E_\lambda(g)$ is an increasing function of λ.*

Proof. An immediate consequence of Corollary V.15 and Lemma V.17. □

A similar argument using Theorem V.20B shows that the gap for the Hamiltonian with Dirichlet B.C. decreases as the space cutoff goes to infinity:

THEOREM V.23. *Let $P(X) = aX^4 + bX^2$; let H_l^D be the Hamiltonian with Dirichlet boundary conditions on the interval $[-l/2, l/2]$, as defined in* (II.93). *Let ΔE_l^D be the gap between the two lowest eigenvalues of H_l^D. Then ΔE_l^D is a decreasing function of l.*

Strictly speaking, this result depends on the monotonicity of $S_{l\times t}^{DF}$, the Schwinger function for the rectangle $[-l/2, l/2] \times [-t/2, t/2]$ with Dirichlet B.C. on the sides $x_1 = \pm l/2$ and *free* B.C. on the sides $x_2 = \pm t/2$. With the methods of §§ II.6 and IV.3 this monotonicity is proved in a similar fashion to that of $S_{l\times t}^D$. The same idea and the inequality (V.20b) yields:

THEOREM V.24. *If $P(X) = aX^4 + bX^2$, the gap ΔE_l^D for the Hamiltonian H_l^D and the gap ΔE_l for the Hamiltonian $H_l = H(\chi_{(-l/2, l/2)})$ are related by*

$$\Delta E_l \leqq \Delta E_l^D . \tag{V.22a}$$

Finally, we have an infinite volume result:

THEOREM V.25. *Let P be an even polynomial and let $\Delta E(P)$ be the gap in the spectrum of the Hamiltonian of the infinite volume theory described by Theorem* V.20. *Then $\Delta E(P(X) + \lambda X^2)$ is a non-decreasing function of λ.*

Proof. The monotonicity of the cutoff two-point Schwinger function, as in Lemma V.17, extends to the infinite volume Schwinger function. The theorem thus follows from Corollary V.15. □

Remarks 1. As in Theorem V.14 this result holds for any infinite volume $P(\phi)_2$ theory satisfying Nelson's axioms and the FKG and Griffiths inequalities.

2. Since $\Delta E(P)$ is interpreted as the physical mass, m_{ph}, we see that m_{ph} is an increasing function of the coefficient of the quadratic term.

3. Combining the ideas of Theorems V.23 and V.24 with Theorem V.12,

we see that for $\lambda(\phi^4)_2$ with small λ,

(V.22b) $$m_{ph} \leqq m_{ph}^D < m \ .$$

A related question is whether m_{ph} is an increasing function of the bare mass m. An argument based on scaling indicates that the answer is yes for small coupling constant (or large m). However, for general m and polynomials P the answer is probably no, except for the case of ϕ^4:

THEOREM V.26. *Let $H(g)$ be the $P(\phi)_2$ Hamiltonian with bare mass $m > 0$, spatial cutoff g, and P an even polynomial of degree 4 or less. Let $\Delta E(m, g, P)$ be the gap in $\sigma(H(g))$ above the ground state energy. Then $\Delta E(m, g, P)$ is an increasing function of m.*

Proof. We wish to show that $\Delta E(m_0, g, P) \leqq \Delta E(m_1, g, P)$ if $m_0 < m_1$. Accordingly we consider the Hamiltonian

$$H(g, h) = H_0(m_0) + \frac{1}{2}(m_1^2 - m_0^2) : \phi^2(h) : + : P(\phi) : (g) - E_{g,h}$$

where $E_{g,h}$ is chosen so that $\inf \sigma(H(g, h)) = 0$. As $h \to 1$, $H(g, h) \to H_1(g)$ in the usual sense, where in the Fock representation for the field ϕ_1 with mass m_1 [82],

$$H_1(g) = H_0(m_1) + : P_1(\phi_1) : (g) - E_{g,1} \ .$$

In this Hamiltonian the Wick dots refer to ordering with respect to the vacuum for mass m_1 and hence $P_1 \neq P$. At any rate, since the gap for $H(g, h)$ increases as h increases by the argument of Theorem V.22, we conclude by lower semi-continuity of the spectrum that $\Delta E(m_0, g, P) \leqq \Delta E(m_1, g, P_1)$.

So far the hypothesis about the degree of P has not entered, but it is needed for the final step, namely,

(V.23) $$\Delta E(m_1, g, P_1) \leqq \Delta E(m_1, g, P) \ .$$

For according to the following lemma, $P_1(X) = P(X) - 6cX^2 +$ const. where the constant $c > 0$, and we see that the inequality (V.23) is just Theorem V.22. □

LEMMA V.27. *For $s = 1$, let $[\phi^n]_0$ and $[\phi^n]_1$ denote the Wick powers of the field in the Fock representations of mass m_0 and m_1 respectively* (see § II.1 for definitions). *Then*

(V.24) $$[\phi^n]_0 = \sum_{j=0}^{[n/2]} \begin{Bmatrix} n \\ j \end{Bmatrix} (-c)^j [\phi^{n-2j}]_1$$

where $c = 1/4\pi \int_{-\infty}^{\infty} [(m_0^2 + k^2)^{-1/2} - (m_1^2 + k^2)^{-1/2}]dk$, and $\begin{Bmatrix} n \\ j \end{Bmatrix}$ is the number of ways of choosing j pairs from n objects,

$$\left\{ \begin{matrix} n \\ j \end{matrix} \right\} = \frac{1}{j!}\binom{n}{2}\binom{n-2}{2}\cdots\binom{n-2j+2}{2} = \frac{n!\,2^{-j}}{(n-2j)!\,j!}\,.$$

Remarks 1. This lemma has already been noted by R. Baumel (private communication).

2. The precise meaning of (V.24) is as follows: if $f \in C_0^\infty(\mathbf{R})$, the smeared Wick powers $[\phi^n]_0(f)$ and $[\phi^n]_1(f)$ are affiliated with the algebra $\mathfrak{M}$ generated by the fields. Thus we can represent the operator $[\phi^n]_0(f)$ on $\mathcal{F}_{m_1}$ where it is given in terms of the $[\phi^n]_1(f)$ by (V.24) on the domain, say, of smooth vectors with a finite number of particles.

Proof. By simple combinatorics, ordinary powers are given in terms of Wick powers by (we omit ultraviolet cutoffs)

(V.25) $$\phi^n = \sum_{j=0}^{[n/2]} \left\{ \begin{matrix} n \\ j \end{matrix} \right\} c_0^j [\phi^{n-2j}]_0$$

where $c_i = 1/4\pi \int (m_i^2 + k^2)^{-1/2} dk$, $i = 0, 1$. It is convenient to write (V.25) as a linear transformation in an $[n/2] + 1$ dimensional space:

(V.26) $$\overrightarrow{\phi^n} = e^{c_0 T_n} [\overrightarrow{\phi^n}]_0\,,$$

where

$$\overrightarrow{\phi^n} = (\phi^n, \phi^{n-2}, \cdots),\ [\overrightarrow{\phi^n}]_0 = \big([\phi^n]_0, [\phi^{n-2}]_0, \cdots\big)$$

and

(V.27) $$T_n = \begin{pmatrix} 0 & \binom{n}{2} & 0 & 0 & \cdots \\ 0 & 0 & \binom{n-2}{2} & 0 & \cdots \\ 0 & 0 & 0 & \binom{n-4}{2} & \cdots \\ \vdots & & & & \end{pmatrix}.$$

Then it is clear that

$$[\overrightarrow{\phi^n}]_0 = e^{-c_0 T_n} \overrightarrow{\phi^n} = e^{(c_1 - c_0) T_n} [\overrightarrow{\phi^n}]_1\,. \qquad \square$$

Remark. The combinatorial relations (V.26) and (V.27) make it clear why the formula for Wick powers in terms of ordinary powers has the same coefficients as (V.25) except for a $(-1)^j$; for $[\overrightarrow{\phi^n}]_0 = e^{-c_0 T_n}\overrightarrow{\phi^n}$.

(C) *Broken Symmetry*

Suppose that the polynomial $P(X) = P_e(X) + \lambda X$, where P_e is even and

λ real. Let $E_p(\cdot)$ denote the expectation for the theory obtained as an infinite valume limit of some spatially cutoff theory with polynomial P, say, for the sake of definiteness, the half-Dirichlet state (cf. Theorem V.20A). Since E_P is translation invariant we know that $E_P(\phi(f)) = c \int f(x)dx$. We denote the constant c by $E_P(\phi(0))$.

There are various meanings to the statement that the $\phi \to -\phi$ symmetry is broken for a $P_e(\phi)$ theory, for instance:

(a) the spontaneous Bogoliubov parameter

$$b(P_e) \equiv \lim_{\lambda\to 0^-} E_{P_e+\lambda X}(\phi(0)) \neq 0 \ ;$$

(b) $\alpha_\infty(P_e + \lambda X)$ is not differentiable in λ at $\lambda = 0$.

We expect that (a) and (b) are equivalent and occur if and only if E_P is not ergodic, i.e., there is a non-unique vacuum. Thus our main result below (the persistence of broken symmetry under a decrease in the coefficient of X^2) is to be expected on the basis of Theorem V.22.

According to Lemma V.17:

THEOREM V.28. *Let P be an even polynomial and let μ, λ be real. Then $E_{P+\mu X^2+\lambda X}(\phi(0))$ is odd in λ and nonnegative if λ is negative, and $|E_{P+\mu X^2+\lambda X}(\phi(0))|$ is a decreasing function of μ and an increasing function of $|\lambda|$.*

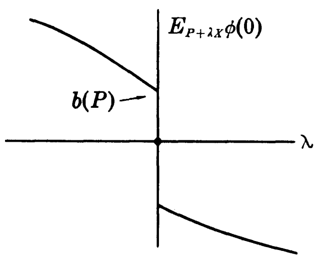


FIGURE V.2

As a result we see that $b(P)$ exists. Moreover:

THEOREM V.29. *If P is an even polynomial, the spontaneous Bogoliubov parameter $b(P)$ is nonnegative and $b(P + \mu X^2)$ is a decreasing function of μ. Thus if P gives rise to a broken symmetry theory (i.e., $b(P) \neq 0$) then so does $P + \mu X^2$, for any $\mu < 0$.*

Remark. The equivalence of (a) and (b) above for $P_e(X) = aX^4 + bX^2$ has been proved in [105].

VI. The basic objects of statistical mechanics

In this section we shall study the pressure associated with an interaction polynomial P, and the entropy density associated with each of a class of states which we describe in § VI.2. The (free boundary condition) pressure, $\alpha_\infty(P)$, which we discuss in § VI.1 is not a new object. It is just the Fock space energy per unit volume which was shown to exist by Guerra [41] (see also [42], [43]). On the other hand, the entropy density $s(f)$ is a new object which is not defined directly in terms of a Hamiltonian theory. It is natural to ask if it is anything more than an object of academic interest. At this point, we cannot give a definitive answer to this question but the program which we outline in the next section (especially § VII.3) suggests that the entropy density will be an important object in future developments of the $P(\phi)_2$ theory.

VI.1 *The Pressure.* Throughout this section $d\mu_0$ will denote the measure for the non-interacting Markov field of mass m (free B.C.) and $d\mu_\Lambda^D$ the measure for the non-interacting Markov field with Dirichlet B.C. in region Λ; P will denote a fixed semibounded, normalized $(P(0) = 0)$ polynomial, $U_\Lambda = \int_\Lambda :P(\phi(x)):d^2x$, $U_\Lambda^D = \int_\Lambda :P(\phi^D(x)):d^2x$ (see (II.25) and (II.63)).

Definitions. The partition functions and pressures associated to a bounded, open region Λ are given by

$$Z_\Lambda(P) = \int e^{-U_\Lambda} d\mu_0 ,$$

$$Z_\Lambda^D(P) = \int e^{-U_\Lambda^D} d\mu_\Lambda^D ,$$

$$p_\Lambda = |\Lambda|^{-1} \ln Z_\Lambda(P) , \qquad \text{(free B.C. pressure)}$$

$$p_\Lambda^D = |\Lambda|^{-1} \ln Z_\Lambda^D(P) , \qquad \text{(Dirichlet B.C. pressure)}$$

where $|\Lambda|$ is the volume of Λ.

Remark. We use the term "pressure" even though we have no particular reason for thinking of $\int \cdot\, d\mu_0$ as a grand canonical ensemble as opposed to a canonical ensemble average.

Our main goal is to prove that p_Λ (resp. p_Λ^D) converges to a limit α_∞(resp. α_∞^D) as $\Lambda \to \infty$ (Fisher). We shall investigate the natural conjecture that $\alpha_\infty = \alpha_\infty^D$ in our forthcoming paper on boundary conditions.

We first note some simple consequences of the theory of conditioning (§ II.5):

LEMMA VI.1. (a) *For any bounded open region* Λ,

$$1 \leqq Z_\Lambda^D \leqq Z_\Lambda .$$

(b) *If* $\Lambda_1 \subset \Lambda_2$ *are bounded open regions, then*

$$Z^D_{\Lambda_1} \leqq Z^D_{\Lambda_2} .$$

(c) *If* $\Lambda_1, \cdots, \Lambda_n$ *are disjoint open regions and* $\Lambda_1 \cup \cdots \cup \Lambda_n \subset \Lambda$, *then*

$$\prod_{j=1}^n Z^D_{\Lambda_j} \leqq Z^D_\Lambda .$$

(d) $\ln Z_\Lambda(P)$ *and* $\ln Z^D_\Lambda(P)$ *are convex functions of* P *for each fixed* Λ.

Proof. (a) Since P is normalized, $\int U^D_\Lambda d\mu^D_\Lambda = 0$, so by Jensen's inequality,

$$Z^D_\Lambda \geqq \exp\left(-\int U^D_\Lambda d\mu^D_\Lambda\right) = 1 .$$

The bound $Z^D_\Lambda \leqq Z_\Lambda$ follows from Lemma II.20 and the fact that the Dirichlet theory is a conditioned theory (Theorem II.28).

(b) The Dirichlet theory in Λ_1 can be obtained from that in Λ_2 by conditioning onto those functions orthogonal to the functions with support in $\Lambda_2 \backslash \Lambda_1$. Thus $Z^D_{\Lambda_1} \leqq Z^D_{\Lambda_2}$, once again by Lemma II.20.

(c) By (b), $Z^D_{\Lambda_1 \cup \Lambda_2 \cdots \cup \Lambda_n} \leqq Z^D_\Lambda$. But, as $\Lambda_1, \cdots, \Lambda_n$ are disjoint, $d\mu^D_{\Lambda_1 \cup \cdots \cup \Lambda_n}$ factors (see Theorem II.29) so that $Z^D_{\Lambda_1 \cup \cdots \cup \Lambda_n} = Z^D_{\Lambda_1} \cdots Z^D_{\Lambda_n}$.

(d) follows from Hölder's inequality. □

COROLLARY VI.2. *For any bounded open region* Λ,

$$p^D_\Lambda \leqq p_\Lambda \leqq \alpha_\infty$$

where α_∞ *is the Fock space energy per unit volume* [42].

Proof. The first inequality follows from Lemma VI.1(a) and the second from Lemma III.13. □

In the special case where the Λ are *rectangles*, we can now take the thermodynamic limit. We write $l \times t$ as shorthand for the rectangle $[-l/2, l/2] \times [-t/2, t/2]$.

THEOREM VI.3. (a) $p_{l\times t}$ *is monotone increasing in* l *and* t *and* $\lim_{l,t\to\infty} p_{l\times t} = \alpha_\infty$, *the Fock space energy per unit volume.*

(b) *For each fixed* l (resp. t) $\ln Z^D_{l\times t}$ *is superadditive in* t (resp. l), $\lim_{l,t\to\infty} p^D_{l\times t}$ *exists and equals* $\sup_{l,t} p^D_{l\times t} \equiv \alpha^D_\infty \geqq 0$.

(c) $\alpha^D_\infty \leqq \alpha_\infty$.

(d) $\alpha_\infty(P)$ *and* $\alpha^D_\infty(P)$ *are convex functions of* P.

Proof. (a) By the FKN formula (II.32), $P_{l\times t} = (1/lt) \ln \langle \Omega_0, e^{-tH_l}\Omega_0 \rangle$; the monotonicity of this expression was proved in [42, Theorem 1]. Since $\lim_{t\to\infty} p_{l\times t} = \alpha_l$ by the transfer matrix formula (II.38), it follows that $\lim_{l,t\to\infty} p_{l\times t} = \alpha_\infty$.

(b) The superadditivity follows directly from Lemma VI.1(c). By a standard argument using superadditivity $\lim_{l,t\to\infty} p^D_{l\times t} = \sup_{l,t} p^D_{l\times t}$; here we use Corollary VI.2 to conclude that $p^D_{l\times t}$ is bounded from above, and Lemma VI.1(a) to conclude that it is bounded from below.

(c) and (d) are obvious consequences of Lemma VI.1. □

At this point, we can say the following about convergence of p_{Λ_n} and $p^D_{\Lambda_n}$ for the more general situation where $\Lambda_n \to \infty$ (van Hove) (see Appendix C or [85] for the definition and notation concerning the van Hove limit):

LEMMA VI.4. *Let* $\Lambda_n \to \infty$ (van Hove). *Then*

(a) $\overline{\lim}_n\, p_{\Lambda_n} \leqq \alpha_\infty$;

(b) $\underline{\lim}_n\, p^D_{\Lambda_n} \geqq \alpha^D_\infty$.

Proof. (a) follows from Corollary VI.2.

(b) Fix a. By Lemma VI.1(c) and the definition of $N_a^-(\Lambda)$ (Appendix C),

$$|\Lambda_n|\, p^D_{\Lambda_n} \geqq a^2 N_a^-(\Lambda_n) p^D_{a\times a} .$$

Since $a^2 N_a^-(\Lambda_n)/|\Lambda_n| \to 1$, we see that

$$\underline{\lim}\, p^D_{\Lambda_n} \geqq p^D_{a\times a} .$$

Since a is arbitrary and $\lim_{a\to\infty} p^D_{a\times a} = \alpha^D_\infty$, by Theorem IV.3, the result follows. □

By Lemmas VI.1 and VI.4, if we could show that $\alpha_\infty = \alpha^D_\infty$, then it would follow that $\lim p_{\Lambda_n} = \lim p^D_{\Lambda_n} = \alpha_\infty$ whenever $\Lambda_n \to \infty$ (van Hove). We hope to show that $\alpha_\infty = \alpha^D_\infty$ in a future paper. For the present we note:

THEOREM VI.5. *Let* $\Lambda'_n \to \infty$ (Fisher). *Then*

(a) $\lim_{n\to\infty} p_{\Lambda'_n} = \alpha_\infty$;

(b) $\lim_{n\to\infty} p^D_{\Lambda'_n} = \alpha^D_\infty$.

Proof. (a) Choose the squares $\Lambda_n \supset \Lambda'_n$ and the regions $\Lambda''_n \subset \Lambda_n \backslash \Lambda'_n$ in accordance with Theorem C.4. Let $R_n = \Lambda_n \backslash \Lambda'_n \cup \Lambda''_n$ so that $|R_n|/|\Lambda_n| \to 0$ by condition (c) of Theorem C.4. Fix $0 < \lambda < 1$. Then, by Hölder's inequality,

$$\text{(VI.1)} \quad |\Lambda_n|\, p_{\Lambda_n} \leqq \lambda\, |\Lambda'_n \cup \Lambda''_n|\, p_{\Lambda'_n \cup \Lambda''_n}(\lambda^{-1}P) + (1-\lambda)\, |R_n|\, p_{R_n}((1-\lambda)^{-1}P) .$$

Now, for any $a > 1$ and all sufficiently large n,

$$\text{(VI.2)} \quad |\Lambda'_n \cup \Lambda''_n|\, p_{\Lambda'_n \cup \Lambda''_n}(\lambda^{-1}P) \leqq a^{-1}\, |\Lambda'_n|\, p_{\Lambda'_n}(a\lambda^{-1}P) + a^{-1}\, |\Lambda''_n|\, p_{\Lambda''_n}(a\lambda^{-1}P) ,$$

by Theorem III.3 and the fact that $d(\Lambda'_n, \Lambda''_n) \to \infty$. We combine (VI.1) and (VI.2), divide by $|\Lambda_n|$, and take $n \to \infty$ through a subsequence so that $b \equiv \lim |\Lambda'_n|/|\Lambda_n|$ exists. By Theorem C.4, $\Lambda''_n \to \infty$ (Fisher) so that $\Lambda''_n \to \infty$ (van Hove) by Proposition C.2. Hence by Theorem VI.3(a) and Lemma VI.4(a)

$$\alpha_\infty(P) \leqq \lambda a^{-1} b\, \underline{\lim}\, p_{\Lambda'_n}(a\lambda^{-1}P) + (1-b)\lambda a^{-1} \alpha_\infty(a\lambda^{-1}P) .$$

Since P has been arbitrary in this argument, we see that for arbitrary $\mu = a\lambda^{-1} > 1$,

$$\mu\alpha_\infty(\mu^{-1}P) \leqq b \underline{\lim}\, p_{\Lambda'_n}(P) + (1-b)\alpha_\infty(P) .$$

But by Theorem VI.3(d), $\alpha_\infty(\mu^{-1}P)$ is continuous in μ so we may set $\mu = 1$. Since by Theorem C.4(b), $b > 0$, $\underline{\lim}\, p_{\Lambda'_n}(P) \geqq \alpha_\infty(P)$. This, together with Lemma VI.4, proves (a).

(b) Choose Λ_n squares and Λ''_n in accordance with Theorem C.4. By Lemma VI.1(c),

$$|\Lambda_n|\, p^D_{\Lambda_n} \leqq |\Lambda'_n|\, p^D_{\Lambda'_n} + |\Lambda''_n|\, p^D_{\Lambda''_n} .$$

Passing to a subsequence so that $b \equiv \lim_{n\to\infty} |\Lambda'_n|/|\Lambda_n|$ exists and using Lemma VI.4 and Proposition C.2, we obtain

$$\alpha^D_\infty \leqq b\, \overline{\lim}\, p^D_{|\Lambda'_n|} + (1-b)\alpha^D_\infty$$

from which (b) follows. □

VI.2. *States and Entropy.* The basic objectives of constructive Euclidean field theory are to prove the existence of states associated to a given interaction and to study their properties. In general, a state is given by some probability measure μ on a Q-space. If the theory associated with μ is non-trivial and translation invariant, then μ cannot be absolutely continuous with respect to μ_0, the free field measure (i.e., cannot be of the form $d\mu = f\,d\mu_0$ with $f \in L^1(Q, d\mu_0)$) because the action of translations on $L^1(Q, d\mu_0)$ is ergodic. This is a Euclidean version of Haag's theorem. On the other hand, there is no general principle to prevent the restriction of μ to a local σ-algebra Σ_Λ (henceforth $\mu \restriction \Sigma_\Lambda$) from being absolutely continuous with respect to $\mu_0 \restriction \Sigma_\Lambda$.

In theories with more than 2 space-time dimensions and deg $P \geqq 4$, it is probably not possible for $\mu \restriction \Sigma_\Lambda$ to be absolutely continuous with respect to $\mu_0 \restriction \Sigma_\Lambda$, but in two dimensions we have the following indications that such local absolute continuity does occur:

(a) It holds for spatially cutoff theories.

(b) It is true for the exactly soluble linear and quadratic models.

(c) It holds in perturbation theory.

(d) Most importantly, it is suggested by the locally Fock property of the Hamiltonian theory [31].

Thus, we define:

Definition. A *state* f is a family $\{f_\Lambda\}$ of functions on the free field Q-space, labelled by bounded open $\Lambda \subset \mathbf{R}^2$, so that

a) Each f_Λ is Σ_Λ-measurable, almost everywhere nonnegative, in $L^1(Q, d\mu_0)$, and normalized

$$\|f_\Lambda\|_1 = \int_Q f_\Lambda d\mu_0 = 1 .$$

b) The system is compatible in the sense that

$$E_\Lambda(f_{\Lambda'}) = f_\Lambda \qquad \text{if} \quad \Lambda \subset \Lambda' ,$$

where E_Λ is the (free field) conditional expectation on Σ_Λ.

We say the state is *$\bar{p}$-smooth* for some fixed $\bar{p} > 1$, if and only if, in addition,

c) For each Λ and $p \in [1, \bar{p}]$

$$f_\Lambda \in L^p(\Sigma_\Lambda, d\mu_0) .$$

Given a state f and any closed bounded set R we can define f_R by $f_R = E_R f_\Lambda$ for any $\Lambda \supset R$. The similarity to the corresponding definition in statistical mechanics [85] is evident. There is one important difference. In statistical mechanics, it is easy to construct translation invariant states. That is not true in the field theory case; in fact, the only translation invariant states we know of in two dimensions are those associated with the exactly soluble theories and those associated [70] with small coupling $P(\phi)_2$. Moreover, it is not even known that the latter are $\bar{p}$-smooth for any $\bar{p} > 1$! As usual, this difference from statistical mechanics is a relation of the fact that disjoint regions are not μ_0-independent. We emphasize that, at the current stage, our considerations in §§ VI.2, VI.3, VII.3 are a trifle abstract; however, we expect that in the future it will be proved that all $P(\phi)_2$ theories have associated $\bar{p}$-smooth states. We also warn the reader of a difference in emphasis from considerations in § V. The Schwinger functions are the moments of μ and the question of convergence (and existence) of moments is distinct (for technical reasons) from convergence of measures.

In analogy with statistical mechanics [85] and information theory, we define:

Definition. Let f be a $\bar{p}$-smooth state. If Λ is a bounded (open or closed) region in $\mathbf{R}^2$, the *entropy of f associated to* Λ is given by

$$S_\Lambda(f) = -\int_Q f_\Lambda \ln f_\Lambda d\mu_0 . \tag{VI.3}$$

When f is fixed, we will often write $S(\Lambda)$ in place of $S_\Lambda(f)$.

THEOREM VI.6. *The following inequalities hold:*

a) (boundedness) $-\infty < S(\Lambda) \leqq 0$;

b) (monotonicity) $S(\Lambda') \leqq S(\Lambda)$ *if* $\Lambda \subset \Lambda'$;

c) (weak subadditivity) *if* $\Lambda_1 \cdots, \Lambda_n$ *are disjoint regions*

$$S(\bigcup_{i=1}^n \Lambda_i) \leqq \sum_{i=1}^n S(\Lambda_i) + \ln \| \prod_{i=1}^n f_{\Lambda_i} \|_1 .$$

Proof. We make extensive use of

$$-\ln x \leqq \frac{1}{x} - 1 \tag{VI.4}$$

and Jensen's inequality

$$\int \ln f d\mu \leqq \ln \int f d\mu \tag{VI.5}$$

where μ is a probability measure, f is a nonnegative L^1 function.

(a) By (VI.5)

$$\begin{aligned}\int f_\Lambda \ln f_\Lambda d\mu_0 &= \frac{1}{\bar{p}-1} \int [\ln f_\Lambda^{\bar{p}-1}] f_\Lambda d\mu_0 \\ &\leqq \frac{1}{\bar{p}-1} \ln \int f_\Lambda^{\bar{p}} d\mu_0 < \infty ,\end{aligned}$$

while on the other hand by (VI.4),

$$-\int f_\Lambda \ln f_\Lambda d\mu_0 \leqq \int (1 - f_\Lambda) d\mu_0 = 0 .$$

(b) Using the consistency condition on f with (VI.5), we see that

$$\begin{aligned}S(\Lambda') - S(\Lambda) &= \int f_\Lambda \ln f_\Lambda d\mu_0 - \int f_{\Lambda'} \ln f_{\Lambda'} d\mu_0 \\ &= \int f_{\Lambda'} \ln \left(\frac{f_\Lambda}{f_{\Lambda'}}\right) d\mu_0 \leqq \ln \int f_{\Lambda'} \frac{f_\Lambda}{f_{\Lambda'}} d\mu_0 = 0 .\end{aligned}$$

(c) Let $\Lambda = \bigcup_{i=1}^n \Lambda_i$. As in the proof of (b),

$$\begin{aligned}S(\Lambda) - \sum_{i=1}^n S(\Lambda_i) &= \int f_\Lambda \ln (f_{\Lambda_1} \cdots f_{\Lambda_n} / f_\Lambda) d\mu_0 \\ &\leqq \ln \int f_{\Lambda_1} \cdots f_{\Lambda_n} d\mu_0 .\end{aligned}$$ □

This last theorem is very similar to the results in the theory of entropy for classical and quantum continuous statistical mechanics [79]. The main difference lies in (c) and is once again due to the non-independence of disjoint regions. For, if $d\mu_0$ did factor (as it does in classical systems), then

$$\ln \int f_{\Lambda_1} \cdots f_{\Lambda_n} d\mu_0 = \ln \left(\int f_{\Lambda_1} d\mu_0 \right) \cdots \left(\int f_{\Lambda_n} d\mu_0 \right) = 0$$

would hold and subadditivity would follow.

VI.3 *Convergence of the Entropy per Unit Volume.* In order to control the limit of $|\Lambda|^{-1} S_\Lambda(f)$ as $|\Lambda| \to \infty$, we shall require the following weak growth condition on $\{f_\Lambda\}$:

Definition. A $\bar{p}$-smooth state f is called *weakly tempered* if there exist $\eta < 1$, $a > 0$ and D so that for all Λ with diameter $d(\Lambda) \geqq D$,

$$\ln \|f_\Lambda\|_{\bar{p}} \leqq \exp[ad(\Lambda)^\eta] \ . \tag{VI.6}$$

We expect that the infinite volume $P(\phi)_2$ states will actually satisfy a stronger growth condition:

Definition. If there are an $A > 0$ and D so that for $d(\Lambda) \geqq D$,

$$\ln \|f_\Lambda\|_{\bar{p}} \leqq Ad(\Lambda)^2 \tag{VI.7}$$

we call f *tempered.*

For a "reasonable" set of Λ, $d(\Lambda)^2$ can be replaced by $|\Lambda|$. The use of $d(\Lambda)$ is seen in:

PROPOSITION VI.7. *In order that f be weakly tempered (resp. tempered) it is sufficient that* (VI.6) (*resp.* (VI.7)) *hold for all discs* Λ.

Proof. Any Λ' can be put inside a disc Λ of radius $d(\Lambda')$. By Lemma II.6, $E_{\Lambda'}$ is a contraction on each L^p; hence

$$\begin{aligned}\ln \|f_{\Lambda'}\|_{\bar{p}} = \ln \|E_{\Lambda'}(f_\Lambda)\|_{\bar{p}} &\leqq \ln \|f_\Lambda\|_{\bar{p}} \\ &\leqq \exp(2^\eta ad(\Lambda')^\eta)\ (\text{resp. } 4A\, d(\Lambda')^2)\ .\end{aligned}$$ □

Weak temperedness together with hypercontractivity (Theorem III.3) allows us to control the "correction term" $\ln \|f_{\Lambda_1}f_{\Lambda_2}\|_1$ in the weak subadditivity condition of Theorem VI.6(c):

LEMMA VI.8. *Let f be a $\bar{p}$-smooth, weakly tempered state. Let $d_n > 0$ with $d_n \to \infty$ and suppose that Λ_n, Λ'_n are regions in $\mathbf{R}^2$ satisfying*

(a) $d(\Lambda_n) \leqq d_n$, $d(\Lambda'_n) \leqq d_n$;

(b) $\text{dist}(\Lambda_n, \Lambda'_n) \geqq d_n^{\eta'}$ *for some $\eta' > \eta$ and all large n.*

Then
$$\lim_{n\to\infty} \ln \|f_{\Lambda_n}f_{\Lambda'_n}\|_1 = 0\ .$$

Proof. By Theorem III.3,

$$\ln \|f_{\Lambda_n}f_{\Lambda'_n}\|_1 \leqq \ln \|f_{\Lambda_n}\|_{\beta_n} + \ln \|f_{\Lambda'_n}\|_{\beta_n} \tag{VI.8}$$

with $\beta_n - 1 \leqq \exp(-cd_n^{\eta'})$ for some $c > 0$ and all large n. For large n, $\beta_n < \bar{p}$ and so by (III.19) ($\|f_\Lambda\|_1 = 0$),

$$\ln \|f_\Lambda\|_{\beta_n} \leqq \frac{\bar{p}(\beta_n - 1)}{\beta_n(\bar{p} - 1)} \ln \|f_\Lambda\|_{\bar{p}}\ . \tag{VI.9}$$

By (VI.8), (VI.9) and (VI.6),

$$\ln \|f_{\Lambda_n}f_{\Lambda'_n}\|_1 \leqq \text{const.} \exp(2ad_n^\eta - cd_n^{\eta'}) \longrightarrow 0\ .$$ □

Our main result in this subsection is

THEOREM VI.9. *Let f be a translation invariant, weakly tempered, $\bar{p}$-smooth state. Then whenever $\Lambda_n \to \infty$* (Fisher) *the following limit exists:*

$$\lim \frac{S_{\Lambda_n}(f)}{|\Lambda_n|} = s(f) \tag{VI.10}$$

and is independent of the particular Fisher sequence. The functional $s(f)$ is affine in f; i.e., *if* $0 \leqq \lambda \leqq 1$,

$$s(\lambda f + (1-\lambda)f') = \lambda s(f) + (1-\lambda)s(f') .$$

Remarks 1. Unlike the situation in classical statistical mechanics [85], we have not yet proved the upper semi-continuity of $s(f)$.

2. This result extends to dimensions $d \neq 2$.

We shall prove Theorem VI.9 in a series of lemmas (VI.10 to VI.13) all of which hold under the hypothesis of the theorem. First using the Checkerboard Theorem of § III.2, we establish the existence of the limit (VI.10) for squares. Then using Theorem C.4 and Lemma VI.8, we extend the result to arbitrary Fisher limits.

LEMMA VI.10. *If Λ is a square of side a, denote $S_\Lambda(f)$ by $S(a)$. Then* $\lim_{a\to\infty} a^{-2}S(a)$ *exists.*

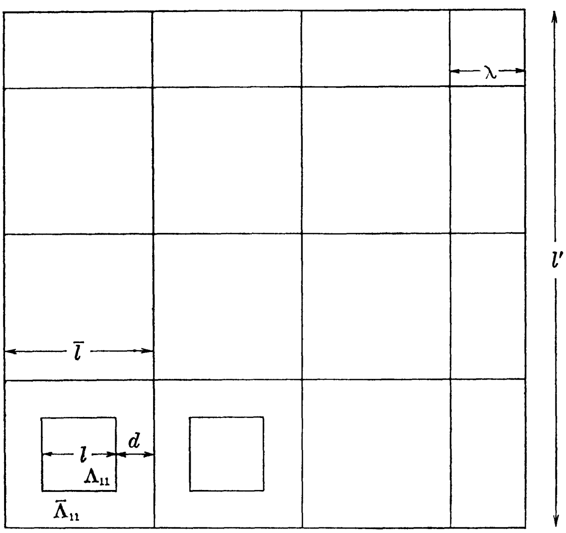


FIGURE VI.1

Proof. Fix $\eta' \in (\eta, 1)$ and for each l, let $d = l^{\eta'}$ and $\bar{l} = l + 2d$. Temporarily fix l, d and $\bar{l}$. Given l', write $l' = n\bar{l} + \lambda$ with $0 \leqq \lambda < \bar{l}$ and n an integer. As in Figure VI.1, the square of side l' can be partially filled by n^2

squares $\{\bar{\Lambda}_{ij}\}_{1\leqq i,j\leqq n}$ of side $\bar{l}$ and each $\bar{\Lambda}_{ij}$ contains a square Λ_{ij} of side l surrounded by a border of width d. By the monotonicity and weak subadditivity of S, and the translation invariance of f,

$$S(l') \leqq S(\bigcup \Lambda_{ij}) \leqq n^2 S(l) + \ln \int \prod_{i,j} f_{\Lambda_{ij}} d\mu_0 .$$

By the Checkerboard Estimate (Theorem III.12)

$$\ln \int \prod_{i,j} f_{\Lambda_{ij}} d\mu_0 \leqq n^2 \ln ||f_\Lambda||_{\beta^2}$$

where

(VI.11a) $$\beta = (e^{2md} + 1)/(e^{2md} - e^{-m\bar{l}})$$

so that

(VI.11b) $$\beta^2 - 1 \leqq \exp(-cl^{\eta'})$$

for large l. Thus

(VI.12) $$(l')^{-2} S(l') \leqq n^2 (l')^{-2} S(l) + n^2 (l')^{-2} \ln ||f_{l\times l}||_{\beta^2} .$$

As $l' \to \infty$, $n^2(l')^{-2} \to \bar{l}^{-2}$ so that

(VI.13) $$\overline{\lim}_{l'\to\infty} S(l')/(l')^2 \leqq l^2 \bar{l}^{-2}(S(l)/l^2) + \bar{l}^{-2} \ln ||f_{l\times l}||_{\beta^2} .$$

Now by (VI.6), (VI.9), and (VI.11)

$$\lim_{l\to\infty} \ln ||f_{l\times l}||_{\beta^2} \longrightarrow 0$$

and clearly $l^2\bar{l}^{-2} \to 1$ as $l \to \infty$. Thus by (VI.13)

$$\overline{\lim}_{l'\to\infty} S(l')/(l')^2 \leqq \underline{\lim}_{l\to\infty} S(l)/l^2 .$$ □

LEMMA VI.11. *If* $\Lambda_n \to \infty$ (van Hove), *then*

$$\overline{\lim}\, S(\Lambda_n)/|\Lambda_n| \leqq s(f) \equiv \lim S(a)/a^2 .$$

Proof. By mimicking the proof of (VI.12), we see that for fixed l, $\bar{l}$, and d:

$$|\Lambda_n|^{-1} S(\Lambda_n) \leqq |\Lambda_n|^{-1} N_{\bar{l}}^{-}(\Lambda_n) S(l) + |\Lambda_n|^{-1} N_{\bar{l}}^{-}(\Lambda_n) \ln ||f_{l\times l}||_{\beta^2} .$$

Taking first $n \to \infty$ and then $l \to \infty$, the result follows as in the proof of Lemma VI.10. □

The proof of the existence of the Fisher limit is clearly completed by:

LEMMA VI.12. *If* $\Lambda'_n \to \infty$ (Fisher), *then* $\underline{\lim}\, |\Lambda'_n|^{-1} S(\Lambda'_n) \geqq s(f)$.

Proof. Pick $\eta' \in (\eta, 1)$, squares Λ_n and regions Λ''_n in accordance with Theorem C.4. By passing to a subsequence, we can suppose that $|\Lambda'_n|/|\Lambda_n|$ has a limit b (necessarily $0 < b < 1$) and that the $\underline{\lim}$ is the same as the $\underline{\lim}$ for the original sequence. By monotonicity and weak subadditivity

(VI.14) $$S(\Lambda_n) \leqq S(\Lambda_n' \cup \Lambda_n'') \leqq S(\Lambda_n') + S(\Lambda_n'') + \ln \int f_{\Lambda_n'} f_{\Lambda_n''} d\mu_0 .$$

By Lemma VI.8, the last factor goes to zero as $n \to \infty$ since $d(\Lambda_n', \Lambda_n'') \geqq d(\Lambda_n')^{\eta'}$. Thus dividing (VI.14) by $|\Lambda_n|$ and letting $n \to \infty$, we have:

$$s(f) \leqq b \underline{\lim}\, |\Lambda_n'|^{-1} S(\Lambda_n') + (1-b)\, \overline{\lim}\, |\Lambda_n''|^{-1} S(\Lambda_n'') \qquad \text{(by Lemma VI.10)}$$
$$\leqq b \underline{\lim}\, |\Lambda_n'|^{-1} S(\Lambda_n') + (1-b) s(f) . \qquad \text{(by Lemma VI.11)}$$

Since $b > 0$, the result follows. □

All that remains in the proof of Theorem VI.9 is:

LEMMA VI.13. *$s(f)$ is affine in f.*

Proof. As in the statistical mechanical case [85, p. 180] one finds

$$\alpha S_\Lambda(f) + (1-\alpha) S_\Lambda(f') \leqq S_\Lambda(\alpha f + (1-\alpha) f')$$
$$\leqq \alpha S_\Lambda(f) + (1-\alpha) S_\Lambda(f') + \ln 2$$

where the first inequality depends on the convexity of $t \ln t$ and the second on the monotonicity of $\ln t$. Dividing by $|\Lambda|$ and taking $\Lambda \to \infty$ (Fisher), we obtain the affine relation. □

VII. Equilibrium and variational equations

The construction of infinite volume field theories in both the Hamiltonian and Euclidean formulations has proceeded by imposing spatial cutoffs which are then removed by appropriate limiting procedures. It is thus of particular interest to obtain an *a priori* characterization of what is meant by "an infinite volume $P(\phi)_2$ field theory" in a way that makes no mention of cutoffs. By doing this, one can hope to isolate those situations where the infinite volume theory is unique independent of the cutoff or removal procedures.

In the Hamiltonian approach, this *a priori* characterization has depended on the use of field equations [30], [81]. The question of uniqueness then has two aspects: the algebraic aspect and the representation aspect. That is, one can discuss separately the uniqueness question for time automorphisms and the uniqueness question for vacuum representations, i.e., invariant states with an induced positive energy. The Guenin-Segal argument can be viewed as a kind of uniqueness theorem for automorphisms with Fock initial conditions. The difficult problem is then the characterization of vacuum representations. While this has been accomplished for free theories ([91], [92], [114]), the method is sufficiently specialized to present no clue about how to approach the problem in general.

In this section, it is our goal to give a characterization of equilibrium states for a given interaction by following methods and ideas from modern statistical mechanics [84], [12]–[15], [56], [86]. Let us emphasize at the outset that our results in this section are very incomplete and are more an indication of the direction in which we feel the theory will go rather than a construction of the theory. The first difficulty in mimicking statistical mechanical ideas involves the non-independence of disjoint regions in the free theory, a difference we have already emphasized. For this reason, we occasionally consider one-dimensional theories where the transfer matrix technique enables us to overcome this difficulty. The second difficulty in handling equilibrium equations is that they do not specify the state completely but must be supplemented by the right boundary conditions at infinity in order to rule out unphysical solutions. There is an analogous situation in continuous classical statistical mechanics [86]—the need for supplementary boundary conditions in both cases is connected with the fact that the basic quantities (fields or densities) can take unbounded values [12].

The organization of the section is as follows. In § VII.1, we discuss equilibrium equations for an interaction polynomial P together with a bare mass, m. We also discuss the transformation laws between pairs (P, m) and $(\widetilde{P}, \widetilde{m})$ which have equivalent equilibrium equations. In § VII.2, we show that the DLR equations alone do not characterize even the free theory, thereby demonstrating the need for boundary conditions at infinity. In § VII.3, we obtain some partial information about the Gibbs' variational principle; in particular, we uncover a striking connection between the one-dimensional (statistical mechanical) Euclidean theory and the associated (quantum) Hamiltonian theory; namely, the Rayleigh-Ritz (quantum) variational principle and the Gibbs (statistical mechanical) principle are essentially identical.

VII.1. DLR *Equations.* In their discussions of the statistical mechanics of lattice gases, Dobrushin [13]–[15] and Lanford-Ruelle [56] proposed a set of relations on an infinite volume state which intuitively express the fact that the state is an equilibrium state for a system with some given interaction at some fixed temperature. These relations are generally called *equilibrium equations* or DLR equations. Ruelle [86] has discussed their analogues for classical continuous systems, and it is generally believed that the KMS boundary condition [45] is their analogue in quantum lattice systems. We now wish to discuss their analogue in Euclidean field theories viewed as classical statistical mechanics. The basic idea behind our version of the DLR equations is this: the Markov property assures us that a change in the interaction outside a bounded region, Λ, affects the state restricted to Λ only by a multiplicative factor concentrated on the boundary.

For the present we fix the bare mass m and the semibounded polynomial P, although later in this subsection we shall discuss changes in m and P. As usual, $U_\Lambda = \int_\Lambda :P(\phi(x)):d^2x$ where ϕ is the free Euclidean field of mass m and : : the associated Wick ordering. We shall discuss Markov fields over $\mathcal{D}(\mathbf{R}^2)$ in the sense of Nelson [67] since there are no technical complications introduced by doing so. All such fields are essentially defined by measures on a fixed σ-algebra, for example, in the realization of Q-space as $\mathcal{D}'$, the σ-algebra of cylinder sets. If Λ is open, Σ_Λ is the σ-algebra generated by the $\{\phi(f) \mid \text{supp } f \subset \Lambda\}$ and, if Λ is closed, $\Sigma_\Lambda = \bigcup_{\substack{\Lambda' \text{open} \\ \Lambda' \supset \Lambda}} \Sigma_{\Lambda'}$. Given a Markov field measure $\bar{\mu}$ on $\Sigma_{\mathbf{R}^2}$, $\bar{\mu} \restriction \Sigma_\Lambda$ will denote the restriction of $\bar{\mu}$ to Σ_Λ. Throughout, μ_0 denotes the mass m free field measure and E_Λ its conditional expectation associated to Σ_Λ. Given a measure $\bar{\mu}$ over $\mathcal{D}'$, $\bar{E}_\Lambda$ denotes its conditional expectations.

Definition. Let $\bar{\mu}$ be the measure associated to a Markov field theory over $\mathcal{D}(\mathbf{R}^2)$. Let Λ be a bounded compact region in $\mathbf{R}^2$. We say that $\bar{\mu}$ is Λ-Gibbsian for P (or, if we wish to emphasize the bare mass, for (P, m)) if and only if

(i) $\bar{\mu} \restriction \Sigma_\Lambda$ is absolutely continuous with respect to $\mu_0 \restriction \Sigma_\Lambda$;

(ii) For every bounded, Σ_Λ-measurable function A

$$\bar{E}_{\partial\Lambda}(A) = \frac{E_{\partial\Lambda}(Ae^{-U_\Lambda})}{E_{\partial\Lambda}(e^{-U_\Lambda})}. \tag{VII.1}$$

If $\bar{\mu}$ is Λ-Gibbsian for P for every compact Λ, we say that $\bar{\mu}$ is *an infinite volume Gibbs state* or a *Gibbsian state for P*. (VII.1) is called the DLR *equation*.

Remarks 1. Because of the differences between field theory and statistical mechanics, (VII.1) has a slightly different form than the usual DLR equations. In fact, because of the strict locality of the interaction and the associated Markov property, (VII.1) is cleaner looking than its statistical mechanical analogue. In the first place, we can write $\bar{E}_{\partial\Lambda}$ and $E_{\partial\Lambda}$ instead of $\bar{E}_{\mathbf{R}^2\setminus\Lambda}$ and $E_{\mathbf{R}^2\setminus\Lambda}$ because of the Markov property. In the statistical mechanical case, this is not possible, although, if the interaction has range k, then a "boundary" of width k can be used in place of $\mathbf{R}^2\setminus\Lambda$. In addition, because of the locality of U, it is not necessary to add an "external field" to U_Λ.

2. Because of condition (i), (VII.1) can be interpreted in the "almost everywhere" (with respect to $\bar{\mu}$) sense and is independent of any concrete realization of Q-space.

3. Since $e^{-U_\Lambda} \in L^1(Q, d\mu_0)$, the denominator of the right side of (VII.1) is

finite almost everywhere.

4. Since $\bar{E}_{\partial\Lambda}(\cdot)$ and $E_{\partial\Lambda}(\cdot e^{-U_\Lambda})/E_{\partial\Lambda}(e^{-U_\Lambda})$ are both positivity preserving and take 1 into 1, (VII.1) extends by continuity to any A which is Σ_Λ-measurable and in $L^1(Q, d\bar{\mu})$ and $L^1(Q, e^{-U_\Lambda}d\mu_0)$.

We shall show that suitable limits of spatially cutoff states with free, Dirichlet or half-Dirichlet B.C. are Gibbsian, but it is first important to develop some general properties of the DLR equations. Given a family of Markov field measures $\{\mu_n\}_{n=1}^\infty$ we write $\mu_n \upharpoonright \Sigma_\Lambda \xrightarrow{w} \mu \upharpoonright \Sigma_\Lambda$ if and only if

$$\int F d\mu_n \longrightarrow \int F d\mu$$

for every bounded Σ_Λ-measurable function F.

PROPOSITION VII.1. *Let μ_n be a sequence of Markov field measures which are Λ-Gibbsian for some fixed compact $\Lambda \subset \mathbf{R}^2$. Suppose that $\mu \upharpoonright \Sigma_\Lambda$ is absolutely continuous with respect to μ_0 and that $\mu_n \upharpoonright \Sigma_\Lambda \xrightarrow{w} \mu \upharpoonright \Sigma_\Lambda$. Then μ is Λ-Gibbsian.*

Proof. Given A, Σ_Λ-measurable and bounded, let $\mathcal{E}_\Lambda(A)$ be the right side of (VII.1). By Remark 4 above, $\mathcal{E}_\Lambda(A)$ is also bounded. To say that a measure ν (with $\nu \upharpoonright \Sigma_\Lambda$ absolutely continuous with respect to μ_0) is Λ-Gibbsian is equivalent to saying that for every pair of bounded measurable functions A and B with A Σ_Λ-measurable and B $\Sigma_{\partial\Lambda}$-measurable

$$\int AB d\nu = \int \mathcal{E}_\Lambda(A) B d\nu .$$

This is clearly left invariant by the stated type of limit. □

THEOREM VII.2. *Let $\Lambda \subset \mathbf{R}^2$ be compact and let $\bar{\mu}$ be a Markov field measure with $\bar{\mu} \upharpoonright \Sigma_\Lambda$ absolutely continuous with respect to μ_0. Suppose that*

$$d(\bar{\mu} \upharpoonright \Sigma_\Lambda) = e^{-U_\Lambda}\psi_{\partial\Lambda} d(\mu_0 \upharpoonright \Sigma_\Lambda) \tag{VII.2}$$

where $\psi_{\partial\Lambda}$ is a positive $\Sigma_{\partial\Lambda}$-measurable function in $L^1(Q, e^{-U_\Lambda}d\mu_0)$. Then $\bar{\mu}$ is Λ-Gibbsian. Conversely, if $\bar{\mu}$ is Λ-Gibbsian, then $d(\bar{\mu} \upharpoonright \Sigma_\Lambda)$ has the form (VII.2) *for some positive $\Sigma_{\partial\Lambda}$-measurable function, $\psi_{\partial\Lambda}$.*

Proof. Suppose that $d(\bar{\mu} \upharpoonright \Sigma_\Lambda)$ has the form (VII.2) and that v is a bounded $\Sigma_{\partial\Lambda}$-measurable function. Then

$$\begin{aligned}\int \bar{E}_{\partial\Lambda}(A)E_{\partial\Lambda}(e^{-U_\Lambda})v d\bar{\mu} &= \int AE_{\partial\Lambda}(e^{-U_\Lambda})ve^{-U_\Lambda}\psi_{\partial\Lambda}d\mu_0 \\ &= \int E_{\partial\Lambda}(Ae^{-U_\Lambda})E_{\partial\Lambda}(e^{-U_\Lambda})v\psi_{\partial\Lambda}d\mu_0 \\ &= \int E_{\partial\Lambda}(Ae^{-U_\Lambda})ve^{-U_\Lambda}\psi_{\partial\Lambda}d\mu_0 \\ &= \int E_{\partial\Lambda}(Ae^{-U_\Lambda})v d\bar{\mu}\end{aligned}$$

by repeated use of the definitions of conditional expectation. Since v is arbitrary and both $\bar{E}_{\partial\Lambda}(A)E_{\partial\Lambda}(e^{-U_\Lambda})$ and $E_{\partial\Lambda}(Ae^{-U_\Lambda})$ are $\Sigma_{\partial\Lambda}$-measurable they are equal. Thus (VII.1) holds.

Conversely, suppose that $\bar{\mu}$ is Λ-Gibbsian. Then, by condition (i) of the definition, $d(\bar{\mu} \upharpoonright \Sigma_\Lambda) = f_\Lambda d(\mu_0 \upharpoonright \Sigma_\Lambda)$ for some Σ_Λ-measurable function $f_\Lambda \in L^1(Q, d\mu_0)$. By (VII.1)

$$\text{(VII.3a)} \qquad \int E_{\partial\Lambda}(Ae^{-U_\Lambda})f_\Lambda d\mu_0 = \int \bar{E}_{\partial\Lambda}(A)E_{\partial\Lambda}(e^{-U_\Lambda})d\bar{\mu}$$

$$\text{(VII.3b)} \qquad = \int AE_{\partial\Lambda}(e^{-U_\Lambda})f_\Lambda d\mu_0$$

for any Σ_Λ-measurable A with both A and Ae^{-U_Λ} bounded. Suppose that B is bounded and that $\{q \mid B(q) \neq 0\} \subset \{q \mid U_\Lambda(q) < c\}$ for some c. Then $A = Be^{+U_\Lambda}$ can be substituted in (VII.3):

$$\text{(VII.4)} \qquad \int E_{\partial\Lambda}(B)f_\Lambda d\mu_0 = \int BE_{\partial\Lambda}(e^{-U_\Lambda})(f_\Lambda e^{+U_\Lambda})d\mu_0 .$$

Any positive B can be realized as a monotone limit of B's for which (VII.4) has been proved so that (VII.4) holds for any $B \geqq 0$ and in particular for $B = 1$. Thus, $\int E_{\partial\Lambda}(e^{-U_\Lambda})(f_\Lambda e^{+U_\Lambda})d\mu_0 = 1$ so that (VII.4) extends to any B in $L^\infty(Q, d\mu_0)$. For this we conclude that $E_{\partial\Lambda}(e^{-U_\Lambda})(f_\Lambda e^{+U_\Lambda})$ is $\Sigma_{\partial\Lambda}$-measurable. For suppose not. Then, since $\{G \in L^1(Q, d\mu_0) \mid C \text{ is } \Sigma_{\partial\Lambda}\text{-measurable}\}$ is closed we can find $B \in L^\infty$ with $\int BCd\mu_0 = 0$ for all these C, but with

$$\int BE_{\partial\Lambda}(e^{-U_\Lambda})(f_\Lambda e^{+U_\Lambda})d\mu_0 \neq 0 .$$

Since this contradicts (VII.4), we conclude that $E_{\partial\Lambda}(e^{-U_\Lambda})f_\Lambda e^{+U_\Lambda}$ and thus $f_\Lambda e^{+U_\Lambda}$ are $\Sigma_{\partial\Lambda}$-measurable. This proves (VII.2). □

COROLLARY VII.3. *If $\bar{\mu}$ is Λ-Gibbsian, it is Λ'-Gibbsian for any $\Lambda' \subset \Lambda$.*

Proof. By Theorem VII.2, $d(\bar{\mu} \upharpoonright \Sigma_\Lambda) = f_\Lambda d(\mu_0 \upharpoonright \Sigma_\Lambda)$ with $f_\Lambda = e^{-U_\Lambda}\psi_{\partial\Lambda}$. It follows that $d(\bar{\mu} \upharpoonright \Sigma_{\Lambda'}) = f_{\Lambda'} d(\mu_0 \upharpoonright \Sigma_\Lambda)$ where

$$f_{\Lambda'} = E_{\Lambda'}(e^{-U_\Lambda}\psi_{\partial\Lambda}) .$$

By the locality of the interaction and the Markov property

$$\begin{aligned} E_{\Lambda'}(e^{-U_\Lambda}\psi_{\partial\Lambda}) &= e^{-U_{\Lambda'}}E_{\Lambda'}(e^{-U_{\Lambda\setminus\Lambda'}}\psi_{\partial\Lambda}) \\ &= e^{-U_{\Lambda'}}E_{\partial\Lambda'}(e^{-U_{\Lambda\setminus\Lambda'}}\psi_{\partial\Lambda}) . \end{aligned}$$

Thus $f_{\Lambda'}$ is of the form (VII.2) with

$$\text{(VII.5)} \qquad \psi_{\partial\Lambda'} = E_{\partial\Lambda'}(e^{-U_{\Lambda\setminus\Lambda'}}\psi_{\partial\Lambda}) .$$

Remark. One can thus attempt to construct Gibbsian states by finding

$\psi_{\partial\Lambda_1}, \cdots, \psi_{\partial\Lambda_n}, \cdots$ satisfying the consistency conditions (VII.5) for a sequence $\Lambda_1 \subset \Lambda_2 \subset \cdots$ with $\bigcup_{n=1}^{\infty} \Lambda_n = \mathbf{R}^2$. This is precisely the method used by Dobrushin [13], [14] in his proof of uniqueness of Gibbsian states for high temperature lattice gases. We have not been able to duplicate this argument in the $P(\phi)_2$ case.

COROLLARY VII.4. *Let $\bar{\mu}$ be the measure associated with a $P(\phi)_2$ spatial cutoff g which equals 1 on Λ. Then $\bar{\mu}$ is Λ-Gibbsian for P. More generally, if $g \restriction \Lambda = f$, then*

$$\bar{E}_{\partial\Lambda}(A) = E_{\partial\Lambda}(Ae^{-U(f)})/E_{\partial\Lambda}(e^{-U(f)}) .$$

Proof. Similar to the proof of Corollary VII.3.

THEOREM VII.5. *Suppose that g_n is a sequence of cutoffs with $g_n \geqq 0$ so that for any compact Λ, $\int_\Lambda |g_n - 1|^2 d^2x \rightarrow 0$ as $n \rightarrow \infty$. Suppose further that the associated measures μ_n converge to a measure $\bar{\mu}$ in local weak L^1 sense* (i.e., $\mu_n \restriction \Sigma_\Lambda \xrightarrow{w} \bar{\mu} \restriction \Sigma_\Lambda$) *with $\bar{\mu} \restriction \Sigma_\Lambda$ absolutely continuous relative to $\mu_0 \restriction \Sigma_\Lambda$. Then $\bar{\mu}$ is Gibbsian for (P, m).*

Remarks 1. For free boundary conditions and for g_n the characteristic function of a rectangle $(-l_n/2, l_n/2)\times(-T_n/2, T_n/2)$, Newman [70] has proved the necessary convergence of measures for small coupling constant theories. His limiting states are thus Gibbsian states.

2. We emphasize once more our remark in § VI.2 that convergence of moments and convergence of measures are not the same and that, at the present stage of knowledge, we have control over the infinite volume states (as opposed to moments) only in the small coupling case or for the exactly soluble models (deg $P \leqq 2$).

As a final corollary of Theorem VII.2, we can show that (essentially by definition) every Gibbsian state is a local weak L^1 limit of spatially cutoff theories with some boundary conditions, where:

Definition. Fix Λ compact in $\mathbf{R}^2$ and $\eta_{\partial\Lambda}$ a nonnegative $\Sigma_{\partial\Lambda}$-measurable function with $\int e^{-U_\Lambda}\eta_{\partial\Lambda} d\mu_0 = 1$. The measure $e^{-U_\Lambda}\eta_{\partial\Lambda} d\mu_0$ is called *the Gibbs state in region Λ with boundary condition $\eta_{\partial\Lambda}$.*

THEOREM VII.6. *An infinite volume state is a Gibbs state for P if and only if it is a limit (in local weak L^1-sense) as $\Lambda \rightarrow \infty$ of Gibbs states in region Λ with some boundary conditions.*

Proof. Such a limit of Gibbs states is Gibbsian by Proposition VII.1 and Theorem VII.2. Conversely, if $\bar{\mu}$ is Gibbsian, $d(\bar{\mu} \restriction \Sigma_\Lambda) = e^{-U_\Lambda}\eta_{\partial\Lambda} d(\mu_0 \restriction \Sigma_\Lambda)$. Letting $d\bar{\mu}_\Lambda = e^{-U_\Lambda}\eta_{\partial\Lambda} d\mu_0$, we see that $\bar{\mu}_\Lambda \rightarrow \bar{\mu}$ in weak local L^1 sense, so $\bar{\mu}$ is a limit of Gibbs states. □

Remarks 1. It should be noted that in statistical mechanics, the infinite volume Gibbsian states are, in general, only in the closed convex hull of the limits of finite volume Gibbs states. This difference is the reflection of the strict locality of U_Λ.

2. Theorem VII.6 is of limited interest since extremely general types of boundary conditions are allowed. It is to be expected that only Gaussian $\eta_{\partial\Lambda}$ (although not necessarily with mean zero) will be needed to obtain all Gibbsian states, but this is a much more difficult question.

Our next goal is to prove that limits of Dirichlet states or half-Dirichlet states obey DLR equations. This is not quite trivial, even in the non-interacting theory since it requires our analysis of § II.6.

LEMMA VII.7. *Let $\Lambda \subset \Lambda'$ with Λ compact and Λ' open. Let $\hat{\mu}$ be the measure on the Σ_Λ-measurable functions induced by the free Euclidean field with Dirichlet boundary conditions on $\partial\Lambda'$. Then $\hat{\mu}$ is Λ-Gibbsian for $P = 0$,* i.e.,

$$\hat{E}_{\partial\Lambda}(A) = E_{\partial\Lambda}(A)$$

for any Σ_Λ-measurable A.

Proof. By Theorem II.35, $\hat{\mu} \restriction \Sigma_\Lambda$ is absolutely continuous with respect to $\mu_0 \restriction \Sigma_\Lambda$ and its Radon-Nikodym derivative is $\Sigma_{\partial\Lambda}$-measurable. □

THEOREM VII.8. *Any weak local L^1-limit of Dirichlet or half-Dirichlet states is Gibbsian.*

Proof. Consider first the half-Dirichlet state for region Λ'. Let $\bar{E}$ denote the expectation value for this state and $\hat{E}$ the expectation value for the free Dirichlet state in Λ'. If Λ is a compact set in Λ', then by mimicking the argument in Corollary VII.3, we see that for any Σ_Λ-measurable bounded function A,

$$\bar{E}_{\partial\Lambda}(A) = \hat{E}_{\partial\Lambda}(Ae^{-U_\Lambda})/\hat{E}_{\partial\Lambda}(e^{-U_\Lambda}) .$$

By Lemma VII.7, $\hat{E}_{\partial\Lambda}(\cdot) = E_{\partial\Lambda}(\cdot)$ when applied to Σ_Λ-measurable functions, so $\bar{E}$ is Λ-Gibbsian.

Similarly, if $E^{(D,\Lambda')}$ is the Dirichlet state for region Λ' and A is Σ_Λ-measurable and bounded, then

$$E^{(D,\Lambda')}_{\partial\Lambda}(A) = E_{\partial\Lambda}\big(A \exp(-U^{(D,\Lambda')}_\Lambda)\big)/E_{\partial\Lambda}\big(\exp(-U^{(D,\Lambda')}_\Lambda)\big) \qquad \text{(VII.6)}$$

where

$$U^{(D,\Lambda')}_\Lambda = \int_\Lambda :P\big(\phi(x)\big):_{D,\Lambda'} d^2x ,$$

as in Theorem II.40. By that theorem, $\exp(-U_\Lambda^{(D,\Lambda')}) \to \exp(-U_\Lambda)$ as $\Lambda' \to \infty$, the convergence being in any $L^p(Q, d\mu_0)$ $(p < \infty)$. Thus, the right hand side of (VII.6) converges to the right hand side of (VII.1) pointwise a.e. □

From the point of view of the DLR equations, equilibrium states of a $P(\phi)_2$ field theory are labelled by a pair (m, P) consisting of the *bare mass* m and the *interaction polynomial* P. This is to be compared with the field equation approach in which theories are labelled by a pair (m_f, Q) consisting of the *field equation mass* and the *field equation polynomial*, determined by requiring that

$$(\Box + m_f^2)\phi = -:Q'(\phi):,$$

where Q has no quadratic term and $:\ :$ is Wick ordering with respect to the physical vacuum [32], [93]. To some extent, the choice of labelling is a theological question, but we prefer to characterize the theory by a pair (m, P) with the equivalence of certain pairs $(m, P) \cong (\tilde{m}, \tilde{P})$, as discussed below. In defense of this view, we note the following:

(1) The field equation mass is not the physical mass in general. For example, we have seen (V.22b) that for $\lambda :\phi^4:_2$ with small coupling constant λ, $m_{ph} < m$. On the other hand, one finds that for $\lambda :\phi^4:_2$, $Q(X) = \lambda X^4$ with

$$m_f^2 = m^2 + 12\lambda\langle :\phi^2:\rangle_{\text{phys}} . \tag{VII.7}$$

Baumel [2] has proved that $\langle :\phi^2:\rangle_{\text{phys}} > 0$ so that (VII.7) implies that $m_f > m_{ph}$.

(2) It is the coefficients of P (together with m), rather than those of Q, which enter into the Feynman perturbation series. For example, consider a q^4 anharmonic oscillator. Define $\alpha_f(\lambda)$ to be the ground state energy for the oscillator yielding a $q(t)$ which obeys

$$\left(-\frac{d^2}{dt^2} + m_f^2\right)q(t) = -4\lambda :q^3(t):$$

with m_f^2 fixed. Then $\alpha_f(\lambda)$ is not given by the Rayleigh-Schroedinger series.

As a final topic in this subsection, we turn to classifying the equivalence of DLR equations for certain pairs, (m, P). Independently, Baumel [2] has found the same transformation law. Our results complement Baumel's in the sense that he discusses the (periodic) pressure and we discuss states.

THEOREM VII.9. *Let (m, P) be given and let $\tilde{m} > 0$. If $P(X) = \sum_{j=0}^N a_j x^j$, define $\tilde{P}$ by*

$$\tilde{P}(X) = \sum_{j=0}^N a_j \sum_{n=0}^{[j/2]} \frac{j!\, d^n}{2^n n!\,(j-2n)!} X^{j-2n} + \frac{1}{2}(m^2 - \tilde{m}^2)X^2 \tag{VII.8}$$

where

(VII.9) $$d = (2\pi)^{-2} \int \frac{(m^2 - \tilde{m}^2)}{(k^2 + m^2)(k^2 + \tilde{m}^2)} d^2k = -\frac{1}{4\pi} \ln\left(\frac{\tilde{m}^2}{m^2}\right).$$

Then a state is an (m, P) infinite volume Gibbs state if and only if it is an $(\tilde{m}, \tilde{P})$ infinite volume Gibbs state.

Since this theorem asserts the equality of two conditional expectations, it is enough to prove that there is one state which is Λ-Gibbsian for (P, m) and $(\tilde{P}, \tilde{m})$. Thus Theorem VII.9 follows from a result which is of independent interest:

THEOREM VII.10. *Let Λ be a regular region and let (P, m) and $(\tilde{P}, \tilde{m})$ be given by* (VII.8) *and* (VII.9). *Then the half-Dirichlet state* (*and Schwinger functions*) *in region Λ for interaction P with bare mass m agrees with the half-Dirichlet state in region Λ for interaction $\tilde{P}$ with bare mass $\tilde{m}$.*

Proof. Let $d_\delta = (2\pi)^{-2} \int_\delta [\mu_\delta(k, \tilde{m})^{-2} - \mu_\delta(k, m)^{-2}] d^2k$, where we have made explicit the dependence of μ_δ (as defined in equation (IV.6)) on m. Let $\tilde{P}_\delta$ be given by (VII.8) where d_δ replaces d. Explicit computation shows that the half-Dirichlet lattice states for (m, P) and $(\tilde{m}, \tilde{P}_\delta)$ agree. Since $d_\delta \to d$ as $\delta \to 0$ and we can control the convergence of the half-Dirichlet lattice states (Theorem IV.13), the result follows. □

Remarks 1. In particular, this theorem implies equality of half-Dirichlet magnetization and Bogoliubov parameters (cf. § V.4) and thus relates the possibility of broken symmetry in the (m, P) and $(\tilde{m}, \tilde{P})$ theories.

2. There is another relation between distinct pairs due to scaling (cf. [35]). If we make the bare mass explicit in the free field, then $\phi(x, m)$ and $\phi(\lambda x, \lambda^{-1}m)$ have identical covariance matrices. This implies not only a covariance of DLR equations, but also of states and Schwinger functions for free, Dirichlet, and half-Dirichlet B.C.

3. R. Baumel [2] has made an interesting observation based on Remarks 1 and 2. Consider $P = aX^4 + bX^2$. If we ignore the constant term in P (which only affects the pressure and not the states), then $(a, b; m^2)$ is equivalent to $(a, \tilde{b}; \tilde{m}^2)$, where

(VII.10) $$\tilde{b} = b + \frac{1}{2}(m^2 - \tilde{m}^2) - \frac{3a}{2\pi} \ln\left(\frac{\tilde{m}^2}{m^2}\right).$$

$\tilde{b}$ is a continuous monotone decreasing function of $\tilde{m}^2$ on $(0, \infty)$ going to $-\infty$ as $\tilde{m} \to \infty$ and to ∞ as $\tilde{m} \to 0$. Thus we can find $\tilde{m}$ such that $\tilde{b} = 0$. By scaling we can take $\tilde{m}$ to m and a to $\tilde{a} = (m/\tilde{m})^4 a$. We conclude that the $(a, b; m)$ theory has many equilibrium states if and only if the $(\tilde{a}, 0; m)$ theory has many equilibrium states. This emphasizes that the conventional wisdom picture [117] has some defects because it ignores Wick ordering.

4. As pointed out by Baumel [2], the polynomial $\hat{P}$ given by

$$\hat{P} = \tilde{P} + f_\infty$$

with

$$f_\infty = \frac{m^2 - \tilde{m}^2}{8\pi} + \frac{m^2}{8\pi} \ln \frac{\tilde{m}^2}{m^2}$$

is more natural than $\tilde{P}$. Of course, the constant f_∞ does not affect the states. Moreover, Baumel shows that the periodic pressures for P and $\hat{P}$ agree (if the periodic pressures exist).

VII.2. *Spurious Solutions of the* DLR *Equations and Boundary Conditions at Infinity.* It is a basic fact that the DLR equations of § VII.1 are not sufficient to characterize "physical" states for the $P(\phi)_2$ field theory. At the very least, they must be supplemented by a growth condition at infinity. Such a situation has appeared already in the classical statistical mechanics of continuous systems [12], [86]. We shall illustrate the phenomenon in a simple example and then propose a suitable boundary condition.

For our example, we take a one-dimensional field on the half-line $[0, \infty)$ with Dirichlet boundary conditions at $t = 0$ and we consider the case $P = 0$. The same method works for P linear or quadratic and in several dimensions. For the free field we have a covariance

$$\langle q(t)q(t')\rangle = \frac{1}{2} g(t)h(t') \qquad t \leqq t'$$

$$= \frac{1}{2} g(t')h(t) \qquad t' \leqq t$$

where

(VII.11a) $$g(t) = e^t - e^{-t}$$

and

(VII.11b) $$h(t) = e^{-t} .$$

We shall attempt to choose a state $\{f_\Lambda\}$ with

(VII.12) $$f_{[0,t]} = \alpha(t) \exp\big(\beta(t)q(t)\big)$$

for suitable real valued functions $\alpha(t)$ and $\beta(t)$. Since $f_{[0,t]}$ is a function of the fields on the boundary, f will clearly obey the zero interaction DLR equations if we can choose $\alpha(t)$ and $\beta(t)$ so that the $f_{[0,t]}$ are normalized and consistent. The normalization condition is equivalent to

(VII.13) $$\alpha(t) = \exp\Big(-\frac{1}{4}\beta(t)^2 h(t)g(t)\Big) .$$

Consistency requires that

$$E_{[0,t]}f_{[0,t']} = f_{[0,t]}$$

whenever $t' \geqq t$, where $E_{[0,t]}$ is the free field conditional expectation. By explicit computation

$$E_{[0,t]}f_{[0,t']} = \gamma(t, t') \exp\big(\delta(t, t')q(t)\big)$$

where

$$\delta(t, t') = \beta(t')h(t')/h(t)$$

and

$$\gamma(t, t') = \alpha(t') \exp\Big(\frac{1}{4}\beta(t')^2h(t')g(t') - \frac{1}{4}\delta(t, t')^2h(t)g(t)\Big).$$

Consistency and normalization then hold if and only if we choose $\alpha(t)$ obeying (VII.13) and $\beta(t)$ obeying $\beta(t)h(t) = \text{constant}$, i.e., $\beta(t) = ce^t$. We summarize by:

PROPOSITION VII.11. *The state of the one-dimensional* (Dirichlet) *field on* $[0, \infty)$ *given by*

$$f_{[0,t]} = \exp\Big(-\frac{1}{4}c^2(e^{2t} - 1)\Big) \exp\big(ce^tq(t)\big) \tag{VII.14}$$

is a normalized state obeying the zero interaction DLR *equations for any c. Moreover*

$$\ln \|f_{[0,t]}\|_p = \frac{1}{4}c^2(p - 1)(e^{2t} - 1) \tag{VII.15a}$$

and

$$-\int(\ln f_{[0,t]})f_{[0,t]}d\mu_0 = -\frac{1}{4}c^2(e^{2t} - 1)\,. \tag{VII.15b}$$

The formulae (VII.15) follow by explicit computation. They show that the states (VII.14) for $c \neq 0$ are not weakly tempered and that they have entropy density $-\infty$. The moral of Proposition VII.11 is that one must supplement the DLR equations with some growth condition at infinity if there is to be any hope for uniqueness theorems in the small coupling constant region. To illustrate that uniqueness is possible under some growth conditions at infinity, we note the following rather weak result:

PROPOSITION VII.12. *The only tempered state for the one-dimensional (Dirichlet) field on* $[0, \infty)$ *which satisfies the zero interaction* DLR *equations is the state given by* $f_\Lambda = 1$.

Proof. Because of the Dirichlet boundary conditions and Theorem VII.2,

$f_{[0,t]}$ must be of the form

$$f_{[0,t]} = J_t\phi_t$$

where $\phi_t \in L^1[\mathbf{R}, (2\pi)^{-1/2} \exp\big((-q^2)/2\big)dq]$ is nonnegative and J_t is the canonical embedding of § II.6. Consistency then says that for $t < t'$

(VII.16) $$\exp[-(t' - t)H_0]\phi_{t'} = \phi_t \, ,$$

while temperedness says that

(VII.17) $$\| \phi_t \|_2 \leqq \exp(at) \, ,$$

at least for t large. Let $E_0 \leqq E_1 \leqq \cdots$ be the eigenvalues of H_0 with corresponding eigenfunctions $\psi_0, \psi_1, \cdots$. Let $a_n(t) = (\psi_n, \phi_t)$. Then by (VII.16), $a_n(t) = \exp\big((t-1)E_n\big)a_n(1)$ so that (VII.17) implies that $a_n(t) = 0$ for $E_n > a$ and thus that

$$\phi_t = \sum_{n=0}^{N} c_n \exp(tE_n)\psi_n \, .$$

If $\phi_t \neq 1$, then we can suppose that $c_N \neq 0$ and $N > 0$. But, this would imply that

$$(\arg c_N)\psi_N = \lim_{t\to\infty} | c_N |^{-1} \exp(-tE_N)\phi_t$$

is a.e. positive. Since $(\psi_0, \psi_N) = 0$ and ψ_0 is strictly positive, this is impossible. □

Remarks 1. The same proof works for an arbitrary $P(q)$ interaction but the proof depends critically on there being Dirichlet B.C. at one end, allowing us to use the transfer matrix formula (VII.16).

2. We see that the positivity of f_Λ must play a major role in studying uniqueness. For example, we feel that by suitable use of positivity, one should be able to extend the above result to the case where f is only known to be weakly tempered.

The "right" boundary condition at infinity should be determined by the following properties:

(1) Any solution of the DLR equations with the boundary condition should satisfy the variational equality of the next section.

(2) Any state satisfying the boundary condition and the variational equality should obey the DLR equations.

(3) For small coupling constant, there should be a unique solution of the DLR equations together with the boundary condition.

We should like to suggest that weak temperedness is the correct boundary condition. The counterexample in Proposition VII.11 says no weaker condition is possible. And, in the next section, we will reduce (1) above for weakly tempered states to a conjecture whose analogue is known to be true in classical statistical mechanics.

VII.3 *Gibbs Variational Principle: Partial Results.* In statistical mechanics, a variational principle for the entropy density [84] provides a very elegant characterization of the infinite volume equilibrium states associated with a given interaction [56]. It is our purpose in this subsection to introduce similar ideas in Euclidean field theory. While our results are far from definitive, they strongly support the idea that a variational principle can be used to characterize infinite volume equilibrium states in Euclidean field theories.

The Gibbs variational principle involves three quantities: The pressure, $\alpha_\infty(P)$, which depends only on the interaction polynomial P; the entropy density, $s(f)$, which depends only on the state $\{f_\Lambda\}$; and the mean interaction, $\rho(f, P)$, which we shortly introduce and which depends on *both* the interaction polynomial and the state. A complete analysis of the Gibbs principle should involve demonstrating:

(a) For all states f from some "acceptable" class $\mathcal{C}$,

$$s(f) - \rho(f, P) \leqq \alpha_\infty(P) .$$

This, we shall call the *Gibbs variational inequality.*

(b) $\sup_{f\in\mathcal{C}} [s(f) - \rho(f, P)] = \alpha_\infty(P)$,

which we shall call the *Gibbs variational equality.*

(c) $f \in \mathcal{C}$ is Gibbsian for P if and only if

$$s(f) - \rho(f, P) = \alpha_\infty(P) .$$

We shall take $\mathcal{C}$ to be the class of translation invariant, weakly tempered states.

The structure of this subsection is the following. After introducing the mean interaction $\rho(f, P)$ we shall prove the variational inequality for $P(\phi)_2$. Next, we shall look at the one-dimensional case, where we shall be able to compute explicitly the entropy of the infinite volume state for an anharmonic oscillator. We shall use this computation to speculate about the situation in the $P(\phi)_2$ field theory. Finally, we shall indicate an approach linking the DLR equations to the variational equality.

Definition. Given a $\bar{p}$-smooth ($\bar{p} > 1$) translation invariant state $\{f_\Lambda\}$ and an interaction polynomial, P, we define the *mean interaction* associated to the region Λ by

$$\rho_\Lambda(f, P) = \frac{1}{|\Lambda|} \int_Q U_\Lambda f_\Lambda d\mu_0 .$$

The strict locality of the interaction makes the control of the infinite volume "limit" of ρ_Λ especially easy:

PROPOSITION VII.13. *$\rho_\Lambda(f, P)$ is a number $\rho(f, P)$ independent of Λ.*

Proof. Since $\Lambda \to U_\Lambda$ is L^p continuous for $p < \infty$ (cf. Theorem II.10), $f_{\Lambda'}$ is in L^p ($q \leqq \bar{p}$) and, by consistency,

$$\rho_\Lambda(f, P) = \frac{1}{|\Lambda|}\int_Q U_\Lambda f_{\Lambda'} d\mu_0$$

if $\Lambda \subset \Lambda'$, $\Lambda \to \rho_\Lambda$ is clearly continuous. Moreover, if $\Lambda_1 \cup \Lambda_2 = \Lambda$ and $\Lambda_1 \cap \Lambda_2 = \phi$, then

$$|\Lambda|\,\rho_\Lambda = \int_Q (U_{\Lambda_1} + U_{\Lambda_2}) f_\Lambda d\mu_0 = |\Lambda_1|\,\rho_{\Lambda_1} + |\Lambda_2|\,\rho_{\Lambda_2} .$$

It now follows easily that ρ_Λ is independent of Λ. □

The following half of the Gibbs variational principle is now straightforward:

THEOREM VII.14 (Gibbs variational inequality). *For any semibounded interaction polynomial P, and any weakly tempered translation invariant state $\{f_\Lambda\}$,*

$$s(f) - \rho(f, P) \leqq \alpha_\infty(P) .$$

Proof. By the concavity of $\ln x$ and the normalization of the f_Λ we have for any finite Λ:

$$\begin{aligned} |\Lambda|^{-1} S_\Lambda(f) - \rho_\Lambda(f, P) &= |\Lambda|^{-1}\int_Q f_\Lambda[-U_\Lambda - \ln f_\Lambda]d\mu_0 \\ &= |\Lambda|^{-1}\int_Q f_\Lambda \ln\,[e^{-U_\Lambda}/f_\Lambda]d\mu_0 \\ &\leqq |\Lambda|^{-1}\ln\left(\int_Q e^{-U_\Lambda}d\mu_0\right) = |\Lambda|^{-1}\ln Z_\Lambda . \end{aligned}$$

Taking $\Lambda \to \infty$ (Fisher), we complete the proof. □

Except for the exactly solvable linear and quadratic interactions (where the variational equality can be verified by direct computation) we have nothing to report about the variational *equality* for two-dimensional theories. This is not surprising, since we know the existence of infinite volume states only for small coupling constant [70] and we do not even know weak temperedness holds in that case. However, for the one-dimensional case, we can verify the variational equality directly by the use of transfer matrix ideas. Let Ω_Q be the ground state for the Hamiltonian $H_Q = H_0 + :Q(q):$ and let E_Q be the ground state energy. Finally let f^Q be the associated Euclidean state so that (cf. Theorem II.18)

$$\text{(VII.18)} \qquad f^Q_{[a,b]} = (J_a\Omega_Q)(J_b\Omega_Q)\exp\left(-\int_a^b :Q(q(t)):dt\right)\exp\left((b-a)E_Q\right) .$$

Explicit computation now yields

THEOREM VII.15. *For the one-dimensional Markov theory, we have*:

(i) $\rho(f^Q, P) = (\Omega_Q, :P(q):\Omega_Q)$;

(ii) $s(f^Q) = -(\Omega_Q, H_0\Omega_Q)$;

(iii) $s(f^Q) - \rho(f^Q, Q) = \alpha_\infty(Q) = -E_Q$.

Proof. If $a < c < b$, then

$$\int :P(q(c)):f^Q_{[a,b]}d\mu_0 = \langle e^{-(c-a)(H_Q-E_Q)}\Omega_Q, :P(q): e^{-(b-c)(H_Q-E_Q)}\Omega_Q\rangle$$
$$= \langle \Omega_Q, :P(q):\Omega_Q\rangle ,$$

which proves (i). By a similar computation using (VII.18), we obtain

$$-\int f^Q_{[a,b]}\ln f^Q_{[a,b]}d\mu_0 = -\int (J_a\Omega_Q)(J_b\Omega_Q)\ln (J_a\Omega_Q)(J_b\Omega_Q)d\mu_0$$
$$+ |b-a|(\langle\Omega_Q, :Q:\Omega_Q\rangle - E_Q) .$$

The first term on the right can be dominated by

$$2|\langle\Omega_Q, e^{-(b-a)H_0}\Omega_Q \ln \Omega_Q\rangle| \leqq 2\|\Omega_Q\|\cdot\|\Omega_Q\ln\Omega_Q\| < \infty .$$

Dividing by $|b-a|$ and going to infinity, we see that

$$s(f^Q) = \langle\Omega_Q, :Q:\Omega_Q\rangle - E_Q ,$$

and (ii) and (iii) follow immediately. □

We emphasize two things about these explicit computations. First, let us translate the Gibbs inequality $s(f^Q) - \rho(f^Q, P) \leqq \alpha_\infty(P)$ to time zero expectations. Using Theorem VII.15 we see it is equivalent to

$$-(\Omega_Q, (H_0 + :P:)\Omega_Q) \leqq -E_P .$$

Thus *the Gibbs variational inequality for the states* $\{f^Q\}$ *is equivalent to the Rayleigh-Ritz variational inequality for the vectors* $\{\Omega_Q\}$. It is also worth emphasizing the content of (ii); namely, *entropy density is the Euclidean version of the free Hamiltonian expectation value.* In the case of non-relativistic quantum mechanics, general results of this nature have been recently obtained by W. Crutchfield [9].

This suggests a variety of alternative pictures for the two-dimensional case. Let us mention one of them. By results of Glimm-Jaffe [33] and Spencer [97], we know that in certain infinite volume (Hamiltonian) limits, both $(\Omega, T_0^{00}\Omega)$ and $(\Omega, T_I^{00}\Omega)$ exist where T_0^{00} is the free energy density and $T_I^{00} = :P(\phi(x)):$. Let f be the associated Euclidean state. Then:

Conjecture. $s(f) = -(\Omega, T_0^{00}\Omega)$ and $\alpha_\infty(P) = -(\Omega, T^{00}\Omega)$ where $T^{00} = T_0^{00} + T_I^{00}$. (Note: If we know that f is weakly tempered, it is easy to prove that $\rho(f, P) = (\Omega, T_I^{00}\Omega)$.) Moreover, one has the following Rayleigh-Ritz principle: the energy density associated with the interaction $:P:$ for translation invariant locally Fock states of the Hamiltonian theory is minimized precisely by the infinite volume (Hamiltonian) states for the $P(\phi)_2$ theory and the minimum value is $-\alpha_\infty(P)$. Were such a variational principle established, it might become a valuable calculational tool.

Finally, we wish to go part way towards a proof that every weakly tempered, translation invariant, Gibbsian state f for interaction P satisfies the variational equality $s(f) - \rho(f, P) = \alpha_\infty(P)$. If f is Λ-Gibbsian then we claim that

(VII.19) $$f_\Lambda = e^{-U_\Lambda} f_{\partial\Lambda}/E_{\partial\Lambda}(e^{-U_\Lambda})$$

where $E_{\partial\Lambda}$ is the free field conditional expectation. For by Theorem VII.2, $f_\Lambda = e^{-U_\Lambda}\psi_{\partial\Lambda}$ from which it follows that $f_{\partial\Lambda} = E_{\partial\Lambda}(f_\Lambda) = E_{\partial\Lambda}(e^{-U_\Lambda})\psi_{\partial\Lambda}$.

Now suppose we define *the conditional partition function*

$$\widetilde{Z}_{\partial\Lambda} = E_{\partial\Lambda}(e^{-U_\Lambda}) ,$$

the *conditional pressure,*

$$\widetilde{p}_{\partial\Lambda} = \frac{1}{|\Lambda|}\ln \widetilde{Z}_{\partial\Lambda} ,$$

and the *boundary entropy,*

$$S_{\partial\Lambda} = -\int_Q f_{\partial\Lambda} \ln f_{\partial\Lambda} d\mu_0 ,$$

where $\widetilde{Z}_{\partial\Lambda}$ and $\widetilde{p}_{\partial\Lambda}$ are $\Sigma_{\partial\Lambda}$-measurable functions. Then

PROPOSITION VII.16. *For any Gibbsian state f,*

(VII.20) $$\frac{S_\Lambda(f)}{|\Lambda|} = \rho_\Lambda(f, P) + \frac{1}{|\Lambda|}S_{\partial\Lambda} + \int_Q f_\Lambda \widetilde{p}_{\partial\Lambda} d\mu_0 .$$

Proof. By (VII.19) $\ln f_\Lambda = -U_\Lambda + \ln f_{\partial\Lambda} - \widetilde{p}_{\partial\Lambda}$. Multiplying by $-|\Lambda|^{-1}f_\Lambda$ and integrating, we obtain (VII.20). □

Thus to prove that a given Gibbsian state f obeys the Gibbs variational equality we need only prove that (i) the conditional pressure converges to the pressure in a suitable sense, and (ii) the boundary entropy density vanishes. At present, we are unable to prove (i). As for (ii), we have

THEOREM VII.17. *Let f be a weakly tempered, translation invariant state. Let $\Lambda_n \to \infty$* (Fisher). *Then $(1/|\Lambda_n|)S_{\partial\Lambda_n}(f) \to 0$ as $n \to \infty$.*

Proof. Let $\eta < 1$ be the exponent in the weak temperedness condition on f. Choose $\eta' \in (\eta, 1)$ and Λ_n' in accordance with Theorem C.3. By the monotonicity (Theorem VI.6b) and weak subadditivity of the entropy (Theorem VI.6c):

$$S_{\Lambda_n}(f) \leqq S_{\Lambda'_n \cup \partial\Lambda_n}(f)$$
$$\leqq S_{\Lambda'_n}(f) + S_{\partial\Lambda_n}(f) + \ln \|f_{\Lambda'_n} f_{\partial\Lambda_n}\|_1 .$$

By Lemma VI.8,

$$\frac{1}{|\Lambda_n|} \ln \|f_{\Lambda'_n} f_{\partial\Lambda_n}\|_1 \longrightarrow 0$$

since $d(\Lambda'_n, \partial\Lambda_n) \geqq d(\Lambda_n)^{\gamma'}$. Thus by Theorem VI.9,

$$s(f) \leqq s(f) + \underline{\lim} \frac{1}{|\Lambda_n|} S_{\partial\Lambda_n}(f) .$$

Since $S_{\partial\Lambda_n}(f) \leqq 0$ (Theorem VI.6a), we see that the limit in question is 0. □

Appendix A. Positive definite matrices with nonpositive off-diagonal elements

In this appendix, we summarize a few properties of two classes of matrices:

Definitions.

(i) M_n will denote the family of $n \times n$ invertible positive definite matrices with nonpositive off-diagonal elements.

(ii) $K_n = \{A \mid A^{-1} \in M_n\}$.

(iii) M_∞ denotes the family of positive invertible operators on l^2 with nonpositive off-diagonal elements.

(iv) $K_\infty = \{A \mid A^{-1} \in M_\infty\}$.

THEOREM A.1. *Any* $A \in K_n$ $(n \leqq \infty)$ *has nonnegative off-diagonal elements.*

Proof. Since $F \equiv A^{-1}$ is invertible, $F \geqq cI$ for some $c > 0$. Write $F = D + B$ where B vanishes on the diagonal and D vanishes off the diagonal. Then $D \geqq cI$ and for any $\lambda \in [0, 1]$, $F(\lambda) \equiv D + \lambda B = \lambda F + (1 - \lambda)D$ is greater than cI. It follows that $F(\lambda)^{-1}$ is analytic in a neighbourhood of [0, 1]. But for λ small, $F(\lambda)^{-1} = D^{-1}\sum_{n=0}^{\infty}(-\lambda BD^{-1})^n$. Since B has only nonpositive elements and D^{-1} only nonnegative elements, each matrix in the Neumann series above has only nonnegative elements. Thus each matrix element of $F(\lambda)^{-1}$ is analytic in a neighbourhood of [0, 1] and has positive Taylor series at $\lambda = 0$. By a standard theorem in complex variables, the series converges at $\lambda = 1$, so $F^{-1} = A$ has nonnegative off-diagonal elements. □

Remarks 1. This theorem is known as the Stieltjes-Ostrowski theorem after the work [107], [75]. Ostrowski's proof also depends upon the convergence of a Neumann series although his argument is more complicated.

2. Since $(F^{-1})_{ij} = (\text{const})(\det F)^{1/2} \int x_i x_j e^{-(x,Fx)} d^n x$, this theorem is also a consequence of Theorem V.I, which, as we have seen, is in turn a consequence of Griffiths' first inequality for ferromagnets.

THEOREM A.2. *Let $A \in K_n$ $(n \leqq \infty)$. Choose a subset, S, of m elements from $\{i \mid 1 \leqq i < n+1\}$. Then the matrix $A_S = \{A_{ij}\}_{i,j \in S}$ is in K_m.*

Proof. Since A is strictly positive definite and bounded, it follows that A_S is strictly positive definite and bounded so that A_S^{-1} exists and is positive definite. Corresponding to the decomposition of $\{i \mid 1 \leqq i < n+1\}$ into S and its complement, we can write:

$$A = \left(\begin{array}{c|c} A_S & B \\ \hline B^* & C \end{array}\right); \qquad A^{-1} = \left(\begin{array}{c|c} D & E \\ \hline E^* & F \end{array}\right).$$

The equations $A_S D + BE^* = 1$ and $A_S E + BF = 0$ imply that $A_S^{-1} = D - EF^{-1}E^*$. But since $A \in K_n$, E has nonpositive elements and $F \in M_{n-m}$. By Theorem A.1, F^{-1} has nonnegative elements so that $EF^{-1}E^*$ has nonnegative elements. It follows that the elements of A_S^{-1} are less than the elements of D which are, by hypothesis, nonpositive off-diagonal. □

Remark. This proof also shows that $(A^{-1})_S - A_S^{-1}$ is positive semi-definite with nonnegative elements (cf. Theorem IV.7).

COROLLARY A.3. *If $A \in K_n$, and $A_{ii} = 1$ for all i, then for any i, j, k*

$$A_{ij} \geqq A_{ik} A_{kj} .$$

Proof. Explicit computation shows that a positive definite matrix of the form

$$\begin{pmatrix} 1 & a & b \\ a & 1 & c \\ b & c & 1 \end{pmatrix}^{-1}$$

has nonpositive off-diagonal elements if and only if $a \geqq bc$, $b \geqq ac$, $c \geqq ab$. The corollary follows from Theorem A.2 upon taking $S = \{i, j, k\}$. □

Appendix B. Correlation inequalities for the anharmonic oscillator: Alternate proofs

In this appendix, we wish to discuss alternate proofs of correlation inequalities for non-dimensional $P(\phi)$ theories, i.e., for the anharmonic oscillator (cf. (II.40)). Because $q(t)$ without any smearing is an L^p function we can avoid the lattice approximation. It is also possible to demonstrate explicitly that the usual ultraviolet cutoff, unlike the lattice cutoff, destroys the ferromagnetic property of the free theory. Since we have already given detailed proofs in Section V which establish the Griffiths inequalities for the anharmonic oscillator, we only sketch the ideas. On $L^2(\mathbf{R}, d\nu)$ where $d\nu(q) = \pi^{-1/2}e^{-q^2}dq$, the harmonic oscillator Hamiltonian ($m=1$) is $H_0 = 1/2(-(d^2/dq^2)+2q(d/dq))$. We denote its vacuum (the function 1) by Ω_0.

1. *Proof of* G-I. This proof avoids all the ideas of Markov field theory. Let C be the cone of all $\psi(q)$ of the form $f(q) + qg(q)$ where f and g are positive even functions in $L^p(\mathbf{R}, d\nu)$ for all $p < \infty$. Then

(i) $\Omega_0 \in C$;

(ii) If $h_1, h_2 \in C$, then $\langle h_1, h_2\rangle \geqq 0$;

(iii) $qC \subset C$;

(iv) If $P(X) = \sum_{n=0}^{2m} a_n X^n$ with $a_{2m} > 0$ and $a_{2k+1} \leqq 0$ for $k = 0, \cdots, m-1$, then $e^{-P(q)}C \subset C$;

(v) $e^{-tH_0}C \subset C$;

(vi) $e^{-t(H_0+P(q))}C \subset C$ if P is of the form given in (iv).

(i)–(iv) are easy to verify. (v) follows from Mehler's formula (II.80) and (vi) from (iv), (v) and the Trotter product formula. (i)–(iii) and (vi) imply that if $H = H_0 + P(q)$, then

$$(\Omega_0, e^{-t_0 H}q^{n_1} \cdots q^{n_r}e^{-t_r H}\Omega_0) \geqq 0 ,$$

which is the first Griffiths inequality.

2. *Proofs of* G-I, II *and* FKG. Let $t_1 < t_2 < \cdots < t_n$ so that

$$a_{ij} \equiv \langle q(t_i)q(t_j)\rangle = 1/2e^{-|t_i-t_j|}$$

obeys

$$a_{ij} = 2^{j-i-1}a_{i,i+1}a_{i+1,i+2}\cdots a_{j-1,j} \qquad \text{if } i<j . \tag{B.1}$$

Using (B.1) it is easy to show by explicit computation that the inverse $\{b_{ij}\}$ of $\{a_{ij}\}$ is tri-diagonal with

$$\begin{aligned}
b_{11} &= 2D^{-1}\textstyle\prod_{j\neq 1}(1-\alpha_j^2) , \\
b_{ii} &= 2D^{-1}(1-\alpha_{i-1}^2\alpha_i^2)\textstyle\prod_{j\neq i,i-1}(1-\alpha_j^2) , && i = 2, \cdots, n-1 , \\
b_{nn} &= 2D^{-1}\textstyle\prod_{j\neq n-1}(1-\alpha_j^2) , \\
b_{i,i+1} &= b_{i+1,i} = -2D^{-1}\alpha_i\textstyle\prod_{j\neq i}(1-\alpha_j^2) , && i = 1, \cdots, m-1
\end{aligned}$$

where

$$D = \textstyle\prod_{j=1}^{n-1}(1-\alpha_j^2)$$

and

$$\alpha_i = \exp(-t_{i+1}+t_i) < 1 ,$$

and the products have $j = 1, \cdots, n-1$.

We thus see explicitly that b has negative off-diagonal elements. If we now approximate $\int_a^b P(q(t))dt$ in a Feynman-Kac formula by

$$\sum_{j=1}^{n} \frac{(b-a)}{n} P(q(t_j))$$

with $t_j = a + j(b-a)/n$, we can prove the correlation inequalities by the method of § V without use of the lattice approximation.

3. *Space Smearing Destroys "Ferromagnetism"*. We claim that if $q(t)$ is replaced by $q_h(t) = \int h(t-s)q(s)ds$ the inverse of a correlation matrix $\langle q_h(t_i)q_h(t_j)\rangle$ $(i, j = 1, \cdots, n)$ will not be negative off-diagonal, at least for h positive. For example if $h(t) = e^{-\alpha|t|}$ with α large, explicit computation shows that

$$\langle q_h(0)q_h(t)\rangle < \langle q_h(0)q_h(t/2)\rangle\langle q_h(t/2)q_h(t)\rangle\ . \tag{B.2}$$

Thus, by Corollary A.3, no covariance matrix

$$\langle q_h(t_i)q_h(t_j)\rangle \quad \text{with } t_1 = 0,\ t_2 = t/2,\ t_3 = t;\ t_4, \cdots, t_n \text{ arbitrary}$$

can have an inverse with negative off-diagonal elements.

4. *Remarks* 1. That the matrix b in the proof of 2 above is tri-diagonal is a reflection of the locality of $-(d^2/dt^2) + 1$.

2. On an intuitive level, it is easy to understand why space smearing destroys "ferromagnetism". Heuristically, $(-(d^2/dt^2) + 1)\delta(t-t')$ has a positive infinity at $t = t'$ and negative infinities at $t = t' \pm \varepsilon$. It is thus positive on-diagonal and negative off-diagonal. Smearing in t tends to produce at least a small region $|t - t'| < \delta$ of positivity. This region includes some off-diagonal elements.

Appendix C. Fisher convergence: Some technical results

Here we want to recall the definition of convergence in the sense of Fisher and van Hove [85] and to prove a few related technical facts of a geometric nature. Although all the results extend to d dimensions, we restrict ourselves to $d = 2$. Given $a \in \mathbf{R}^+$, the lattice $a\mathbf{Z}^2$ induces in a natural way a decomposition of $\mathbf{R}^2$ into closed squares with side a and centers in $a\mathbf{Z}^2$. Given $\Lambda \subset \mathbf{R}^2$ bounded we let $N_a^-(\Lambda)$ be the number of such squares inside Λ and $N_a^+(\Lambda)$ be the number of such squares which intersect Λ. $|\Lambda|$ is the volume of Λ and $d(\Lambda)$ its diameter. Finally

$$V_a(\Lambda) = |\{x \in \mathbf{R}^2 \mid d(x, \Lambda) < a\}|\ . \tag{C.1}$$

Definition. Let Λ_n be a family of regions of $\mathbf{R}^2$ with $|\Lambda_n| \to \infty$. We say that $\Lambda_n \to \infty$ in *van Hove* sense, if and only if for each $a \in \mathbf{R}^+$,

$N_a^-(\Lambda_n)/N_a^+(\Lambda_n) \to 1$ as $n \to \infty$. Our notion of Fisher convergence is slightly weaker than Ruelle's:

Definition. Let Λ_n be a family of regions in $\mathbf{R}^2$ with $|\Lambda_n| \to \infty$. We say $\Lambda_n \to \infty$ in *Fisher* sense if and only if there is an $\varepsilon > 0$ and a function π on $(0, \varepsilon)$ with $\lim_{\alpha\to 0} \pi(\alpha) = 0$ so that for each $\alpha \in (0, \varepsilon)$, there is an N so that

$$V_{\alpha d(\Lambda_n)}(\Lambda_n) \leqq \pi(\alpha)\,|\Lambda_n| \tag{C.2}$$

for all $n > N$.

Clearly, $|\Lambda| \leqq \pi d(\Lambda)^2$ and $V_a(\Lambda) \geqq \pi a^2$ if $\Lambda \neq \phi$. Thus:

PROPOSITION C.1. *If* $\Lambda_n \to \infty$ (Fisher), *then there are constants* c_1, c_2, N *such that for all* $n > N$,

$$c_1 d(\Lambda_n)^2 \leqq |\Lambda_n| \leqq c_2 d(\Lambda_n)^2 .$$

It is also clear that $a^2 N_a^+(\Lambda) \geqq |\Lambda|$ and that $a^2\big(N_a^+(\Lambda) - N_a^-(\Lambda)\big) \leqq V_{a\sqrt{2}}(\Lambda)$ so that:

PROPOSITION C.2. *If* $\Lambda_n \to \infty$ (Fisher), *then* $\Lambda_n \to \infty$ (van Hove).

The converse of this is false [85]; e.g., let Λ_n be the rectangle of sides n and n^2. In § VII.3, we used the following:

THEOREM C.3. *Let* $\Lambda_n \to \infty$ (Fisher). *Suppose* $\eta' < 1$ *is given. Then there exists* $\Lambda_n' \subset \Lambda_n$ *so that* (Fig. C.1)

(a) $\Lambda_n' \to \infty$ (Fisher);
(b) $|\Lambda_n'|/|\Lambda_n| \to 1$ *as* $n \to \infty$;
(c) $d(\Lambda_n', \partial\Lambda_n) \geqq d(\Lambda_n)^{\eta'}$.

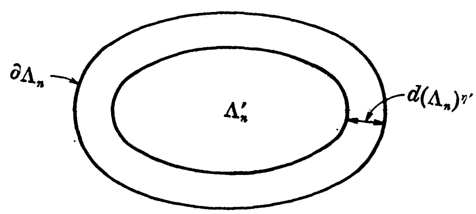


FIGURE C.1

Proof. Let $\Lambda_n' = \{x \in \Lambda_n \mid d(x, \partial\Lambda_n) \geqq d(\Lambda_n)^{\eta'}\}$. Since $|\Lambda_n'| \leqq |\Lambda_n| \leqq |\Lambda_n'| + V_{d(\Lambda_n)^{\eta'}}(\Lambda_n)$, (b) follows. (c) is trivial. Let $\varepsilon' = \varepsilon/2$ and $\pi'(\alpha) = 2\pi(2\alpha)$. Given α, choose N so that $n > N$ implies that (i) $|\Lambda_n'| \geqq |\Lambda_n|/2$, (ii) $d(\Lambda_n)^{\eta'} \leqq \alpha d(\Lambda_n)$, and (iii) $V_{2\alpha d(\Lambda_n)}(\Lambda_n) \leqq \pi(2\alpha)\,|\Lambda_n|$. Then, if $n > N$,

$$V_{\alpha d(\Lambda_n')}(\Lambda_n') \leqq V_{\alpha d(\Lambda_n)}(\Lambda_n') \leqq V_{\alpha d(\Lambda_n) + d(\Lambda_n)^{\eta'}}(\Lambda_n) \leqq V_{2\alpha d(\Lambda_n)}(\Lambda_n) \leqq 2\pi(2\alpha)\,|\Lambda_n'| .$$

This proves (a). □

Finally in § VI we used:

THEOREM C.4. *Suppose* $\Lambda'_n \to \infty$ (Fisher). *Let* $\eta' < 1$ *be given. Then we can find sets* Λ''_n *and squares* Λ_n *so that* (Fig. C.2)

(a) $\Lambda'_n \subset \Lambda_n$, $\Lambda''_n \subset \Lambda_n$, $\Lambda'_n \cap \Lambda''_n = \emptyset$;

(b) *for some* $c > 0$ *and all sufficiently large* n, $c < |\Lambda'_n|/|\Lambda_n| < (1/2)$;

(c) $(|\Lambda'_n| + |\Lambda''_n|)/|\Lambda_n| \to 1$ *as* $n \to \infty$;

(d) $\Lambda''_n \to \infty$ (Fisher);

(e) $d(\Lambda'_n, \Lambda''_n) \geqq d(\Lambda'_n)^{\eta'}$ *for all* n.

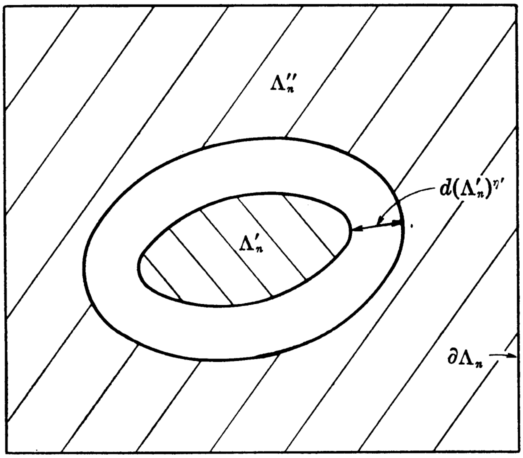


FIGURE C.2

Proof. Choose any square Λ_n of side $3d(\Lambda'_n)$ and center in Λ'_n. Let $\Lambda''_n = \{x \in \Lambda_n \mid d(x, \Lambda'_n) \geqq d(\Lambda'_n)^{\eta'}\}$. (a) and (e) are obvious. (b) follows from Proposition C.1 and its proof. (c) follows from

$$|\Lambda'_n| + |\Lambda''_n| \leqq |\Lambda_n| \leqq |\Lambda'_n| + |\Lambda''_n| + V_{d(\Lambda'_n)^{\eta'}}(\Lambda'_n)$$

and the fact that $\Lambda'_n \to \infty$ (Fisher). Finally, to prove (d) we note that for all large n, $c_1 d(\Lambda'_n) \leqq d(\Lambda''_n) \leqq c_2 d(\Lambda'_n)$ ($c_1, c_2 > 0$) and that

$$V_a(\Lambda''_n) \leqq V_a(\Lambda_n) + V_{a+d(\Lambda'_n)^{\eta'}}(\Lambda'_n)$$

since $\partial\Lambda''_n \subset \partial\Lambda_n \cup \{x \mid d(x, \partial\Lambda'_n) = d(\Lambda'_n)^{\eta'}\}$. (d) follows by mimicking the proof of Theorem C.3. □

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