


Remarks on periodic Jacobi matrices on trees

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ABSTRACT

We look at periodic Jacobi matrices on trees. We provide upper and lower bounds on the gap of such operators analogous to the well-known gap in the spectrum of the Laplacian on the upper half-plane with a hyperbolic metric. We make some conjectures about antibound states and make an interesting observation for the so-called rg -model where the underlying graph has r red and g green vertices and where any two vertices of different colors are connected by a single edge.

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I. INTRODUCTION

This paper is a contribution to growing recent literature on periodic Jacobi matrices on trees. Specifically, we were motivated by the work of Avni, Breuer, and Simon¹ (see also Ref. 2) whose notation and ideas we will follow; in particular, we refer the reader to that paper for the definitions from graph theory that we will use. We also note the relevance of some recent preprints by a group at Berkeley.^{3,4}

One starts out with a finite leafless graph, \mathcal{G} , and a Jacobi matrix on that graph. By this, we mean a matrix with indices labeled by the vertices in the graph whose diagonal elements are a function, b , on the vertices and off diagonal elements, which are only non-zero for pairs of vertices, which are the two ends of some edge, with matrix elements determined by a function, a , on the edges. The universal cover, \mathcal{T} , of \mathcal{G} is always a leafless tree. There is a unique and natural lift of any Jacobi matrix, J , on \mathcal{G} to an operator, H (still called a Jacobi matrix), on $\ell^2(\mathcal{G})$. Because H is invariant under a group of deck transformations on \mathcal{T} (see Ref. 5, Sec. 1.7 for the covering space language we exploit), we call H a *periodic Jacobi matrix*.

There are three big general theorems in the subject: (1) a result of Sunada⁶ on labeling of gaps in the spectrum that implies that the spectrum of H is at most p bands, where p is the number of vertices in the underlying finite graph \mathcal{G} ; (2) a result of Aomoto⁷ stating that if \mathcal{G} has a fixed degree, then H has no point spectrum, and (3) a result of Avni, Breuer, and Simon¹ that there is no singular spectrum because matrix elements of the resolvent are algebraic functions. Besides a very few additional theorems, the subject at this point is mainly some interesting examples and lots of conjectures and questions.

This paper has some additional results, observations, and an interesting fact about one fascinating example.

Typically, σ , the largest eigenvalue of J , does not lie in the spectrum of H . Two of the subjects we treat involve σ . Section II discusses quantitative estimates on the distance from σ to the top of the spectrum of H . Section III will show that in a number of examples, σ can be viewed as an antibound state and we make a general conjecture that it is a pole of the analytically continued Green's function. Section IV studies an example of a finite graph with a non-fixed degree for which Aomoto⁷ showed that H has a point eigenvalue and ABS¹ found an explicit eigenfunction. We prove that this eigenfunction and its translates span the eigenspace.

II. GAP COMPARISON THEOREM

In this section, we want to fix a connected graph, \mathcal{G} , with p vertices, $V(\mathcal{G})$, and q edges, $E(\mathcal{G})$. We are going to fix two sets of Jacobi parameters on \mathcal{G} , namely, $\{a_\alpha, b_j\}_{\alpha \in E(\mathcal{G}), j \in V(\mathcal{G})}$ and $\{\tilde{a}_\alpha, \tilde{b}_j\}_{\alpha \in E(\mathcal{G}), j \in V(\mathcal{G})}$, with corresponding Jacobi matrices J and \tilde{J} on $\ell^2(\mathcal{G})$, and to look at the induced periodic Jacobi matrices H and \tilde{H} on $\ell^2(\mathcal{T})$, with \mathcal{T} being the universal cover of \mathcal{G} .

By the Perron–Frobenius theorem (Ref. 8, Theorem 1.4.4), J has a simple largest eigenvalue, $\sigma(a, b)$, with a unique strictly positive normalized eigenvector, $\psi(a, b, j)_{j \in V}$. (Below in the bipartite case, where we use σ_- , we will sometimes write σ_+ for σ .) We will use $\Sigma(a, b)$ for sup of the spectrum of H . It is a fundamental result of Sy and Sunada⁹ that if the graph \mathcal{G} has more than one independent closed loop (so the fundamental group is non-amenable), then the gap

$$G(a, b) \equiv \sigma(a, b) - \Sigma(a, b) \tag{2.1}$$

is strictly positive. In this section, our goal is to prove comparison inequalities for the gaps $G(a, b)$ and $G(\tilde{a}, \tilde{b})$ based on comparison results in distinct but related contexts by Kirsch and Simon¹⁰ and Frank, Simon, and Weidl.¹¹

These results rely on a ground state representation for the quadratic form of J , a formula that for continuum Sturm–Liouville operators goes back to Jacobi.¹² For operators, $H \equiv -\Delta + V$, on $L^2(\mathbb{R}^v)$ for which there is a strictly positive solution of $(-\Delta + V)\psi = E_0\psi$, it takes the form

$$\langle f\psi, (H - E_0)f\psi \rangle = \int |\nabla f(x)|^2 \psi(x)^2 d^v x. \tag{2.2}$$

This was first used in spectral theory by Birman¹³ and made popular in constructive Hamiltonian quantum field theory after its use by Nelson¹⁴ and then by Glimm, Jaffe, Sigal, and others.^{15,16} Remarkably, the version for Jacobi-type matrices seems to have only been written down in 2008 by Frank, Simon, and Weidl¹¹ (see Ref. 17). For our situation, it takes the following form:

Theorem 2.1. *Let J be a Jacobi matrix on a finite graph, \mathcal{G} , or a periodic Jacobi matrix on an infinite tree, \mathcal{T} . Suppose that ψ is a non-negative sequence $\{\psi_i\}_{i \in V}$ that obeys $J\psi = \sigma\psi$. Then, for any real sequence $\{f_i\}_{i \in V}$, with $f\psi \in \ell^2(V)$, we have that*

$$\langle f\psi, (\sigma - J)f\psi \rangle = \sum_{\alpha=(j,k)} a_\alpha \psi_j \psi_k (f_j - f_k)^2. \tag{2.3}$$

- Remarks.*
1. The sum in (2.3) is over all edges in the graph or tree, and the notation indicates that j and k are the two ends of the edge.
 2. For the finite graph, ψ is just the eigenfunction for the largest eigenvalue of the graph. For the tree, ψ is the (periodic) lift of the underlying graph ψ to its universal cover.
 3. We use V as a shorthand notation for either $V(\mathcal{G})$ or $V(\mathcal{T})$.
 4. The proof is as in Ref. 11, Theorem 3.2. If M_f is multiplication by a function f of finite support, one verifies that

$$[M_f, (J(a, b) - \sigma)] = J(a_\alpha(f_j - f_k), b \equiv 0), \tag{2.4}$$

which then implies

$$[M_f, [M_f, (J(a, b) - \sigma)]] = J(a_\alpha(f_j - f_k)^2, b \equiv 0). \tag{2.5}$$

Letting ψ^\sharp be ψ restricted to the support of f and all vertices connected to this finite support then yields $M_f(J(a, b) - \sigma)\psi^\sharp = 0$, and hence,

$$\langle \psi^\sharp, [M_f, [M_f, (J(a, b) - \sigma)]]\psi^\sharp \rangle = -2\langle f\psi, (J(a, b) - \sigma)f\psi \rangle, \tag{2.6}$$

so (2.3) for finitely supported f follows from (2.5) and (2.6). To get (2.3) for general f with $f\psi \in \ell^2(V)$, one approximates f by finitely supported f_n of increasing support such that $\|(f - f_n)\psi\|_{\ell^2(V)} \rightarrow 0$ and takes limits of both sides. The LHS converges since $J(a, b)$ is a bounded operator, and the RHS converges since we can view it as the difference of

$$S_1(f_n) = \sum_{\alpha=(j,k)} a_\alpha \psi_j \psi_k [(f_n)_j^2 + (f_n)_k^2] \tag{2.7}$$

and

$$S_2(f_n) = 2 \sum_{\alpha=(j,k)} a_\alpha (f_n \psi)_j (f_n \psi)_k = 2 \langle f_n \psi, J(a, 0) f_n \psi \rangle, \tag{2.8}$$

where $S_1(f_n) \rightarrow S_1(f)$ by monotone convergence and $S_2(f_n) \rightarrow S_2(f)$ since $J(a, 0)$ is a bounded operator.

We care about (2.3) because with $\|f\|_\psi = \|f\psi\|_{\ell^2(V(\mathcal{G}))}$ and \mathcal{F} being the sequences of finite support, we have that

$$G(a, b) = \inf_{f \in \mathcal{F}} \left[(\|f\|_\psi)^{-2} \sum_{\alpha=(j,k)} a_\alpha \psi_j \psi_k (f_j - f_k)^2 \right]. \tag{2.9}$$

This variational principle immediately implies our basic comparison theorem:

Theorem 2.2. Fix a finite graph, \mathcal{G} , and two sets of Jacobi parameters, $\{a_\alpha, b_j\}_{\alpha \in E(\mathcal{G}), j \in V(\mathcal{G})}$ and $\{\tilde{a}_\alpha, \tilde{b}_j\}_{\alpha \in E(\mathcal{G}), j \in V(\mathcal{G})}$. Let ψ and $\tilde{\psi}$ be the positive solutions of $J\psi = \sigma\psi$ and $\tilde{J}\tilde{\psi} = \tilde{\sigma}\tilde{\psi}$, respectively, and set

$$S = \sup_{\alpha=(i,j) \in E(\mathcal{G})} \left[\frac{a_\alpha \psi_i \psi_j}{\tilde{a}_\alpha \tilde{\psi}_i \tilde{\psi}_j} \right], \quad \tilde{S} = \sup_{\alpha=(i,j) \in E(\mathcal{G})} \left[\frac{\tilde{a}_\alpha \tilde{\psi}_i \tilde{\psi}_j}{a_\alpha \psi_i \psi_j} \right], \tag{2.10}$$

$$I = \inf_{i \in V(\mathcal{G})} \frac{\psi_i^2}{\tilde{\psi}_i^2}, \quad \tilde{I} = \inf_{i \in V(\mathcal{G})} \frac{\tilde{\psi}_i^2}{\psi_i^2}. \tag{2.11}$$

Then,

$$\frac{\tilde{I}}{\tilde{S}} G(\tilde{a}, \tilde{b}) \leq G(a, b) \leq \frac{S}{I} G(\tilde{a}, \tilde{b}). \tag{2.12}$$

Remark. It is easy to see that one need not take normalized ψ or $\tilde{\psi}$ in these definitions since normalization constants drop out of the ratios of S/I .

We can also define a lower gap by letting $\sigma_-(a, b)$ to be the lowest eigenvalue of J , $\Sigma_-(a, b)$ to be inf of the spectrum of H , and

$$G_-(a, b) \equiv \Sigma_-(a, b) - \sigma_-(a, b). \tag{2.13}$$

A graph \mathcal{G} is called *bipartite* if $V(\mathcal{G})$ can be written as a disjoint union of V_1 and V_2 so that every edge $\alpha \in E(\mathcal{G})$ has one end in V_1 and one end in V_2 . In that case, if U is the unitary operator that multiplies components of vector, u_i , with $i \in V_1$ (respectively, $i \in V_2$), by 1 (respectively, -1), then

$$U(-J(a, b))U^{-1} = J(a, -b), \tag{2.14}$$

which implies that

$$\sigma_-(a, b) = -\sigma(a, -b), \quad \Sigma_-(a, b) = -\Sigma(a, -b), \quad G_-(a, b) = G(a, -b). \tag{2.15}$$

Moreover, for bipartite graphs, (2.14) implies that there is a vector $\psi^{(-)} \in \ell^2(V(\mathcal{G}))$ with $U\psi^{(-)}$ being positive so that $J(a, b)\psi^{(-)} = \sigma_-(a, b)\psi^{(-)}$. Thus, if we define $S^{(-)}$ and $I^{(-)}$ with ψ replaced by $\psi^{(-)}$, we find the following.

Theorem 2.3. Fix a finite bipartite graph, \mathcal{G} , and two sets of Jacobi parameters, $\{a_\alpha, b_j\}_{\alpha \in E(\mathcal{G}), j \in V(\mathcal{G})}$ and $\{\tilde{a}_\alpha, \tilde{b}_j\}_{\alpha \in E(\mathcal{G}), j \in V(\mathcal{G})}$. Define $S^{(-)}$, $I^{(-)}$, $\tilde{S}^{(-)}$, and $\tilde{I}^{(-)}$ as mentioned above. Then,

$$\frac{\tilde{I}^{(-)}}{\tilde{S}^{(-)}} G_-(\tilde{a}, \tilde{b}) \leq G_-(a, b) \leq \frac{S^{(-)}}{I^{(-)}} G_-(\tilde{a}, \tilde{b}). \tag{2.16}$$

These are especially interesting if we have one comparison operator where we can compute everything. If \mathcal{G} is a finite degree d graph so that \mathcal{T} is a degree d homogeneous tree, we can take $\tilde{a}_\alpha = 1$ for all $\alpha \in E(\mathcal{G})$ and $\tilde{b}_j = 0$ for all $j \in V(\mathcal{G})$. Then, $\tilde{\psi}_j = 1$ for all j is an unnormalized positive eigenvector with $\tilde{\sigma} = d$. By Ref. 1, Example 7.1, we have that $\tilde{\Sigma} = \sqrt{4(d-1)}$, so $\tilde{G} = d - \sqrt{4(d-1)}$. Thus, we have what we regard as the main result of this section:

Corollary 2.4. Fix a finite graph, \mathcal{G} , of constant degree d and a set, $\{a_\alpha, b_j\}_{\alpha \in E(\mathcal{G}), j \in V(\mathcal{G})}$, of Jacobi parameters. Let

$$S = \sup_{\alpha=(ij) \in E(\mathcal{G})} [a_\alpha \psi_i \psi_j], \quad \tilde{S} = \sup_{\alpha=(ij) \in E(\mathcal{G})} \left[\frac{1}{a_\alpha \psi_i \psi_j} \right], \quad (2.17)$$

$$I = \inf_{i \in V(\mathcal{G})} \psi_i^2, \quad \tilde{I} = \inf_{i \in V(\mathcal{G})} \frac{1}{\psi_i^2}. \quad (2.18)$$

Then,

$$\frac{\tilde{I}}{\tilde{S}} \left(d - \sqrt{4(d-1)} \right) \leq G(a, b) \leq \frac{S}{I} \left(d - \sqrt{4(d-1)} \right). \quad (2.19)$$

Not all finite graphs of homogeneous degree d are bipartite (the complete graph with $d + 1$ vertices, for instance, is non-bipartite as it contains a triangle), but the universal cover, \mathcal{T}_d , which is the homogeneous tree of degree d , is always bipartite (since every tree is bipartite). If \mathcal{G} is a bipartite degree d finite graph, it is easy to see that $\tilde{\sigma} = d$ (the constant function is an eigenvector), so $\tilde{G}_+ = d - \sqrt{4(d-1)}$. Since \mathcal{G} is bipartite, $\tilde{G}_- = d - \sqrt{4(d-1)}$, and so there is a result similar to (2.19) for $G_-(a, b)$.

Another explicit example is the rg -model of Ref. 1, Example 7.2, where $r > g$ are two positive integers. The underlying graph, \mathcal{G} , has r red vertices and g green vertices, so $p = r + g$. There are rg edges, one between each pair of differently colored vertices. The natural comparison Jacobi parameters have all $\tilde{a} = 1$ and all $\tilde{b} = 0$, which is the model studied in Ref. 1, Example 7.2, and in Sec. IV. If you look at a vector where all red vertices have value u and all green value v and ask it be an eigenvector of the Jacobi matrix \tilde{J} with eigenvalue λ , then the equations are

$$ru = \lambda v, \quad gv = \lambda u \quad (2.20)$$

whose solutions are easily seen to be (unique up to normalization)

$$\lambda = \pm \sqrt{rg}, \quad v = \sqrt{r}, \quad u = \pm \sqrt{g}. \quad (2.21)$$

It is easily seen that \tilde{J} has rank 2, so the orthogonal complement of these eigenvectors is $\ker(\tilde{J})$, and thus,

$$\tilde{\sigma} = \sqrt{rg}, \quad \tilde{\sigma}_- = -\sqrt{rg}. \quad (2.22)$$

By Ref. 1, Example 7.16, we have

$$\tilde{\Sigma} = \left(\sqrt{r-1} + \sqrt{g-1} \right)^2, \quad \tilde{\Sigma}_- = -\left(\sqrt{r-1} + \sqrt{g-1} \right)^2, \quad (2.23)$$

which yields explicit formulas for \tilde{G} and \tilde{G}_- and an explicit comparison formula.

We end this section by noting that these comparison results imply non-zero gaps for all the associated periodic tree Jacobi matrices without using the result of Sy and Sunada and with explicit bounds.

III. ANTIBOUND CONJECTURE

As discussed above, there is a positive periodic eigensolution for any periodic Jacobi matrix on a tree, but the corresponding energy, σ , is not in the spectrum if the tree is not \mathbb{Z} . The norm of the eigenfunction on a ball of radius R is exponentially growing, and it is only polynomial growth that implies connection to the spectrum. In non-relativistic quantum scattering theory,^{18,19} such purely exponentially growing solutions are called *anti-bound states* and correspond to second sheet poles at real energies not in the spectrum. We think the same might be true in this case. We start with some explicit examples for trees.

Example 3.1. The free Laplacian on a degree d homogeneous tree has a Green's function (diagonal matrix element of the resolvent) that is well-known and computed, for example, in Ref. 1, Example 7.1. Reference 1, Example 7.3, says that

$$G(z) = \frac{(2-d)z + d\sqrt{z^2 - 4(d-1)}}{2(d^2 - z^2)}. \quad (3.1)$$

Initially, this function is defined on $\mathbb{C} \setminus \text{spec}(H)$, where

$$\text{spec}(H) = [-s_d, s_d], \quad s_d = \sqrt{4(d-1)}, \quad (3.2)$$

is determined as the set where G has a boundary value with a non-zero imaginary part. As noted in Sec. II, $\sigma = d$ is the largest eigenvalue of the underlying finite Jacobi matrix. It appears that $G(z)$ has a pole at $z = \pm d$, but, in fact, since we take the branch of square root that is positive on (s_d, ∞) , that square root at $z = d$ is $(d - 2)$ and the numerator vanishes, so there is no pole at $z = \pm d$, where G is originally defined. However, G has a meromorphic continuation to a two-sheeted Riemann surface, and on the second sheet, the square root takes the value $\mp(d - 2)$ at $z = \pm d$, so G has second sheet poles at $\pm\sigma$.

Example 3.2. The rg -model mentioned at the end of Sec. II is studied in Ref. 1, Example 7.2. Equations (7.13) and (7.14) in Ref. 1 compute the Green's functions G_r, G_g at red and green sites,

$$G_r(z) = \frac{(2 - g)z^2 - g[(r - g) - \sqrt{\Phi(z)}]}{2z(rg - z^2)}, \tag{3.3}$$

$$G_g(z) = \frac{(2 - r)z^2 - r[(g - r) - \sqrt{\Phi(z)}]}{2z(rg - z^2)}, \tag{3.4}$$

$$\Phi(z) = z^4 + 2(2 - (r + g))z^2 + (r - g)^2. \tag{3.5}$$

If $\eta_{\pm} = \sqrt{r - 1} \pm \sqrt{g - 1}$, then $\text{spec}(H) = [-\eta_+, -\eta_-] \cup [\eta_-, \eta_+] \cup \{0\}$ and the Green's functions are initially defined on $\mathbb{C} \setminus \text{spec}(H)$. As we saw above, $\sigma_{\pm} = \pm\sqrt{rg}$. The denominators of (3.3) or (3.4) vanish at σ_{\pm} , but a calculation shows that the numerators also vanish there on the original domain of definition, so the only pole is at $z = 0$ for G_r . Again, there is a meromorphic continuation to a two-sheeted Riemann surface with poles at σ_{\pm} .

Example 3.3. The last example where Ref. 1 computes the Green's functions is the Jacobi matrix over the graph in Fig. 1, whose covering tree is the homogeneous degree 3 tree with all $a = 1$ and alternate values b and $-b$ for the potential. According to Ref. 1, Example 7.25, the Green's functions are given by

$$G_{\pm}(z) = \frac{(b^2 - z^2) + 3\sqrt{\Delta}}{2(z \mp b)(9 - z^2 + b^2)}, \quad \Delta = (z^2 - b^2)^2 - 8(z^2 - b^2). \tag{3.6}$$

The spectrum of H is

$$[-\sqrt{b^2 + 8}, -b] \cup [b, \sqrt{b^2 + 8}]. \tag{3.7}$$

J is the 2×2 matrix $\begin{pmatrix} b & 3 \\ 3 & -b \end{pmatrix}$, which has eigenvalues $\sigma_{\pm} = \pm\sqrt{b^2 + 9}$. Since $\Delta(\sigma_{\pm}) = 9$, we see that the numerator and denominator of (3.6) vanish at σ_{\pm} , so on the original domain of definition, $\mathbb{C} \setminus \text{spec}(H)$, G has no poles, but, once again, there are second sheet poles at σ_{\pm} .

Given all these examples, we state the following.

Conjecture 3.4. For any periodic Jacobi matrix, H , on a tree (that is not \mathbb{Z}), the Green's functions analytically continued from $\mathbb{C} \setminus \text{spec}(H)$ have poles on a higher sheet at $z = \sigma$ and, in the bipartite case, also at σ_- .

Recall²⁰ that certain finite graphs are called Ramanujan because σ or $\pm\sigma$ are singled out from the other eigenvalues. We wonder if there is a class of finite graph Jacobi matrices for which the only higher sheet poles of the Green's functions of its covering tree are at $\pm\sigma$.

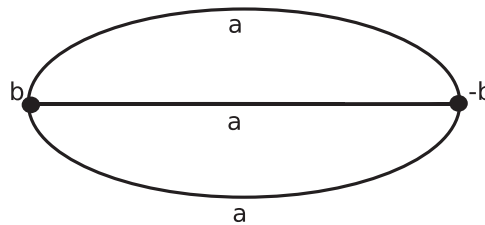


FIG. 1. Example 3.3.

IV. THE rg -MODEL

The rg -model (Ref. 1, Example 7.2) mentioned at the end of Sec. II was originally emphasized by Aomoto⁷ because he showed, by proving one of its Green's functions had a pole, that it has a point eigenvalue at 0, which is outside the continuous spectrum of H . Earlier, Godsil and Mohar²¹ had also noted that the model had a spectral measure with a pure point at zero and computed the weight it contributes to the integrated density of states. The authors of Ref. 1 wrote down an explicit zero energy eigenvector, namely, view the corresponding tree $\mathcal{T}_{r,g}$ (with $r > g$) as a tree centered at a single red vertex at level 0, with g vertices at level 1, then $g(r-1)$ vertices at level 2, each level 1 vertex linked to $r-1$ level 2 vertices, etc. Thus, level $2k-1$; $k = 1, 2, \dots$ has $g[(r-1)(g-1)]^{k-1}$ green vertices, while level $2k$; $k = 1, 2, \dots$ has $g(r-1)[(r-1)(g-1)]^{k-1}$ red vertices. The ABS eigenfunction is radially symmetric (i.e., of constant value on each level) with

$$\begin{aligned} u(2k-1) &= 0, \quad k = 1, 2, \dots, \\ u(2k) &= (-1/(r-1))^k, \quad k = 0, 1, \dots \end{aligned} \tag{4.1}$$

This u has the ℓ^2 -norm,

$$\begin{aligned} \|u\|_2^2 &= 1 + \sum_{k=1}^{\infty} g(r-1)[(r-1)(g-1)]^{k-1} (-1/(r-1))^{2k} \\ &= 1 + \frac{g}{r-1} \sum_{n=1}^{\infty} \left(\frac{g-1}{r-1}\right)^{n-1} = 1 + \frac{g}{r-g} = \frac{r}{r-g}, \end{aligned} \tag{4.2}$$

and it is easy to confirm that $Hu = 0$.

This model has a symmetry group, \mathcal{S} , that includes translations that map any red vertex into any other red vertex. Our goal in this section is to prove the following.

Theorem 4.1. *u and its images under the symmetry \mathcal{S} span the eigenspace, $\ker(H)$, in that the null vector is the only vector in this space, which is orthogonal to u and all its translates.*

Our proof relies on (4.2) and two formulas from Ref. 1, namely, that G_g is regular at 0, and [Ref. 1, Example 7.17] that near $z = 0$,

$$G_r(z) = -\frac{r-g}{rz} + O(1). \tag{4.3}$$

Remark. As noted in Ref. 1, Example 8.5, the Aomoto index theorem implies that the DOS measure of the eigenvalue is $(r-g)/(r+g)$. As also noted in Ref. 1, this follows from the regularity of G_g at 0 and (4.3). One can turn this around and show that (4.3) follows from Aomoto's calculation of the DOS weight of the eigenvalue.

Lemma 4.2. *The residue of G_r at $z = 0$ is $-|u(0)|^2 / \|u\|_2^2$.*

Proof. Immediate from (4.2), (4.3), and $u(0) = 1$. □

Proof of Theorem 4.1. Let $e_1 = u / \|u\|_2$, and pick $\{e_j\}_{j=2}^{\infty}$ an orthonormal basis for the orthogonal complement of e_1 in $\ker(H)$. By the spectral theorem, the residue of the pole of $\langle \delta_\ell, (H-z)^{-1} \delta_\ell \rangle$ at $z = 0$ is just $-\sum_{j=1}^{\infty} |e_j(\ell)|^2$. Since G_g has no pole at $z = 0$, we conclude that every e_j vanishes at every green vertex, which implies that every $\varphi \in \ker(H)$ vanishes at all green vertices. By the above-mentioned lemma, we conclude that for every $j = 2, 3, \dots$, $e_j(0)$ vanishes. It follows that if $\varphi \in \ker(H)$ is orthogonal to u , then $\varphi(0) = 0$.

Now, let $\varphi \in \ker(H)$ be orthogonal to u and all its translates under \mathcal{S} . Since \mathcal{S} acts transitively on the red vertices, the argument in the last paragraph implies that φ vanishes on the red vertices. Since $\varphi \in \ker(H)$, it vanishes at the green vertices. Thus, it is the null vector as claimed.

As we were preparing this paper, which represents work mainly done in June 2019, we received an early draft of a very interesting paper of Banks *et al.*³ that provided a lot of information about the point spectrum of periodic Jacobi matrices on trees. It seemed possible to us that with the methods of Ref. 3, one can extend Theorem 4.1 to a much more general context.

DEDICATION

Jean Bourgain made many deep contributions to the spectral theory of Jacobi matrices, and we dedicate this paper to his memory.

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DATA AVAILABILITY

Data sharing is not applicable to this article as no new data were created or analyzed in this study.

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