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# *Essential Self-Adjointness of Schrödinger Operators with Singular Potentials*

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## 1. Introduction

In this note we wish to study Schrödinger operators $-\Delta+q$ on $L^2(R^m)$, where $q$ is the operator of multiplication by a real-valued measurable function, $q(x)$, on $R^n$. We show that $-\Delta+q$ is essentially self-adjoint on either $C_0^\infty(R^m)$, the $C^\infty$ functions of compact support, or on $C_0^\infty(R^m/\{0\})$, the $C^\infty$ functions of compact support in $R^m/\{0\}$. Upon occasion we shall denote these sets by $C_0^\infty$ and $C_{00}^\infty$, respectively.

The self-adjointness of such operators is an extensively studied problem, but until recently all results have at least supposed that $q$ is in a local Stummel space (slightly weaker than $q\in(L^p)_{\mathrm{loc}}$ with $p>m/2$, $p\geqq 2$). This is considerably stronger than the condition $q\in(L^2)_{\mathrm{loc}}$ (resp. $q\in(L^2(R/\{0\}))_{\mathrm{loc}}$) needed for $-\Delta+q$ to be well-defined on $C_0^\infty$ (resp. $C_{00}^\infty$). In general, $-\Delta+q$ will not be self-adjoint on $C_0^\infty$ if $q\in L^2$ and $m\geqq 4$ (see [4] for an explicit example). However, we recently showed that if $q\geqq 0$, then $q\in L^2$ is sufficient for essential self-adjointness on $C_0^\infty$ [4]. Extensions of this result have been obtained by Kato [2], some of whose methods we shall use below. The two theorems we prove below are:

**Theorem 1** (Kato). *Let $q=q_1+q_2$ with $q_1\in(L^2)_{\mathrm{loc}}$, $q_2\in L^p$ (where $p=2$ if $m<4$, $p>2$ if $m=4$ and $p=m/2$ if $m>4$). Suppose that $q_1$ is bounded from below. Then $-\Delta+q$ is essentially self-adjoint on $C_0^\infty$.*

**Theorem 2.** *Let $q=q_1+q_2$, with $q_1\in(L^2(R^m/\{0\}))_{\mathrm{loc}}$ and $q_2\in L^\infty$ and suppose that*

$$q_1(\vec{r})\geqq -[(m-1)(m-3)-3]/4r^2.$$

*Then $-\Delta+q$ is essentially self-adjoint on $C_{00}^\infty$.*

Theorem 1 was conjectured in [4] and proved by Kato in [2]; we present a partly alternative proof in § 2. Theorem 2 generalizes results of Kalf & Walter [1] and Schmincke [3].

Our method of proof is very different from the methods used in [4] but is closely related to Kato's method in [2]. The basic tool is the use of distributional inequalities. If $T$ and $S$ are distributions, we write $T\geqq S$ if and only if $T(\phi)\geqq S(\phi)$



for every positive test function $\phi\geqq 0$. We shall use the following inequality of Kato ([2], Lemma A):

**Kato's Inequality.** *Suppose $u$ and $\Delta u$ are in $(L^1(R^m))_{\mathrm{loc}}$. Then*

$$\Delta|u|\geqq(\operatorname{sgn}u)\Delta u.$$

**Remarks.** 1. We suppose that $u$ is real-valued although there is a similar inequality if $u$ is complex-valued. The symbol sgn $u$ is used to denote $+1$, $-1$, $0$ according to whether $u>0$, $u<0$ or $u=0$.

2. The inequality $\geqq$ holds in the sense of elements of the distribution space dual to $C_0^\infty$.

3. This inequality is local; thus if $u$ and $\Delta u$ are in $L^1(R^m/\{0\})_{\mathrm{loc}}$, the inequality holds in the sense of the distribution space dual to $C_{00}^\infty$.

4. The proof of the inequality is rather simple (Kato's Lemma A is more general, so its proof is notationally more complex). If $u$ is smooth and $u_\varepsilon=\sqrt{u^2+\varepsilon}$, the formulae $u_\varepsilon\operatorname{grad}u_\varepsilon=u\operatorname{grad}u$, $u_\varepsilon\Delta u_\varepsilon+|\operatorname{grad}u_\varepsilon|^2=u\Delta u+|\operatorname{grad}u|^2$ and $u_\varepsilon\geqq|u|$ imply $u_\varepsilon\Delta u_\varepsilon\geqq u\Delta u$. Dividing by $u$ and letting $\varepsilon$ tend to 0 yields the inequality. For general distributions $u$ (obeying the same conditions), mollified $u$'s obey the inequality $u_\varepsilon\Delta u_\varepsilon\geqq u\Delta u$. Taking limits, we find that $u$ obeys Kato's inequality.

## 2. Proof of Theorem 1

Besides Kato's inequality, the proof relies on the fact that if $u$ is a distrubution with $u\in L^2$ and $(-\Delta+q_2+E)|u|\leqq 0$ for a suitable real $E$, then $u=0$. Here $-\Delta+q_2+E$ is viewed as an operator from $L^2$ to $(C_0^\infty)'$. This is formally connected with the fact that $(-\Delta+q_2+E)^{-1}$, the inverse of the operator from $D(\Delta)$ to $L^2$, is positivity preserving. Rather than using the fact that $H=-\Delta+q_2$ has a positivity preserving resolvent, we use the related result that when

$$\Sigma=\inf\operatorname{spec}(-\Delta+q_2)<0,$$

then $\Sigma$ is a non-degenerate eigenvalue with strictly positive eigenvector.

**Lemma.** *Let $q_2\in L^p(R^m)$ ($p=2$ if $m<4$, $p>2$ if $m=4$ and $p=\frac{1}{2}m$ if $p>4$) and suppose that $\Sigma=\inf\operatorname{spec}(-\Delta+q_2)<0$. Let $E<\Sigma$. Suppose that $u\in L^2$ and that $(-\Delta+q_2+E)|u|\leqq 0$ in the sense of distributions ($-\Delta|u|$ being a distributional derivative). Then $u=0$.*

**Proof.** A simple argument (see [4] for reference) shows that $q_2$ is a relatively compact perturbation of $-\Delta$, so $-\Delta+q_2$ has $[0,\infty)$ as its essential spectrum. Since $\Sigma<0$, it is an eigenvalue. By a result in [5], it follows that $\Sigma$ is nondegenerate and $(-\Delta+q_2)\Psi=\Sigma\Psi$ for some $\Psi\in D(-\Delta+q_2)$ which is strictly positive a.e. Since $D(-\Delta+q_2)=D(-\Delta)$, we can find $\Psi_n\in C_0^\infty$ such that $\Psi_n\to\Psi$ in $L^2$, $-\Delta\Psi_n\to-\Delta\Psi$ in $L^2$, and $\Psi_n\geqq 0$. Since $q_2$ is $-\Delta$-bounded, $(-\Delta+q_2+E)\Psi_n\to(\Sigma+E)\Psi$. Thus

$$\begin{aligned}\langle(\Sigma+E)|u|,\Psi\rangle&=\lim_{n\to\infty}\langle|u|,(-\Delta+q_2+E)\Psi_n\rangle\\&=\lim\langle(-\Delta+q_2+E)|u|,\Psi_n\rangle\\&\leqq 0\ \text{(distributional sense).}\end{aligned}$$

Since $\Sigma+E\geqq 0$, $\langle |u|, \Psi\rangle \leqq 0$. But $\Psi$ is strictly positive a.e.; hence $|u|=0$.

**Proof of Theorem 1.** We assume without loss of generality that $q_1\geqq 0$. Moreover we can suppose that $\Sigma=\inf \operatorname{spec}(-\Delta+q_2)<0$. For choose any $\Phi$ in $C_0^\infty$ with support in unit ball $B$ and let $\tilde{q}_2=q_2-c\chi_B$[1], where $c=1+\|\Phi\|^{-2}(\langle\Phi,(-\Delta+q_2)\Phi\rangle)$ and $\tilde{q}_1=q_1+c\chi_B$. Then $\tilde{q}_1\geqq 0$, $\tilde{q}_1\in(L^2)_{\text{loc}}$, $\tilde{q}_2\in L^p$ and $\tilde{\Sigma}=\inf \operatorname{spec}(-\Delta+\tilde{q}_2)\leqq -1$.

Let $\Sigma=\inf(-\Delta+q_2)<0$. It is sufficient to prove that $(-\Delta+q-\Sigma+1)\,[C_0^\infty]$ is dense in $L^2$. If it is not dense, then there exists an element $u\in L^2$ orthogonal to $(-\Delta+q-\Sigma+1)\,[C_0^\infty]$ which can be assumed to be real-valued, since $q$ is real-valued and $C_0^\infty$ is self-conjugate. But then $(-\Delta+q-\Sigma+1)\,u=0$ (distributional sense). Since $u\in L^2$ we have $u\in(L^1)_{\text{loc}}$, and $\Delta u=(q-\Sigma+1)\,u\in(L^1)_{\text{loc}}$ since $q\in(L^2)_{\text{loc}}$. Thus Kato's inequality is applicable, and we find

$$-\Delta\,|u|\leqq -(\operatorname{sgn} u)\,\Delta u=(-q+\Sigma-1)\,|u|$$

or

$$-\Delta+q_2+(-\Sigma+1)\,|u|\leqq -q_1\,|u|\leqq 0$$

since $q_1\geqq 0$. By the lemma, $u=0$.

### 3. Proof of Theorem 2 (Generalized Kalf-Walter-Schmincke Theorem)

Let $B$ be the ball of radius 1 and let $q_3=-\frac{1}{4}[m(m-4)]\,r^{-2}-c\chi_B$, where $c$ is a positive constant which we shall adjust below. Let $q_4=q_1-q_3$. We shall prove that $-\Delta+q_1$ is essentially self-adjoint on $C_0^\infty$, from which Theorem 2 will follow since $q_2$ is bounded.

The key to the proof is to construct an auxiliary function $\Psi_0$ which is spherically symmetric, $C^1$ on $\mathbb{R}^m\backslash\{0\}$, $C^\infty$ on $\mathbb{R}^m\backslash\{0\}\cup\{\vec{r}\,|\,r=1\}$, and moreover such that: (1) $\Psi_0$ is strictly positive, (2) $\Psi_0\in L^2(\mathbb{R}^m)$, (3) $\Psi_0\cong r^{(4-m)/2}$ as $r\to 0$, (4) at any point $\vec{r}$ with $r\neq 0, 1$, $\Psi_0$ obeys $(-\Delta+q_3)\,\Psi_0=-\Psi_0$ near and $\vec{r}$, where $-\Delta$ is the classical derivative.

Notice that we do not claim that $(-\Delta+q_3)\,\Psi_0=-\Psi_0$ in terms of an operator equation when $-\Delta+q_3$ is interpreted as an operator in some sense. (Once the theorem is proved, it follows that $\Psi_0$ is an eigenfunction of $\overline{-\Delta+q_3\restriction C_{00}^\infty}$, but a priori we neither know nor need that fact.)

To construct $\Psi_0(\vec{r})$, we choose a function $f(r)$ which is $C^1$ on $(0,\infty)$ and $C^\infty$ on $(0,1)$ and on $(1,\infty)$ and obeys (1′) $f(r)>0$ for all $r$; (2′) $f(r)$ decays exponentially at $\infty$; (3′) $f(r)\sim r^{3/2}$ at $r=0$; and (4′)

$$-f''+\tfrac{3}{4}r^{-2}f=(C-1)f \quad \text{on} \quad (0,1)$$

and

$$-f''+\tfrac{3}{4}r^{-2}f=-f \quad \text{on} \quad (1,\infty).$$

We then let $\Psi_0=r^{(1-m)/2}f$. The differential equations $-f''+\frac{3}{4}r^{-2}f=(C-1)f$ are exactly solvable for any $C$ in terms of Bessel functions. When $C=0$ (i.e., $r\in(1,\infty)$), there is a solution which is strictly positive and which decays exponentially at $\infty$. For any real $C$, there is a solution asymptotic to $r^{3/2}$ near $r=0$. By

[1] $\chi_B$ denotes the characteristic function of the ball.



adjusting the positive constant $C$ in the definition of $q_3$, we can arrange that the two solutions and their derivatives match at $r=1$ and also that the resulting function $f$ is strictly positive.

Let $\eta$ be a $C^\infty$ function which is 1 outside the ball of radius 1/2, and 0 inside the ball of radius 1/4. Let $\Psi_n(r)=\eta(rn)\,\Psi_0(r)$. Then $\Psi_n\to\Psi_0$ in $L^2$ and

$$(-\Delta+q_3+1)\Psi_n=-n[\vec{\nabla}\eta(rn)]\cdot\vec{\nabla}\Psi_0-n^2[(\Delta\eta)(rn)]\Psi_0$$

is bounded in $L^2$ and has support shrinking to 0. We conclude that $(-\Delta+q_3+2)\,\Psi_n$ converges to $\Psi_0$ weakly in $L^2$. By modifying $\Psi_n$ near $\infty$ and on the unit sphere, we can find $\phi_n\in C_{00}^\infty$ such that (*i*) $\phi_n\geqq 0$, (*ii*) $\phi_n\to\Psi_0$, and (*iii*) $(-\Delta+q_3+2)\,\phi_n\to\Psi_0$ weakly.

To prove that $-\Delta+q_1$ is essentially self-adjoint on $C_{00}^\infty$, we need only show that $(-\Delta+q_1+2)[C_{00}^\infty]$ is dense. If $u$ is orthogonal to $(-\Delta+q_1+2)[C_{00}^\infty]$, then $(-\Delta+q_1+2)\,u=0$ in the sense of $(C_{00}^\infty)'$. Thus $u$ and $\Delta u=(q_1+2)\,u$ are in $L^1(R^m/\{0\})_{\text{loc}}$ and Kato's inequality is applicable. We conclude that $-\Delta|u|\leqq(-q_1-2)|u|$ (in the sense of $(C_{00}^\infty)'$). Since $q_4\geqq 0$ by hypothesis, we have

$$(-\Delta+q_3+2)\,|u|\leqq -q_4\,|u|\leqq 0.$$

As a result

$$\begin{aligned}\langle\Psi_0,|u|\rangle&=\lim_{n\to\infty}\langle(-\Delta+q_3+2)\,\phi_n,|u|\rangle\\&=\lim_{n\to\infty}\langle\phi_n,(-\Delta+q_3+2)\,|u|\rangle\leqq 0\ \ (C_{00}^\infty-\text{sense}).\end{aligned}$$

Since $\Psi_0$ is strictly positive, it follows that $u=0$. We conclude that $-\Delta+q_1$ is essentially self-adjoint.

**Remarks.** 1. In particular, $-d^2/dx^2+ar^{-2}$ is essentially self-adjoint on $C_{00}^\infty(R)$ if and only if $a\geqq 3/4$. It is interesting to compare our proof with the usual proof employing WEYL's limit-point limit-circle method. The two solutions of $-f''+3/4r^{-2}f=df$ for $d$ constant behave at $r=0$ like $r^{-1/2}$ and $r^{3/2}$. In WEYL's method, the key fact is that $r^{-1/2}$ is not in $L^2$. In our method, certain properties of $r^{3/2}$ are critical, namely the estimates

$$\int_0^R (r^{3/2})^2\,dr=O(R^4) \quad\text{and}\quad \int_0^R (d/dr(r^{3/2}))^2\,dr=O(R^2).$$

2. T. KATO (private communication) has found an alternate proof of Theorem 2 which also allows $q_1$ to vary slightly below the $r^{-2}$ bound at 0 and the 0 bound at $\infty$.

3. The constant $[(m-1)(m-3)-3]/4=c_m$ is best possible in the sense that if $\alpha>c_m$, then $-\Delta-\alpha r^{-2}$ is not essentially self-adjoint on $C_{00}^\infty$.

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