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Correlation Inequalities and the Mass Gap in $P(\varphi)_2$

I. Domination by the Two Point Function

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Abstract. For the $P(\varphi)_2$ field theory, we prove that the falloff of the (vacuum subtracted) two point Schwinger function dominates the higher order (vacuum subtracted) Schwinger functions. As applications, we prove that for even polynomials, the first excited state is odd, and that when there is a one particle state in the infinite volume limit, it is coupled to the vacuum by a single power of the field. The main inputs are the theory of Markov fields and the F.K.G. inequalities.

1. Introduction

In this note, we discuss certain properties of the $P(\varphi)_2$ theory of Glimm, Jaffe, and Rosen (see [2, 3, 20] for references and background). We will employ the statistical mechanical techniques [22, 4] made available by Nelson's Markov field theory [12, 13, 15]. In fact, our line of approach is suggested by some recent work of Lebowitz [9] on the properties of ferromagnetic Ising models. Lebowitz proved that the rate of decrease of the two point spin correlations, $\langle\sigma_i\sigma_j\rangle - \langle\sigma_i\rangle\langle\sigma_j\rangle$, as $|i-j|\to\infty$ is no faster than that of any other spin-spin correlation $\langle\sigma_{i_1}\dots\sigma_{i_m}\sigma_{j_1}\dots\sigma_{j_k}\rangle - \langle\sigma_{i_1}\dots\sigma_{i_m}\rangle\langle\sigma_{j_1}\dots\sigma_{j_k}\rangle$ as $\min|i_p - j_q|\to\infty$. In the language of transfer matrices [16, 10, 11], this says that a single spin must have a non-vanishing matrix element between the two lowest eigenvectors. Our goal here will be to prove analogous results for the $P(\varphi)_2$ field theory. As explained in [5], the (spatially cutoff or infinite volume) Hamiltonian plays the role of a transfer matrix in Markov field theory. In fact, this analogy is the basis of (and is implicit in) the proof of Nelson's reconstruction theorem [12].

Our main theorems appear in Sections 5–7. They are essentially corollaries of a technical result in Section 4 which is very similar to Lebowitz'main technical estimate (Lemma 1 of [9]). Lebowitz relied on the correlation inequalities proven for ferromagnetics by Fortuin,

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Kasteleyn, and Ginibre [1]; we will rely on the analogous inequalities proven by Guerra, Rosen, and Simon [5] using the abstract F.K.G. inequalities [1, 8]:

Theorem 1. ([5]). *Let $\langle\ \rangle$ denote the expectation value with respect to some $P(\varphi)_2$ spatially cutoff Gibbs states or some limit of such states. Let $f_1, \ldots, f_n$ be non-negative space-imaginary time test functions. Let $F(x_1, \ldots, x_n)$ and $G(x_1, \ldots, x_n)$ be two real-valued functions on $\mathbb{R}^n$ which are monotone non-decreasing as any coordinate is increased. Then:*

$$\begin{aligned}&\langle F(\varphi(f_1), \ldots, \varphi(f_n))\, G(\varphi(f_1), \ldots, \varphi(f_n))\rangle \\ &\geqq \langle F(\varphi(f_1), \ldots, \varphi(f_n))\rangle \langle G(\varphi(f_1), \ldots, \varphi(f_n))\rangle .\end{aligned}$$

Sections 2 and 3 contain technical preliminaries. We note that we expect the objects introduced in Section 3 to have a variety of other uses in constructive field theory.

We use freely the notation and results of Ref. [5].

2. Truncated Correlations

In this section, as a preliminary, we discuss some elementary properties of expectations, $\langle fg\rangle - \langle f\rangle\langle g\rangle$, and their quantum analogue.

Definition. Given a probability measure space $(M, d\mu)$ and $f \in L^1(M, d\mu)$, we write $\langle f\rangle = \int f\, d\mu$. If f, g and fg are in $L^1(M, d\mu)$ we write

$$\langle fg\rangle_T \equiv \langle fg\rangle - \langle f\rangle\langle g\rangle .$$

If $\langle fg\rangle_T \geqq 0$, we say that *the pair (f, g) is positively correlated.*

Lemma 1 (Implicit in Lebowitz [9]). *Let f_1, f_2, g_1, g_2 be functions on a probability measure space and let $h_i = f_i - g_i$ $(i = 1, 2)$. Suppose that the pairs (h_1, g_2) and (f_1, h_2) are positively correlated. Then:*

$$\langle g_1 g_2\rangle_T \leqq \langle f_1 f_2\rangle_T .$$

Proof. Adding the inequalities

$$\langle (f_1 - g_1) g_2\rangle \geqq (\langle f_1\rangle - \langle g_1\rangle)\langle g_2\rangle$$

$$\langle f_1 (f_2 - g_2)\rangle \geqq \langle f_1\rangle (\langle f_2\rangle - \langle g_2\rangle)$$

we obtain

$$\langle f_1 f_2\rangle - \langle g_1 g_2\rangle \geqq \langle f_1\rangle\langle f_2\rangle - \langle g_1\rangle\langle g_2\rangle . \quad \Box$$

Next consider the general situation of a self adjoint operator, H, on a separable Hilbert space, $\mathscr{H}$. Suppose that H is positive and that Ω is a normalized eigenvector for H with $H\Omega = 0$. Let $E \equiv \inf(\sigma(H \upharpoonright \{\Omega\}^{\perp}))$.

Given $\Psi \in \mathscr{H}$, we define

$$\Psi_\perp = \Psi - (\Omega, \Psi)\Omega$$

and we use $d\mu_\Psi$ to denote the spectral measure for H associated with Ψ [17; p. 225].

Definition. $(\Psi, e^{-tH}\varphi)_\perp \equiv (\Psi_\perp, e^{-tH}\varphi_\perp)$

$$= (\Psi, e^{-tH}\varphi) - (\Psi, \Omega)(\Omega, \varphi).$$

Definition. $M(\varphi) \equiv \inf(\mathit{Supp}\, d\mu_{\varphi_\perp})$.

Definition. Let S be a subset of $\mathscr{H}$. If $\operatorname{Inf}\{M(\Psi) \mid \Psi \in S\} = E$, we say that S *is coupled to the first excited state.*

Lemma 2. (a) $M(\Psi) = -\lim_{t \to \infty} t^{-1} \ln(\Psi, e^{-tH}\Psi)_T$.

(b) $\ln(\Psi, e^{-tH}\varphi)_T \leqq \ln\|\Psi\| + \ln\|\varphi\| - t\max(M(\Psi), M(\varphi))$.

Proof. A direct consequence of the spectral theorem. □

Lemma 3. *Let T be a total set of vectors in $\mathscr{H}$ and let S be a set of vectors in $\mathscr{H}$. Suppose that for each $\varphi \in T$, there exist $\Psi_1, \ldots, \Psi_n \in S$ and constants $(C_{ij})_{1 \leqq i \leqq j \leqq n}$ so that*

$$(\varphi, e^{-tH}\varphi)_T \leqq \sum_{1 \leqq i \leqq j \leqq n} C_{ij}(\Psi_i, e^{-tH}\Psi_j)_T \tag{1}$$

for all t. Then S is coupled to the first excited state.

Proof. By Lemma 2, $-M(\varphi) \leqq \sup_{i=1,\ldots,n}(-M(\Psi_i))$ whenever (1) holds. Since $E = \inf_{\varphi \in T} M(\varphi)$, the lemma follows. □

Remark. If (1) holds and if E is an eigenvalue, it is easy to prove that some $\Psi_i \in S$ is not orthogonal to some eigenvector with eigenvalue E.

3. Field Theoretic Spins and Occupation Numbers

The applications of the F.K.G. inequalities to spin systems depends heavily on the fact that the spins are bounded so we introduce a cutoff field as a spin-analogue.

Definition. Let $Y(x)$ be the function from $\mathbb{R}$ to $\mathbb{R}$ given by

$$\begin{aligned} Y(x) &= +1 \quad \text{if} \quad x \geqq 1 \\ &= x \quad \text{if} \quad |x| \leqq 1 \\ &= -1 \quad \text{if} \quad x \leqq -1 . \end{aligned}$$

Definition. Given f a positive test function on space-imaginary time, let

$$\sigma(f) = Y(\varphi(f))$$

$$\varrho(f) = \tfrac{1}{2}(1 + \sigma(f)).$$

The *non-linear* functionals ϱ and σ play the role of the occupation number in a lattice gas and the spin in a magnetic spin system. Introduce the shorthand:

Definition. A random variable on the Markov field space which is of the form $F(\varphi(f_1), \ldots, \varphi(f_n))$ with $f_1, \ldots, f_n$ positive test functions and F a monotone increasing function on $\mathbb{R}^n$ in the sense of Theorem 1 is called *a monotone function of the fields.*

Theorem 2. (i) *Let f be a positive test function. Then $\varphi(f)$, $\varrho(f)$, $\sigma(f)$, and $\varphi(f)-\sigma(f)$ are monotone functions of the fields.*

(ii) *Let $f_1, \ldots, f_n$ be positive test function and let*

$$\varrho(f_1, \ldots, f_n) = \prod_{i=1}^{n} \varrho(f_i)$$

$$\Sigma(f_1, \ldots, f_n) = \sum_{i=1}^{n} \varrho(f_i).$$

Then $\varrho(f_1, \ldots, f_n)$, $\Sigma(f_1, \ldots, f_n)$ and $\Sigma(f_1, \ldots, f_n) - \varrho(f_1, \ldots, f_n)$ are monotone functions of the fields.

Proof. (i) $Y(x)$ and $x - Y(x)$ are monotone functions of x.

(ii) Since each $\varrho(f_i)$ is monotone, $\Sigma(f_1, \ldots, f_n)$ is monotone. Since each $\varrho(f_i)$ is also positive, $\varrho(f_1, \ldots, f_n)$ is also monotone. Finally, for each i,

$$\Sigma(f_1, \ldots, f_n) - \varrho(f_1, \ldots, f_n) = \sum_{j \neq i} \varrho(f_j) + \left(1 - \prod_{j \neq i} \varrho(f_j)\right) \varrho(f_i)$$

is a monotone function of $\varrho(f_i)$ because $\varrho(f) \leqq 1$. □

In the applications of Sections 5–7, we will need certain cyclicity properties of the ϱ's. We prove slightly different results in the spatially cutoff and infinite volume situations:

Lemma 4. *Let Ω be the vacuum for a spatially cutoff $P(\varphi)_2$ Hamiltonian. Then $\{\varrho(f_1, \ldots, f_n)\Omega\}$ is a total set in (the relativistic free field) Fock space if $f_1, \ldots, f_n$ runs through all positive time-zero test functions and $n = 1, 2, \ldots$.*

Remark. This lemma and the results in Sections 5 and 6 hold also if the space cutoff is replaced with either a Dirichlet or periodic box.

Proof. Let $(Q, d\mu)$ denote the usual free field Q-space. Since

(i) $\lim_{x \to 0} \lambda^{-1} Y(\lambda x) = x$ for each x,

(ii) $|\lambda^{-1} Y(\lambda x)| \leqq x$ for all λ,

(iii) $\varphi(f) \in L^p(Q, d\mu)$,

for each $p < \infty$ and for each time-zero test function f, we conclude that

$$\lambda^{-1} \sigma(\lambda f) \to \varphi(f)$$

in each $L^p(Q, d\mu)$ ($p<\infty$) as $\lambda \to 0$. Thus since $\Omega \in \bigcap_{p<\infty} L^p(Q, d\mu)$ [21], for each $f_1, \ldots, f_n$ positive time zero test functions, $\varphi(f_1) \ldots \varphi(f_n)\Omega$ is an L^2-limit of a monomial in the $\sigma(\lambda f_i)$'s and so of a polynomial in the $\varrho(\lambda f_i)$'s. Since Ω is cyclic for $\{\varphi(f_i) | f_i \text{ arbitrary}\}$ [21] and any $\varphi(f)$ is a difference of $\varphi(g)$'s with g positive, $\{\varrho(f_1, \ldots, f_n)\Omega\}$ is total. □

Lemma 5. *Let (φ, M, μ) be a Euclidean Markov field over $\mathscr{H}_{-1}(\mathbb{R}^2)$ obeying all the axioms of Nelson [12] including the $\pm\varphi(f) \leqq a(H+1)$ bound. Let P be the projection from $L^2(M, d\mu)$ onto the Wightman field space, $\mathscr{H}$ (i.e. $\mathscr{H}$ is the set of elements of $L^2(M, d\mu)$ measurable with respect to the σ-field generated by $\{\varphi(f) | \operatorname{supp} f \subset \mathbb{R} \times \{0\}\}$). Then $\{P[\varrho(f_1, \ldots, f_n)]\}$ is total in $\mathscr{H}$ if $f_1, \ldots, f_n$ run through all positive space imaginary time test functions with support in $\mathbb{R} \times (-\infty, 0)$ and $n = 1, 2, \ldots$.*

Remarks. 1) Since $\varrho(f_1, \ldots, f_n) \in L^\infty$, $P\varrho \in L^2$.

2) This lemma holds in higher dimensional Markov field theories and for theories over $\mathscr{D}$ provided $a \pm \varphi(f) \leqq a(H+b)^n$ bound holds. The results in Section 7 thus extend to any such theory for which F.K.G. inequalities hold.

Proof. Suppose that $\Psi \in \mathscr{H}$ is orthogonal to $\{P(\varrho(f_1, \ldots, f_n))\}$. Then, in particular, for any $t_1, \ldots, t_n > 0$ and any positive time zero test functions $g_1, \ldots, g_n \in \mathscr{S}(\mathbb{R}^1)$:

$$(\Psi, e^{-t_1 H} \varrho(g_1) \ldots e^{-t_n H} \varrho(g_n)\Omega) = 0 \tag{2}$$

where Ω is the vacuum $P_1 \in \mathscr{H}$. By methods of Nelson [14], for any $s_1, s_2 > 0$ $e^{-s_1 H}\varphi(g_i)e^{-s_2 H}$ is a bounded operator on $\mathscr{H}$ and

$$s\text{-}\lim_{\lambda \to 0} \lambda^{-1} e^{-s_1 H} \varrho(\lambda g_i) e^{-s_2 H} = e^{-s_1 H} \varphi(g_i) e^{-s_2 H}.$$

Thus, (2) implies

$$(\Psi, e^{-t_1 H} \varphi(g_1) \ldots e^{-t_n H} \varphi(g_n)\Omega) = 0. \tag{2'}$$

By analytic continuation, this extends to complex $t_1, \ldots, t_n$ so long as $\operatorname{Re} t > 0$ and thus to the distributional boundary values. We conclude that the distributions

$$(\Psi, \varphi_w(x_1, t_1) \ldots \varphi_w(x_n, t_n)\Omega) = 0$$

where φ_w is the Wightman field. By cyclicity of the vacuum, $\Psi = 0$. □

We feel that the ϱ's and σ's may be useful cutoffs in contexts different from the one we study here. For example, in dealing with broken symmetries, the following might be useful:

Proposition. *Let* $\langle\ \rangle_\alpha$ $0<\alpha\leqq 1$ *be a sequence of Markov field theory expectations. Suppose that for each* $f\in\mathcal{S}(\mathbb{R}^2)$:

(i) $$\sup_{0<\alpha\leqq 1}\langle(\varphi(f))^2\rangle_\alpha<\infty\,.$$

(ii) $$\lim_{\alpha\downarrow 0}\langle\sigma(f)\rangle_\alpha=0\,.$$

Then, for each $f\in S(\mathbb{R}^2)$:

$$\lim_{\alpha\downarrow 0}\langle\varphi(f)\rangle_\alpha=0\,.$$

Proof. Since

$$|\langle\lambda^{-1}\sigma(\lambda f)\rangle_\alpha-\langle\varphi(f)\rangle_\alpha|\leqq\int_{|\varphi(f)|\geqq\lambda^{-1}}|\varphi(f)|\,d\mu_\alpha\leqq\lambda\langle\varphi(f)^2\rangle_\alpha$$

we conclude that $\lim_{\lambda\to 0}\langle\lambda^{-1}\sigma(\lambda f)-\varphi(f)\rangle_\alpha=0$ uniformly in α. The proposition follows from an $\varepsilon/3$ argument.

4. The Basic Estimate

We are now prepared for the proof of the field theoretic analogue of Lemma 1 of [9].

Theorem 3. *Let A and B be finite index sets. For each $i\in A\cup B$, let f_i be a positive test function in space imaginary time. Let $\varrho_A=\prod_{i\in A}\varrho(f_i)$, $\varrho_B=\prod_{j\in B}\varrho(f_j)$ and let $\langle\ \rangle$ denote expectation in some $P(\varphi)_2$ Gibbs state (of the type described in Theorem 1). Then:*

$$0\leqq\langle\varrho_A\varrho_B\rangle_T\leqq\tfrac{1}{4}\sum_{i\in A,\,j\in B}\langle\varphi(f_i)\,\varphi(f_j)\rangle_T\,. \tag{3}$$

Proof. Let $\Sigma_A=\sum_{i\in A}\varrho(f_i)$ and $\Sigma_B=\sum_{j\in B}\varrho(f_j)$. By Theorems 1 and 2, any pair of Σ_A, Σ_B, ϱ_A, ϱ_B, $\Sigma_A-\varrho_A$ and $\Sigma_B-\varrho_B$ are positively correlated. Thus, by Lemma 1,

$$0\leqq\langle\varrho_A\varrho_B\rangle_T\leqq\langle\Sigma_A\Sigma_B\rangle_T\,. \tag{4a}$$

Again using Theorems 1 and 2 and Lemma 1

$$\langle\sigma(f_i)\,\sigma(f_j)\rangle_T\leqq\langle\varphi(f_i)\,\varphi(f_j)\rangle_T\,. \tag{4b}$$

But

$$\langle\Sigma_A\Sigma_B\rangle_T=\sum_{i\in A,\,j\in B}\langle\varrho(f_i)\,\varrho(f_j)\rangle_T \tag{5}$$

and

$$\langle\varrho(f_i)\,\varrho(f_j)\rangle_T=\tfrac{1}{4}\langle\sigma(f_i)\,\sigma(f_j)\rangle_T \tag{6}$$

(4)–(6) imply (3). □

5. Spatially Cutoff Theories: General Results

In this section and the next, H will denote a spatially cutoff $P(\varphi)_2$ Hamiltonian, renormalized so that $H \geqq 0$ and $H\Omega = 0$ for a unique vector Ω. Thus $H = H_0 + \int g(x) :P(\varphi(x)): dx - E_g$ where $g \in L^{1+\varepsilon} + L^2$ [6]. In this section P will be an arbitrary semibounded, normalized (i.e. $P(0) = 0$) polynomial and in the next it will be even.

Theorem 4. *Let H be a spatially cutoff Hamiltonian for a $P(\varphi)_2$ theory with $P(X) = \sum_{m=1}^{2n} a_m X^m$; $a_{2n} > 0$. Then:*

(i) *$\{\varphi(f)\Omega \mid f \geqq 0; f \in \mathscr{S}(\mathbb{R})\}$ is coupled to the first excited state.*

(ii) *If $E \equiv \inf \sigma(H \upharpoonright \Omega^\perp)$ is an eigenvalue with corresponding spectral projection P and if g_1, g_2 are strictly positive functions in $\mathscr{S}(\mathbb{R})$, then*

$$(\Omega, \varphi(g_1)\, P\, \varphi(g_2)\Omega) > 0 .$$

Remark. Suppose for simplicity that E is simple. Then our methods below show that $(\Psi, \varphi(x)\Omega)$ is a measure where Ψ is a suitably normalized eigenvector. It is a consequence of semiboundedness of $H \pm \varphi(f)$ that this measure has the form $F(x)\,dx$ with $F \in \bigcap_{2 \leqq p < \infty} L^p(\mathbb{R}, dx)$. We expect that F is continuous and everywhere strictly positive. Using Rosen's momentum estimates [18] it may be possible to show F is continuous (or even smooth) but we do not know how to prove it strictly positive.

Proof. (i) We know that the spatially cutoff $P(\varphi)_2$ theory is associated with a Markov theory which is a limit of space-imaginary time cutoff Gibbs states [15, 5]. Let $\langle\ \rangle$ denote expectation value for this Gibb's state. Given time 0 test functions $f_1, \dots, f_n$, let $f_1^t, \dots, f_n^t$ denote the same functions translated to imaginary time $t > 0$. By the Feynman-Kac-Nelson formula [5]:

$$\langle \varrho(f_1^t, \dots, f_n^t)\, \varrho(f_1, \dots, f_n)\rangle = (\varrho(f_1, \dots, f_n)\Omega, e^{-tH} \varrho(f_1, \dots, f_n)\Omega)$$

and similarly

$$\langle \varphi(f_i^t)\, \varphi(f_j)\rangle = (\varphi(f_i)\Omega, e^{-tH} \varphi(f_j)\Omega) .$$

It follows from Theorem 3 that

$$(\varrho(f_1, \dots, f_n)\Omega, e^{-tH} \varrho(f_1, \dots, f_n)\Omega)_T \leqq \tfrac{1}{4} \sum_{i,j=1}^{n} (\varphi(f_i)\Omega, e^{-tH} \varphi(f_j)\Omega)_T$$

(i) now follows from Lemmas 3 and 4.

(ii) By (i), we conclude that $(\Omega, \varphi(f)\, P\varphi(f)\Omega) \neq 0$ for some $f \in \mathscr{S}$. By a limiting argument we can suppose that $f \in \mathscr{D}$.

Since g_1, g_2 are strictly positive we can replace f by a multiple so that (i) $g_i \geqq f \geqq 0$ (ii) $(\varphi(f)\Omega, P\varphi(f)\Omega) \neq 0$. But for any $h_1, h_2 \geqq 0$

$$(\varphi(h_1)\Omega, P\varphi(h_2)\Omega) = \lim_{t\to\infty} e^{tE}(\varphi(h_1)\Omega, e^{-tH}\varphi(h_2)\Omega)_T = \lim_{t\to\infty} e^{tE}\langle\varphi(h_1^t)\,\varphi(h_2)\rangle_T$$

is positive by the $P(\varphi)_2$ Griffiths' inequalities [4, 5]. Thus

$$(\varphi(g_1)\Omega, P\varphi(g_2)\Omega) \geqq (\varphi(f)\Omega, P\varphi(f)\Omega) > 0. \quad \square$$

6. Spatially Cutoff Theories: Even Interactions

As an immediate corollary of Theorem 4, we have:

Theorem 5. *Let H be the Hamiltonian of a spatially cutoff $P(\varphi)_2$ theory with P even, i.e. $P(X) = \sum_{n=1}^{m} a_n X^{2n}$ with $a_m > 0$. Let $\mathscr{H}_{\text{odd}} = \bigoplus_{n=0}^{\infty} \mathscr{F}_{2n+1}$ $\equiv \operatorname{Ran} \frac{1}{2}(I-(-I)^N)$. Then $\inf\sigma(H \upharpoonright \mathscr{H}_{\text{odd}}) = \inf\sigma(H \upharpoonright \Omega^\perp)$. In particular, if H has an eigenvalue in $(0, m_0)$, the lowest one must possess an eigenvector in $\mathscr{H}_{\text{odd}}$.*

Remarks. 1) This theorem combines with the Griffiths inequalities to imply monotonicity statements about the mass gap under certain changes of the interaction; see [4, 5].

2) Because the first excited state of a one degree of freedom system has a single node, it is a general theorem for one degree of freedom systems that the first excited state is odd if the interaction is even. One might think that such a result might hold in general, but it is not true for a general Schrödinger operator on $\mathbb{R}^n$ with $V(x) = V(-x)$. For example, if $n=3$ and V is the sum of a Coulomb potential and a negative spherical square well sharply peaked near $r=0$, then the first excited state is a $2s$ state and hence even. We expect that there are two degree anharmonic oscillators whose first excited state is odd. What is special about the $P(\varphi)_2$ Hamiltonian is that the interaction is local and the free theory is ferromagnetic.

7. Infinite Volume Theories

Perhaps our most interesting result is:

Theorem 6. *Let (φ, M, μ) be an infinite volume $P(\varphi)_2$ Markov field theory for which the F.K.G. inequalities hold (in particular it can be a limit of Gibbs states with free, Dirichlet or periodic boundary conditions).*

Then:

(1) *A necessary and sufficient condition for the vacuum to be unique is that* $\langle \varphi(f^t)\,\varphi(f)\rangle_T \to 0$ *as* $t\to\infty$ *for every function* $f\in\mathscr{D}(\mathbb{R}^2)$ *where* $f^t(x,s)=f(x,s-t)$.

(2) *A necessary and sufficient condition for there to be a mass gap of size at least* m_0 *is that for each* $f\in\mathscr{D}(\mathbb{R}^2)$ *and* $\varepsilon>0$

$$\exp(m_0-\varepsilon)t\langle \varphi(f^t)\,\varphi(f)\rangle_T \to 0 \quad as \quad t\to\infty\,.$$

Proof. Using the techniques of Sections 5 and 6, this is a direct consequence of Lemma 5 and Theorem 3. □

Finally, the following is of technical use in employing the Haag Ruelle scattering theory [19, 7].

Theorem 7. *Suppose that the hypotheses of Theorem 7 hold and that, in addition, the corresponding Wightman theory has an isolated one particle state of mass* $m=\inf\sigma(H\restriction\Omega^\perp)$. *Then the spectral weight for the two point function gives non-zero measure to the mass m. In particular, if P is the projection on the one particle states of mass m and if f is any space (real) time function whose Fourier transform is not identically zero on the hyperboloid of mass, m then*

$$P\,\varphi_w(f)\Omega \neq 0\,. \tag{8}$$

Proof. If $(\Omega,\varphi_w(f)\,P\,\varphi_w(f)\Omega)$ were zero for all space (real) time functions, then by polarization and analytic continuation this would hold for space imaginary time functions. But, by Theorem 3 and Lemma 5, there is some Euclidean test function g so that $\varphi(g)$ couples Ω to the first excited state. We conclude that the two point function spectral weight gives non-zero weight to m. (8) follows from the Källen-Lehmann representation. □

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