JOURNAL OF FUNCTIONAL ANALYSIS 12, 335–339 (1973)

Ergodic Semigroups of Positivity Preserving Self-Adjoint Operators

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Communicated by G. A. Hunt

Received November 9, 1972

We prove that an ergodic semigroup of positivity preserving self-adjoint operators is positivity improving. We also present a new proof (using Markov techniques) of the ergodicity of semigroups generated by spatially cutoff $P(\varphi)_2$ Hamiltonians.

1. INTRODUCTION

Recall the following definitions:

DEFINITION. Let (M, μ) be a σ-finite measure space. A function $f \in L^2(M, d\mu)$ is called *positive* if f is nonnegative a.e. and is not identically 0. f is called *strictly positive* if f is positive and is a.e. nonzero.

DEFINITION. Let (M, μ) be a σ-finite measure space. A bounded operator, A, on $L^2(M, d\mu)$ is called *positivity preserving* if Af is positive whenever f is positive. A is called *ergodic* if it is positivity preserving and for all positive $u, v \in L^2$, there is an n with $\langle u, A^n v\rangle \neq 0$. Finally (following Faris [1]), we say that A is *positivity improving* if Af is strictly positive whenever f is positive. A semigroup, P_t, of bounded operators is called positivity preserving (resp. improving) if P_t is positivity preserving (resp. improving) for each $t > 0$. P_t is called ergodic if for all positive $u, v \in L^2$, there is a t with $\langle u, P_t v\rangle \neq 0$.

Remarks. (1) Since the function 0 is not positive by our definition $\ker A \cap \{f \mid f \text{ is positive}\} = \varnothing$ if A is positivity preserving.

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(2) A is strictly positive if and only if $\langle u, Av \rangle > 0$ for all positive u, v. Thus strict positivity implies ergodicity.

Our main result, which we prove in Section 2 is Theorem 1.

THEOREM 1. *Let* $\exp(-tH)$ *be a semigroup of* self-adjoint *positivity preserving operators. Then the semigroup is ergodic if and only if it is positivity improving.*

Remarks. (1) The self-adjointness of the semigroup is crucial. For let T_t be any ergodic measure preserving flow on a probability measure space. The induced semigroup of unitaries is ergodic, positivity preserving and not positivity improving.

(2) The fact that we have a semigroup indexed by the reals and not the positive integers is important. That is the statement "every positive, ergodic, positivity preserving operator is positivity improving" is false. It is true for 2×2 matrices but already fails for 3×3 matrices, e.g.

$$\begin{pmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{pmatrix}$$

Our interest in the general situation described in Theorem 1 was aroused by developments in the $P(\varphi)_2$ field theory (this theory is reviewed in [2, 3]). For a spatially cutoff field theory it was proven by Glimm and Jaffe [4] that the semigroup generated by the Hamiltonian is ergodic (see also [10, 11]). For the free Hamiltonian in a finite number of degrees of freedom, it was known that the semigroup was positivity improving—it was, thus, natural to try to prove this in general. Of course, Theorem 1 and the Glimm–Jaffe result (see Section 3) imply the following corollary.

COROLLARY. *Let H be the Hamiltonian for a spatially cutoff $P(\varphi)_2$ field theory and view Fock space as L^2 of Q-space. Then* $\exp(-tH)$ *is a positivity improving semigroup.*

Remark. In [4], it is required that the spatial cutoff, g, be smooth. In [11], only $g \in L^1 \cap L^2$ is required. Using the methods in Section 3 below together with [5], the corollary can be proven for $g \in L^{1+\epsilon} + L^2$.

We originally attempted to prove the corollary using the Markov method of Nelson [7], rather then merely abstract consideration. The

Markov theory does, in fact, provide a simple proof of ergodicity which we present in Section 3. This supplements the "Markov proofs" of self-adjointness and semiboundness [9, 6].

2. Proof of Theorem 1

Lemma 1. *Let A be a positivity preserving operator. Suppose u, v are positive vectors with $\langle u, v\rangle \neq 0$. Then $\langle Au, Av\rangle \neq 0$.*

Proof. Let $w = \min(u, v)$. If w were 0 a.e., then either u or v would be 0 a.e. so $uv = 0$ a.e. Since $\langle u, v\rangle \neq 0$, w is not identically 0. But then

$$\begin{aligned}\langle Au, Av\rangle &= \langle Au, Aw\rangle + \langle Au, A(v - w)\rangle \\ &\geqslant \langle Au, Aw\rangle = \langle Aw, Aw\rangle + \langle A(u - w), Aw\rangle \\ &\geqslant \| Aw \|^2 > 0.\end{aligned}$$

Lemma 2. *Let $\exp(-tH)$ be a self-adjoint, positivity preserving ergodic semigroup. Let u, v be positive vectors. Then $\langle u, \exp(-t_nH)v\rangle \neq 0$ for a sequence $t_1, ..., t_n, ...$ converging to 0.*

Proof. Since H is self-adjoint and semibounded, $\exp(-zH)$ is analytic in the right half plane. In particular, $f(z) = \langle u, \exp(-zH)v\rangle$ is analytic in the right half plane. By ergodicity, f is not identically 0. Thus, by analyticity, f has only finitely many zeros in each interval $((n + 1)^{-1}, n^{-1})$.

Proof of Theorem 1. We must show that $\langle u, \exp(-tH)v)\rangle \neq 0$ for all $t > 0$. By Lemma 1, if $\langle u, \exp(-T_0H)v\rangle \neq 0$, then

$$\langle u, \exp(-(T_0 + s)H)v\rangle = \langle \exp(-\tfrac{1}{2} sH) u, \exp(-\tfrac{1}{2} sH) \exp(-T_0H)v\rangle \neq 0.$$

By Lemma 2, we conclude $\langle u, \exp(-tH)v\rangle \neq 0$ for t in

$$\bigcup_{n=1}^{\infty} (t_n, \infty) = (0, \infty).$$

3. Markov Proof of Ergodicity of the $P(\varphi)_2$ Semigroup

We use the Γ, $d\Gamma$ notation of Segal [11]. We first note the following theorem.

Theorem 2. *Let A be an operator on the one particle space with*

$\| A \| < 1$ and with A reality preserving (i. e., commuting with the distinguished complex conjugation). Then $\Gamma(A)$ is ergodic. In particular, if ω is self-adjoint with $\omega > c1 > 0$, then $\exp(-t\, d\Gamma(\omega))$ *is a positivity improving semigroup.*

Proof (following Faris [1]). $\Gamma(A)$ is positivity preserving by a general theorem [8]. Moreover, since $\| A \| < 1$, $\sup_n \| \Gamma(A)^n \| < \infty$. In addition $s - \lim_{n\to\infty} A^n = 0$ so $s - \lim_{n\to\infty} \Gamma(A)^n = \Gamma(0) = (\Omega_0, .)\Omega_0$. Thus, for any positive u, v:

$$\lim_{n\to\infty} \langle u, \Gamma(A)^n v\rangle = (\Omega_0, v)(u, \Omega_0) > 0$$

We conclude that for large n, $\langle u, \Gamma(A)^n v\rangle > 0$. In particular, $\exp(-t\, d\Gamma(\omega)) = \Gamma(\exp(-t\omega))$ is an ergodic semigroup, so by Theorem 1, it is a positivity improving semigroup.

THEOREM 3. *Let H be a spatially cutoff $P(\varphi)_2$ Hamiltonian. Then, viewed as operators on Q-space,* $\exp(-tH)$ *is a positivity improving semigroup.*

Remarks. (1) Our proof uses the Markov techniques of Nelson [7]. Our notation follows [6].

(2) Theorem 1 enters our proof only through its use in Theorem 2.

Proof. Let J_t be the imbedding of Fock space into the Markov field space. Let u, v be positive vectors in Fock space. Then, we claim that for any $t > 0$, $(J_t u)(J_0 v)$ is a positive vector in the Markov field space. It is clearly nonnegative valued and it is not identically zero since

$$\int (J_t u)(J_0 v)\, d\mu_0 = \langle u, e^{-tH_0} v\rangle > 0$$

by Theorem 2. Consider the function

$$f = \int_0^t dt \int g(x){:}\, P(\varphi(x)){:}\, dx.$$

Since $f \in L^2$, f is finite a.e. Thus, $\exp(-f)$ is strictly positive. We conclude that $\int \exp(-f)(J_t u)(J_0 v)\, d\mu_0 > 0$. By the Feynman–Kac–Nelson formula, $\langle u, \exp(-tH)v\rangle > 0$.

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Printed in Belgium