Math. Z. 131, 361–370 (1973)


# Schrödinger Operators with Singular Magnetic Vector Potentials

Barry Simon*

## § 1. Introduction

In this paper we wish to consider domains of essential self-adjointness for Schrödinger operators

$$H = -\sum_{j=1}^{n} (\partial_j - i a_j)^2 + V \tag{1}$$

where $\partial_j = \partial/\partial x_j$ and where $a_j$, $V$ are measurable functions on $\mathbb{R}^n$ with additional conditions which will vary from theorem to theorem. Throughout we employ the notation:

$$D_j = \partial_j - i a_j$$
$$V_+ = \max(V, 0)$$
$$V_- = \max(-V, 0).$$

The interest in operators of the form (1) comes, of course, from the theory of non-relativistic quantum mechanics but we emphasize that for all operators of direct physical interest, we know that $C_0^\infty(\mathbb{R}_n)$ is a core due to the work of Kato [8], Stummel [19] and Ikebe-Kato [6]. We are interested in the mathematical question of determining minimal conditions which assure essential self-adjointness on $C_0^\infty$.

Let us begin by summarizing the "general principles" which have been discovered in the study of operators of the form (1). The first two are classical going back to the work of Weyl and others on the case $n=1$:

(1) Almost all theorems are stated in terms of separate hypotheses on $a$, $V_+$, $V_-$. This is an empirical fact. In case $\vec{a}=0$, this is partly expressed by the recent theorem of Davies and Faris [3, 5] that if $V_-$ is $-\Delta$ bounded with bound less than 1 and $-\Delta + V_+$ is essentially self-adjoint on $D(-\Delta) \cap D(V)$, then $-\Delta + V$ is essentially self-adjoint on $D(-\Delta) \cap D(V)$.

(2) A global condition is needed on $V_-$. Explicitly, a suitable local average of $V_-$ should grow no faster than $|x|^2$ at $\infty$.

* A. Sloan Fellow.

(3) [2, 19] Only local conditions are needed on $V_+$.

(4) [6] Only local conditions are needed on $a$.

(5) [19, 1, 4, 18] Call $p$ *n-canonical* if $p=2$ when $n \leqq 3$, $p>2$ when $n=4$ and $p=\frac{1}{2}n$ if $n \geqq 5$. Local $L^p$ (or, at least, local weak $L^p$ [18]) estimates are needed on $V_-$.

(6) [15, 9] Only local $L^2$ conditions are needed on $V_-$.

In this paper, we wish to explore the question of minimal local regularity conditions on $\vec{a}$, the magnetic vector potential. Since the work of Ikebe and Kato [6], it has been standard to suppose that $\vec{a} \in C^1$. At first sight, it does not appear that one can weaken this much, for terms of the form $(\partial_j a_j)\Phi$ appear when $(\partial_j - i a_j)^2 \Phi$ is expanded. For this to be in $L^2$ when $\Phi \in C_0^\infty$, it seems natural to demand that $a_j$ be in $(H^1)_{\text{loc}}$, the local Sobolev space of functions with locally $L^2$ (distributional) derivatives of first order. However, this view does not take into account the fact that the $a_k$ are not "physical" but only $\vec{b} = \operatorname{curl} \vec{a}$ is "physical", i.e. it ignores the freedom of gauge transformations. To be more explicit, we make two remarks. First, we have the formal relation

$$\exp(i\lambda(x)) \left[ -\sum_{j=1}^{n} D_j^2 + V \right] \exp(-i\lambda(x)) = -\sum_{j=1}^{n} \tilde{D}_j^2 + V$$

where $\tilde{D}_j = \partial_j - i\tilde{a}_j$ with $\tilde{a}_j = a_j + \partial_j \lambda$. Secondly, the only term in $(-\sum D_j^2 + V)\Phi$ which involves derivatives of $a$ is $(\operatorname{div} \vec{a})\Phi$ and $\operatorname{div} \vec{a}$ is not invariant under the gauge transformation $a \to \tilde{a}$. In fact, it is quite common in quantum theory to work in the *Coulomb Gauge* where $\operatorname{div} \vec{a} = 0$. Most of our results (§§ 2, 5) will impose the condition $\operatorname{div} \vec{a} = 0$.

If $\operatorname{div} \vec{a} = 0$, $\vec{a} \in (L^4)_{\text{loc}}$ and $V \in (L^2)_{\text{loc}}$, then we define the operator $H$ on $C_0^\infty$ by:

$$H\Phi = -\Delta\Phi + 2i \sum_{j=1}^{n} a_j \partial_j \Phi + (a^2 + V)\Phi. \tag{2}$$

It is a simple excercise to see that (2) defines a symmetric operator since $\operatorname{div} \vec{a} = 0$.

Our main results concerning the operator (2) when $\operatorname{div} \vec{a} = 0$ appear in § 5. They depend on an idea of Kato [9] and a purely technical argument found in §§ 3, 4. To illustrate the simplicity of working in Coulomb gauge, we first show that when $\vec{a} \in L^4(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$ and $V \in L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$, then $\operatorname{div} \vec{a} = 0$ and a perturbation theory argument proves that (2) is self-adjoint on $D(-\Delta)$. This argument, which appears in § 2 is included solely for motivational purpose and may be skipped by the reader interest in our main results. In § 6, we briefly consider results when $\operatorname{div} \vec{a} \neq 0$.

We conclude this appendix by saying a few words about the definition of Schrödinger operators with singular magnetic potentials by the method of quadratic forms [13, 14]. Suppose $V \geqq 0$, $V \in (L^1)_{\mathrm{loc}}$. Define a quadratic form on $C_0^\infty \times C_0^\infty$ by

$$h(\Psi, \Phi) = \sum_{j=1}^{n} \langle (\partial_j - i a_j)\Psi, (\partial_j - i a_j)\Phi \rangle + \langle v^{\frac{1}{2}} \Psi, v^{\frac{1}{2}} \Phi \rangle$$

$h$ is a positive closable quadratic form (because $\partial_j - i a_j$ and $v^{\frac{1}{2}}$ are closable operators) whose closure is thus the form of a unique self-adjoint operator $H$ [10, 11]. Thus, if we only seek a "natural" meaning for $H$ and don't care about operator cores, $\vec{a} \in (L^2)_{\mathrm{loc}}$ is all that is required. This was first commented by E. Nelson (unpublished) who based his comments on the use of stochastic and path integrals rather than forms.

It is a pleasure to thank E. Nelson for stimulating my interest in the question of singular magnetic potentials, to thank O.E. Lanford III, J. Lions and L. Tartar for discussion of the technical material in § 4, to thank M. O'Carroll for the hospitality of PUC-Rio de Janeiro where this work was begun and to thank N. H. Kuiper for the hospitality of I. H. E. S. where this work was concluded.

## § 2. Coulomb Gauge, I: Theorem with a Global Hypotheses

The Kato-Rellich theory of regular perturbations can be applied to prove a simple result on Schrödinger operators with singular magnetic fields:

**Theorem 1.** *Let $\vec{a}$ be an $\mathbb{R}^3$-valued function on $\mathbb{R}^3$ with $\vec{a} \in L^4 + L^\infty$ and $\operatorname{div} \vec{a} = 0$. Let $V \in L^2 + L^\infty$. Then $H = -\sum_{j=1}^{3} (\partial_j - i a_j)^2 + V$ is essentially self-adjoint on $C_0^\infty$ (and self-adjoint on $D(-\Delta)$).*

*Proof.* Since $H = -\Delta + 2i\vec{a} \cdot \vec{\nabla} + a^2 + V$ and $a^2 \in L^2 + L^\infty$, we need only prove an estimate of the form:

$$\|(\vec{a} \cdot \vec{\nabla})\Phi\| \leqq \alpha \|\Delta \Phi\| + \beta \|\Phi\| \tag{3}$$

for some $\alpha < 1$ and all $\Phi \in C_0^\infty$. On the one hand for any $\varepsilon$:

$$\|\partial_j \Phi\|_2 \leqq \varepsilon \|\Delta \Phi\|_2 + \varepsilon^{-1} \|\Phi\|_2 \tag{4}$$

and on the other hand for any $\varepsilon$, these is a $c$ so that

$$\|\partial_j \Phi\|_4 \leqq \varepsilon \|\Delta \Phi\|_2 + c \|\Phi\|_2 . \tag{5}$$

(4) follows from the Plancherel theorem and the bound

$$\|p_j \Phi\|^2 \leqq \|p_j^2 \Phi\| \, \|\Phi\| \leqq \varepsilon^2 \|p_j \Phi\|^2 + \varepsilon^{-2} \|\Phi\|^2 \leqq \varepsilon^2 \|p^2 \Phi\|^2 + \varepsilon^{-2} \|\Phi\|^2 .$$

(5) comes from the Plancherel theorem, the Hausdorff-Young inequality and $\|p_j\Phi\|_{\frac{4}{3}} \leq \|p_j(1+\alpha p^2)^{-1}\|_4 [\|\Phi\|_2 + \alpha \|p^2 \Phi\|_2]$. (3) follows immediately from (4) and (5). □

*Remarks.* 1. In some ways, it is more natural to place restrictions on $\vec{a}$ and $a^2+V$ rather than $\vec{a}$ and $V$. One sees that it is then only necessary to have $a^2+V \in L^2+L^\infty$ and $\vec{a} \in L^p+L^\infty$ with $p>3$, for the proof to go through.

2. It is quite easy to discuss scattering theory for operators of the type dealt with in Theorem 1 (see also [7]). For example, by using Cook's method, one can prove $s$-$\lim e^{iHt} e^{-iH_0 t}$ exist if $V \in L^2+L^r$ $(2 \leq r < 3)$ and $\vec{a} \in L^4 \cap L^s$ $(s<3)$.

3. The condition $\operatorname{div} \vec{a}=0$ may be replaced with $\operatorname{div} \vec{a} \in L^2+L^\infty$.

## § 3. Kato's Inequality and the Reduction to Condition $K_p$

In Kato's recent discussion of Schrödinger operators with $(L^2)_{\text{loc}}$ potentials [9], a major role is played by:

**Proposition** *(Kato's Inequality). If $u \in (L^1)_{\text{loc}}$ and $-\Delta u \in (L^1)_{\text{loc}}$, then*

$$\Delta |u| \geq \operatorname{Re}((\operatorname{sgn} u)\, \Delta u) \tag{6}$$

*where* (6) *is understood as a distributional inequality and* sgn $u$ *is defined by:*

$$(\operatorname{sgn} u)(x) = \begin{cases} 0 & \text{if } \; u(x)=0 \\ u^*(x)/|u(x)| & \text{if } \; u(x) \neq 0. \end{cases}$$

In our extension of Kato's method, we make heavy use of the fact that in applications $u$ is in $(L^2)_{\text{loc}}$ rather than merely in $(L^1)_{\text{loc}}$.

**Definition.** Let $p \in [4, \infty]$ and let $q=(1-p^{-1})^{-1}$. We say that a function $\vec{a}$ from $\mathbb{R}^n$ to $\mathbb{R}^n$ obeys *condition $K_p$* if and only if

(i) $\vec{a} \in (L^p)_{\text{loc}}$.

(ii) If $u \in (L^2)_{\text{loc}}$ and $-\Delta u + 2i \operatorname{div}(\vec{a} u) \in (L^1)_{\text{loc}}$, then grad $u \in (L^q)_{\text{loc}}$.

**Lemma 1.** *Suppose* $\operatorname{div} \vec{a}=0$. *If $p^{-1}+q^{-1}=1$, if $\vec{a} \in (L^p)_{\text{loc}}$, $u \in (L^q)_{\text{loc}}$ and* grad $u \in (L^q)_{\text{loc}}$, *then*

$$\operatorname{div}(\vec{a} u) = \vec{a} \operatorname{grad} u.$$

*Proof.* Let $j_\delta$ be an approximate identity, i.e. $j_\delta(x)=\delta^{-1} j(x/\delta)$ where $j \in C_0^\infty$, $j \geq 0$ and $\int j(x)\, dx = 1$. Let $a^\delta = a * j_\delta$ so that $\operatorname{div} a^\delta = 0$ and let $u^\varepsilon = u * j_\varepsilon$. Then by Lebnitz' rule,

$$\operatorname{div}(\vec{a}^\delta u^\varepsilon) = \vec{a}^\delta \operatorname{grad} u^\varepsilon. \tag{7}$$

Using the given $L^p$ estimates and Hölders inequality, it is easy to prove that the two sides of (7) converge in $(C_0^\infty)'$ to $\operatorname{div}(\vec{a} u)$ and $\vec{a}$ grad $u$ as $\delta$, $\varepsilon \to 0$. □

**Lemma 2.** *Suppose $\vec{a}$ obeys condition $K_p$, that* $\operatorname{div}\vec{a}=0$ *and that* $-\Delta u+2i\operatorname{div}(\vec{a}u)\in(L^1)_{\text{loc}}$. *Then* $-\Delta u\in(L^1)_{\text{loc}}$.

*Proof.* This follows from Lemma 1. □

**Theorem 2.** *Let $\vec{a}$ obey condition $K_p$ for some $p\in[4,\infty]$ and suppose* $\operatorname{div}\vec{a}=0$. *Let $\tilde{H}$ be the operator from $(L^2)_{\text{loc}}$ to $(C_0^\infty)'$ given by:*

$$\tilde{H}u=-\Delta u+2i\sum_{j=1}^{n}\partial_j(a_j u)+a^2 u.$$

*Then for any $u\in(L^2)_{\text{loc}}$ with $Hu\in(L^1)_{\text{loc}}$ we have*

$$\Delta|u|\geqq-\operatorname{Re}[(\operatorname{sgn}u)(\tilde{H}u)] \tag{8}$$

*as a distributional inequality.*

*Proof.* Following Kato [9], we first prove a modified version of (8) when $u$ is smooth and then obtain (8) by a double limiting proceedure. So let $u$ be $C^\infty$ and define $u_\varepsilon=(|u|^2+\varepsilon^2)^{\frac{1}{2}}$. Since $u_\varepsilon^2=u^*u+\varepsilon^2$:

$$\begin{aligned}2u_\varepsilon\partial_j u_\varepsilon&=u^*\partial_j u+(\partial_j u)^*u\\&=u^*(D_j u)+(D_j u)^*u\\&=2\operatorname{Re}(u^*(D_j u)).\end{aligned} \tag{9}$$

In particular, since $u_\varepsilon\geqq|u|$, we obtain $|\partial_j u_\varepsilon|\leqq|D_j u|$. Letting

$$\|\vec{v}\|=\left(\sum_{i=1}^{n}|v_i|^2\right)^{\frac{1}{2}},$$

we obtain

$$\|\operatorname{grad}u_\varepsilon\|\leqq\|\vec{D}u\|. \tag{10}$$

Moreover, by (8),

$$\begin{aligned}2\partial_j(u_\varepsilon\partial_j u_\varepsilon)&=(\partial_j u)^*(D_j u)+u^*(\partial_j D_j u)+\text{complex conjugate}\\&=(D_j u)^*(D_j u)+u^*(D_j^2 u)\quad+\text{complex conjugate}.\end{aligned}$$

Summing over $j$, and using $\operatorname{div}\vec{a}=0$, we obtain

$$\|\operatorname{grad}u_\varepsilon\|^2+u_\varepsilon\Delta u_\varepsilon=\|Du\|^2-\operatorname{Re}[u^*(\tilde{H}u)]. \tag{11}$$

Define

$$\operatorname{sgn}_\varepsilon u\equiv\frac{u^*}{u_\varepsilon}.$$

Then (10) and (11) imply

$$\Delta u_\varepsilon\geqq-\operatorname{Re}((\operatorname{sgn}_\varepsilon u)(\tilde{H}u)). \tag{19}$$

Now suppose that $u\in(L^2)_{\text{loc}}$ and $\tilde{H}u\in(L^1)_{\text{loc}}$. Let $j_\delta$ be an approximate identity and let $u^\delta\equiv u*j_\delta$. By Lemmas 1 and 2, $\operatorname{div}(\vec{a}u)=\vec{a}\operatorname{grad}u$,

$\Delta u \in (L^1)_{\mathrm{loc}}$ and grad $u \in (L^q)_{\mathrm{loc}}$. Thus $\tilde{H} u^\delta \to \tilde{H} u$ in $(L^1)_{\mathrm{loc}}$. By passing to a subsequence we can also be certain that $u^\delta \to u$ pointwise a.e. so $\mathrm{sgn}_\varepsilon(u^\delta) \to \mathrm{sgn}_\varepsilon(u)$ pointwise a.e. As a result, for any $\varphi \in C_0^\infty$:

$$\begin{aligned}&\int \varphi\,[\mathrm{sgn}_\varepsilon(u^\delta)(\tilde{H} u^\delta) - \mathrm{sgn}_\varepsilon(u)(\tilde{H} u)]\\ &\quad = \int \varphi\,[\mathrm{sgn}_\varepsilon(u^\delta) - \mathrm{sgn}_\varepsilon(u)]\,(\tilde{H} u) + \int \varphi\,[\mathrm{sgn}_\varepsilon(u^\delta)]\,(\tilde{H} u^\delta - \tilde{H} u)\end{aligned}$$

goes to zero (by the dominated convergence theorem). We conclude that $(\mathrm{sgn}_\varepsilon u^\delta)\,\tilde{H} u^\delta$ converges to $(\mathrm{sgn}_\varepsilon u)\,\tilde{H} u$ in $(C_0^\infty)'$. Moreover, since

$$\left|\sqrt{|a|^2+\varepsilon^2} + \sqrt{|b|^2+\varepsilon^2}\right| = \left|\int_{|a|}^{|b|} x(x^2+\varepsilon^2)^{-\frac{1}{2}}\,dx\right| \leqq |b-a|.$$

$u_\varepsilon^\delta \to u_\varepsilon$ in $(L^1)_{\mathrm{loc}}$ and so in $(C_0^\infty)'$. Thus (12) holds for $u$ since it holds for $u^\delta$. Taking $\varepsilon \to 0$, (8) follows. □

**Theorem 3.** *Let $\vec{a}$ obey condition $K_p$ for some $p \in [4, \infty]$ and suppose that* $\operatorname{div} \vec{a} = 0$. *Let $V_+ \in L^2(\mathbb{R}^n)_{\mathrm{loc}}$, $V_- \in L^p(\mathbb{R}^n)$ with $p$ n-canonical. Then the operator $H$ from $C_0^\infty$ to $L^2$ defined by*

$$H\varphi = -\Delta\varphi + 2i\sum_{i=1}^{n} a_i \partial_i \varphi + a^2\varphi + V\varphi$$

*is essentially self-adjoint.*

*Proof.* Since $V_-$ is $\Delta$-bounded with arbitrarily small bound, we can find $E_0$ with $\|V_-(-\Delta+E_0)^{-1}\| < 1$. We will prove that $H + E_0$ has a dense range. If not, we can find $u \in L^2$ with $\langle u, (H+E_0)\varphi\rangle = 0$ for all $\varphi \in C_0^\infty$. Since the operator $\tilde{H}$ of Theorem 2 is defined with distributional derivatives,

$$\tilde{H} u + Vu + E_0 u = 0.$$

Since $V \in (L^2)_{\mathrm{loc}}$, we conclude that $\tilde{H} u = -Vu - E_0 u \in (L^1)_{\mathrm{loc}}$ so Theorem 2 is applicable. We conclude that

$$\Delta|u| \geqq -\mathrm{Re}\big((\mathrm{sgn}\,u)(-Vu - E_0 u)\big) \geqq V|u| + E_0|u| \geqq -V_-|u| + E_0|u|.$$

Thus

$$(-\Delta + E_0)|u| \leqq V_-|u|. \tag{13}$$

But both sides of (13) lie in $\mathcal{S}(\mathbb{R}^3)'$ and $(-\Delta+E_0)^{-1}$ is a positivity preserving map of $\mathcal{S}$ into $\mathcal{S}$ and so of $\mathcal{S}'$ into $\mathcal{S}'$. Thus

$$|u| \leqq (-\Delta + E_0)^{-1} V_-|u|$$

so that

$$\|u\| \leqq \|(-\Delta+E_0)^{-1} V_-\|\,\|u\|.$$

Since $\|(-\Delta+E_0)^{-1} V_-\| = \|[V_-(-\Delta+E_0)^{-1}]^*\| < 1$, $\|u\| = 0$. □

*Remarks.* 1. $V_- \in L^p$ was not essential. All that was needed is that $V_- \in (L^2)_{\text{loc}}$ and that $V_-$ is $-\Delta$-bounded with bound strictly less than 1.

2. By following arguments in [9] or [16], one can prove a whole class of other results when a obeys $K_p$. For example there are theorems when $V = O(-r^{2-\varepsilon})$ at $\infty$ [9] or $O(-cr^{-2})$ at 0 for suitable $c$ [16].

## § 4. Investigation of Condition $K_p$

In order to apply Theorem 3, we need to know when condition $K_p$ holds. If $n<4$, the situation is very simple:

**Lemma 3.** *If $n \leqq 3$ and $\vec{a}$ is a function from $\mathbb{R}^n$ to $\mathbb{R}^n$ with $\vec{a} \in (L^4(\mathbb{R}^n))_{\text{loc}}$, then $\vec{a}$ obeys condition $K_4$.*

*Proof.* We must show that if $u \in (L^2)_{\text{loc}}$ and $f \equiv -\Delta u + 2i \operatorname{div}(\vec{a} u) \in (L^1)_{\text{loc}}$ then grad $u \in (L^{\frac{4}{3}})_{\text{loc}}$. Without loss, we need only prove grad $u$ is in $(L^{\frac{4}{3}})_{\text{loc}}$ near 0. Let $\varphi$ be a function which is one in a neighborhood of 0. Let $g$ be defined by:

$$g = (1-\Delta)^{-1}[(f+u)\varphi] - 2i \sum_{j=1}^{n} [(1-\Delta)^{-1}\partial_j](a_j u \varphi)$$

where $(1-\Delta)^{-1}$ and $[(1-\Delta)^{-1}\partial_j]$ are the usual integral operators on $L^1$. Then $g$ is in $L^1$ and

$$(1-\Delta)g = (f+u)\varphi - 2i \operatorname{div}(\vec{a} u \varphi)$$

so that $(1-\Delta)(g-u)=0$ in a neighborhood of 0. By the elliptic regularity theorem (see, e.g. [12]) $g-u$ is $C^\infty$ near zero so we need only prove that grad $g \in L^{\frac{4}{3}}$ to conclude that grad $u$ is in $(L^{\frac{4}{3}})_{\text{loc}}$ near 0. Now

$$\partial_i g = [\partial_i(1-\Delta)^{-1}](f+u)\varphi - 2i \sum_{j=1}^{n} [\partial_i(1-\Delta)^{-1}\partial_j](a_j u \varphi). \tag{14}$$

Since $n \leqq 3$, $\partial_i(1-\Delta)^{-1}$ is a convolution operator with a function in $L^{\frac{4}{3}}$ so the first term is in $L^{\frac{4}{3}}$ by Young's inequality. Moreover, on each $L^p$ $(p \neq 1, \infty)$ $\partial_i(1-\Delta)^{-1}\partial_j$ is a bounded (singular) integral operator (see, e.g. [17]). Since $a_j u \varphi \in L^{\frac{4}{3}}$ by Hölder's inequality, the second term in (14) is in $L^{\frac{4}{3}}$. Thus grad $g \in L^{\frac{4}{3}}$. $\square$

The following result is all we have been able to prove when $n \geqq 4$; as we will explain, we feel this is far from optimal if $n>4$:

**Lemma 4.** *If $p > n \geqq 4$ and $\vec{a}$ is a function from $\mathbb{R}^n$ to $\mathbb{R}^n$ so that $\vec{a} \in (L^p)_{\text{loc}}$, then $\vec{a}$ obeys condition $K_p$.*

*Proof.* As in the proof of Lemma 2, we need only prove that

$$[\partial_i(1-\Delta)^{-1}](f+u)\varphi - 2i \sum_{j=1}^{n} (\partial_i(1-\Delta)^{-1}\partial_j)(a_j u \varphi) \tag{15}$$

is in $(L^q)_{\text{loc}}$ with $q=p/p-1$ when $u\in(L^2)_{\text{loc}}$, $\varphi\in C_0^\infty$ and $f\in(L^1)_{\text{loc}}$. Since $a_j\varphi\in L^p$, $a_j u\varphi\in L^r$ with $r=2p/p+2$ by Hölder's inequality. Since $p>4$, $r>4/3$ and $q<4/3$ so $L^r\subset(L^q)_{\text{loc}}$. Since $\partial_i(1-\Delta)^{-1}\partial_j$ is bounded from $L^r$ to $L^r$, the second term in (15) is in $(L^q)_{\text{loc}}$. Moreover $\partial_i(1-\Delta)^{-1}$ is convolution with a function in $L^s$ for any $s<n/n-1$, so by Young's inequality, the first term is in $L^s$ for any $s<n/n-1$. But since $p>n$, $q<n/n-1$, so the first term in (15) is in $(L^q)_{\text{loc}}$. □

We conjecture that if $p>4$ and $\vec{a}\in L^p_{\text{loc}}(\mathbb{R}^n)$, then a obeys condition $K_p$. This conjecture is based on the fact that if $-\Delta u\in(L^1)_{\text{loc}}$ and $u\in(L^2)_{\text{loc}}$, then grad $u\in(L^q)_{\text{loc}}$ for any $q>4/3$. In fact (L. Tartar, private communication), if $-\Delta u\in(L^1)_{\text{loc}}$ and $u\in(L^2)_{\text{loc}}$, then grad $u\in(L^{\frac{4}{3}}_W)_{\text{loc}}$, the weak $L^{\frac{4}{3}}$ space (but not in general in $L^{\frac{4}{3}}$).

## § 5. Coulomb Gauge, II: Theorem with a Local Hypothesis

By combining Theorem 3 and Lemmas 3, 4, we obtain

**Theorem 4.** *Let* $a_j\in L^q(\mathbb{R}^n)_{\text{loc}}$, $j=1,\dots,n$. *Let* $\sum_{j=1}^n \partial_j a_j=0$. *Suppose* $q>n$; $q\geqq 4$. *Let* $V_+\in\big(L^2(\mathbb{R}^n)\big)_{\text{loc}}$ *and* $V_-\in L^p$ *with p n-canonical. Then:*

$$-\sum_{j=1}^n(\partial_j-i\,a_j)^2+V$$

*is essentially self-adjoint on* $C_0^\infty$.

*Remarks.* 1. By a good deal of extra work, Lemma 3 can be extended to the case $p=n$ if $n\geqq 5$ so that the condition in Theorem 4 becomes $a_j\in(L^q)_{\text{loc}}$ with $\frac{1}{2}q$ $n$-canonical.

2. If our conjecture at the end of § 4 is correct, $q>n$, $q\geqq 4$ can be replaced with $q\geqq 4$ if $n\leqq 3$ and $q>4$ if $n\geqq 4$.

## § 6. Other Gauges

We first note that all the arguments in § 3 remain valid if div $\vec{a}\in L^2_{\text{loc}}$ rather than div $\vec{a}=0$. Thus:

**Theorem 5.** *The conclusion of Theorem* 4 *remain valid if the hypothesis* $\sum_{j=1}^n \partial_j a_j=0$ *is replaced with* $\sum_{j=1}^n \partial_j a_j\in(L^2)_{\text{loc}}$.

In particular we recover the Ikebe-Kato $C^1$ result and a result when $\vec{a}\in(H^1)_{\text{loc}}$. Our attitude about other gauges is better expressed by the following which suggests that $C_0^\infty$ is the "wrong" domain to consider:

**Theorem 6.** *Let $a_j \in L^q(\mathbb{R}^n)$ with $q > n$ and $q \geqq 4$. Let $V_+ \in (L^2(\mathbb{R}^n))_{\text{loc}}$ and $V_- = 0$. Let $H$ be the operator $-\sum_{j=1}^{n} (\partial_j - i a_j)^2 + V$ defined as a quadratic form. Then there exists a measurable function $\lambda$ so that*

1) *For every $\varphi \in C_0^\infty$, $e^{-i\lambda(.)}\varphi(.) \in D(H)$*
2) *$\{e^{-i\lambda}\varphi \mid \varphi \in C_0^\infty\}$ is an operator core for $H$.*

*Remark.* By more work, it may be possible to extend this result to allow $a_j \in (L^q(\mathbb{R}^n))_{\text{loc}}$ and $V_- \neq 0$.

*Proof.* Let $\lambda = \sum_{j=1}^{n} [(-\Delta)^{-1}\partial_j] a_j$ where $(-\Delta)^{-1}\partial_j$ is the operator defined by the theory of Reisz potentials and transforms [17]. Then $\lambda \in L^q(\mathbb{R}^n)$ and $\partial_i \lambda \in L^q(\mathbb{R}^n)$, again by the theory of Reisz potentials. Let $\hat{a}_j = a_j + \partial_j \lambda$. Then $\sum_{j=1}^{n} \partial_j \hat{a}_j = \sum_{j=1}^{n} \partial_j a_j + \Delta\lambda = 0$. As a result the operator $\hat{H} = -\sum_{j=1}^{n} (\partial_j - i\hat{a}_j)^2 + V$ is essentially self-adjoint on $C_0^\infty$ by Theorem 4. If we can show that

$$H = e^{-i\lambda} \hat{H} e^{+i\lambda} \tag{16}$$

we will obtain (1) and (2) and so prove the theorem. But it is easy to prove that if $\varphi \in C_0^\infty$, then $e^{i\lambda}\varphi \in D(\overline{\partial_j - i\hat{a}_j \upharpoonright C_0^\infty})$ (by approximiting $\lambda$ with smooth functions) and that

$$e^{-i\lambda}(\partial_j - i\hat{a}_j)\, e^{i\lambda}\varphi = (\partial_j - i a_j)\,\varphi. \tag{17}$$

(17) immediatly implies the operator equality

$$e^{-i\lambda} \hat{D}_j e^{i\lambda} = D_j \tag{18}$$

where $D_j = \overline{\partial_j - i a_j \upharpoonright C_0^\infty}$; (18) implies the equality of the quadratic forms

$$e^{-i\lambda}\left(\sum_{j=1}^{n} \hat{D}_j^* \hat{D}_j + V\right) e^{+i\lambda} = \sum_{j=1}^{n} D_j^* D_j + V$$

which is (16). □

## References


1. Brownell, F.H.: A Note on Kato's Uniqueness Criterion for Schrödinger Operator Self-Adjoint Extensions. Pac. J. Math. **9**, 953–973 (1959).
2. Carleman, T.: Sur la theorie mathèmatique de l'èquation de Schrödinger, Arkiv för Matematik, Astronome och Fysik **24B**, N° 11 (1934).
3. Davies, E.B.: Properties of the Green's functions of some Schrödinger operators. J. London Math. Soc., to appear.
4. Faris, W.: The Product Formula for Semigroups Defined by Friedrichs Extensions. Pac. J. Math. **22**, 47–20 (1967).

5. Faris, W.: Essential Self-Adjointness of Operators in Ordered Hilbert Space. To appear in Commun math. Phys.
6. Ikebe, T., Kato, T.: Uniqueness of the Self-Adjoint Extension of Singular Elliptic Differential Operators. Arch. rat. Mech. Analysis **9**, 77–92 (1962).
7. Ikebe, T., Tayoshi, T.: Wave and Scattering Operators for Second Order Elliptic Operators in $\mathbb{R}^3$. Publ. Res. Inst. Math. Sci. Kyoto Univ. **4**, 483–496 (1968/69).
8. Kato, T.: Fundamental Properties of Hamiltonian Operators of Schrödinger Type. Trans. Amer. math. Soc. **70**, 195–211 (1951).
9. Kato, T.: Schrödinger Operators with Singular Potentials, Proc. Jerusalem Conf. Functional Analysis (to appear).
10. Kato, T.: Perturbation Theory for Linear Operators. Berlin-Heidelberg-New York: Springer 1966.
11. Reed, M., Simon, B.: Methods of Modern Mathematical Physics, I. Functional Analysis. New York, London: Academic Press 1972.
12. Reed, M., Simon, B.: Methods of Modern Mathematical Physics, II. Analysis of Operators. New York-London: Academic Press (to appear).
13. Simon, B.: Hamiltonians Defined by Quadratic Forms. Commun. math. Phys. **21**, 192–210 (1971).
14. Simon, B.: Quantum Mechanics for Hamiltonians Defined by Quadratic Forms. Princeton, N,J.: Princeton University Press 1971.
15. Simon, B.: Essential Self-Adjointness of Schrödinger Operators with Positive Potentials. Math. Ann. **201**, 211–220 (1973).
16. Simon, B.: Essential Self-Adjointness of Schrödinger Operators with Singular Potentials: A Generalized Kalf-Walter-Schmincke Theorem. To appear in Arch. rat. Mech. Analysis.
17. Stein, E.: Singular Integrals and Differentiability Properties of Functions. Princeton, N.J.: Princeton University Press 1970.
18. Strichartz, R.S.: Multipliers on Fractional Sobolov Spaces. J. Math. Mech. **16**, 1031–1060 (1967).
19. Stummel, F.: Singuläre elliptische Differentialoperatoren in Hilbertschen Räumen. Math. Ann. **132**, 150–176 (1956).

Barry Simon
Depts. of Mathematics and Physics
Princeton University
Princeton, N.J. 08540
USA

*(Received January 19, 1973)*