it follows that the construction given above represents a solution to the problem posed in this Letter.

The necessary and sufficient nature of the condition, Eq. (13), also allows one to correct, in a simple way, wave-function models[4] of the half-off-shell T matrix for the fact that these wave functions may not be menbers of a complete orthogonal set. An additional adjustable parameter in $|\Phi(E_l)$ would allow one to satisfy Eq. (13) with $|\Psi(E_l)\rangle$ taken to be the solution of a known potential.[5] We have seen that this is both sufficient and necessary for $|\Phi(E_l)\rangle$ to be a member of a complete orthonormal set.

A more detailed discussion of such constructions and their consequences off the energy shell is in preparation.

*Work supported in part by the U.S. Atomic Energy Commission and the National Science Foundation.

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Griffiths-Hurst-Sherman Inequalities and a Lee-Yang Therorem for the $(\varphi^4)_2$ Field Theory

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(Received 19 March 1973)

The Griffiths-Hurst-Sherman inequalities and the Lee-Yang zero theorem in the theory of Ising ferromagnets are shown to hold in a two-dimensional self-coupled Bose quantume field theory with interaction $:a\varphi^4+b\varphi^2-\mu\varphi:$. Applications include the continuity of the infinite-volume "magnetization," $\langle\varphi(0)\rangle$, away from $\mu=0$. Our results should carry over to three or four dimensions once it is known how to control the ultraviolet divergences in these theories.

The past year has seen remarkable progress[1] in constructuve quantum field theory because of the exploitation of Euclidean techniques advocated by Nelson.[2] One of several advantages of the Euclidean approach is that the Euclidean Bose field is commutative, so that time ordering is unnecessary in the Gell-Mann–Low formula which becomes[3]

$$\langle\varphi(x_1)\cdots\varphi(x_n)\rangle = \lim_{|\Lambda|\to\infty}\left[\frac{\langle\varphi(x_1)\cdots\varphi(x_n)\exp[-\int_\Lambda :P(\varphi(y)):dy]\rangle_0}{\langle\exp[-\int_\Lambda :P(\varphi(y)):dy]\rangle_0}\right]. \tag{1}$$

Equation (1) has a remarkable similarity to the formula for correlation functions in statistical mechanics and suggests that one attempt to carry over the techniques of rigorous statistical mechanics[4] to constructive quantum field theory. Such a program has been begun by Guerra, Rosen, and Simon[5] with further developments by Nelson[6] and Simon.[7] In this note, we wish to announce some further results within this program.

We feel that the techniques we present here (combined with those in Ref. 5) represent a new tool in understanding nontrivial quantum field theories, and, in particular, in studying the validity of the Goldstone picture of dynamical instability. By its very nature, dynamical instability is a strong-coupling phenomenon, and previous attempts at studying it have been hampered by relying basically on a perturbative approach. The recent techniques of Glimm, Spencer, Jaffe, and Dimock[1] are also restricted to a small coupling constant, and thus, presumably are only applicable away from the region of dynamical instability. On the other hand, statistical-mechanical techniques are not limited to small coupling. Our main applications (theorems 3–6 below) prove that certain quantities are continuous in various coupling constants precisely in regions where the Goldstone picture "predicts" continuity.

In Ref. 5, correlation inequalities of Griffiths-Kelly-Sherman (GKS)[8] and Fortuin-Kastelyn-Ginibre (FKG)[9] type were proven for the $P(\varphi)_2$ field theory. These inequalities are known to hold for general kinds of ferromagnets: with many-body interactions, with arbitrary spins, and with arbitrary single-spin distributions. Our results, on the other hand, are field-theory analogs of certain theorems which have only been proven *directly*[10] for spin-$\frac{1}{2}$ Ising ferromegnetics with pair interactions, namely the zero theorem of Lee and Yang[11] and the correlation inequalities of Griffiths-Hurst-Sherman (GHS) type.[12] We are only able to treat $P(\varphi)_2$ interactions with P of the form $P(X)=aX^4+bX^2-\mu X$. Our main results are as follows:

Theorem 1 (GHS inequality).—Let $\langle\ \rangle$ be a $P(\varphi)_2$ expectation value[13] for $P(X)=aX^4+bX^2-\mu X$ with $\mu \geqslant 0$. Then

$$\langle\varphi(x)\varphi(y)\varphi(z)\rangle+2\langle\varphi(x)\rangle\langle\varphi(y)\rangle\langle\varphi(z)\rangle-\langle\varphi(x)\varphi(y)\rangle\langle\varphi(z)\rangle-\langle\varphi(x)\varphi(z)\rangle\langle\varphi(y)\rangle-\langle\varphi(x)\rangle\langle\varphi(y)\varphi(z)\rangle\leqslant 0$$

for all x, y, z.

Theorem 2 (Lee-Yang theorem).—Let Λ be a finite region in $\mathbb{R}^2$. Fix $a>0$ and b real. For any complex μ, define

$$F_\Lambda(\mu)=\langle\exp\{-\int_\Lambda[a{:}\varphi^4(x){:}+b{:}\varphi^2(x){:}-\mu\varphi(x)]\,dx\}\rangle_0.$$

Then $F_\Lambda(\mu)\neq 0$ if $\operatorname{Re}\mu\neq 0$.

The proofs of these theorems (which will be described in full elsewhere[14]) is by a double-approximation procedure. First, we follow Ref. 5 and approximate the $P(\varphi)_2$ field theory by a nearest-neighbor Ising ferromagnet with continuous spins having a single-spin distribution of the form $C\exp(-\alpha s^4+\beta s^2+\gamma s)$ [if $P(X)=aX^4+bX^2-\mu X$]. We then[15] approximate each of the continuous spins by a ferromagnetic array of spin-$\frac{1}{2}$ Ising spins.

These theorems have a variety of applications[16] modeled after those in statistical mechanics. Let $\langle\ \rangle_{a,b,\mu}$ denote the infinite-volume state[17] for the $aX^4+bX^2-\mu X$ field theory. Since it is translation invariant, $\langle\varphi(x)\rangle_{a,b,\mu}$ is a number $M(a,b,\mu)$ independent of x. In Ref. 5 it is shown that M is non-negative if $\mu>0$. By tradition, dynamical instability (and, in particular, spontaneous broken symmetry) is supposed to be accompanied by a discontinuity in M as a function of μ. The following can be proven using theorem 1.

Theorem 3.—Fix $a>0$, b real. In the region $\mu>0$, $M(a,b,\mu)$ is a strictly positive, strictly monotonic, concave, continuous function of μ.

Theorem 1 also implies the following:

Theorem 4.—Fix $a>0$, b real. The mass gap for the ${:}a\varphi^4+b\varphi-\mu\varphi{:}$ theory is a monotonic nondecreasing function of μ in the region $\mu>0$.

Theorem 4 holds for either the spatially cut off theories or for infinite-volume theories arrived at by some fixed-limit procedure. The following is an application of theorem 2:

Theorem 5.—Let $\alpha_\infty(a,b,\mu)$ be the energy per unit volume[18] for the ${:}a\varphi^4+b\varphi^2-\mu\varphi{:}$ theory. Fix a and b. Then $\alpha_\infty(a,b,\mu)$ is real analytic in the region $\mu>0$, possesses an analytic continuation into the region $\operatorname{Re}\mu>0$, and for any $\mu>0$

$$d\alpha_\infty(a,b,\mu)/d\mu=M(a,b,\mu).$$

All three theorems suggest that dynamical instability can only occur at $\mu=0$. Since $aX^4+bX^2-\mu X$ has a unique minimum if $\mu\neq 0$, this fits in nicely with the Goldstone picture of dynamical instability.

We are also able, by following some Ising-model arguments of Lebowitz,[19] to prove theorem 6.

Theorem 6.—If the ${:}a\varphi^4+b\varphi^2{:}$ theory in infinite volume has a mass gap, then $M(a,b,\mu)$ is continuous in μ at $\mu=0$ and, in particular,

$$\lim_{\mu\to 0^+} M(a,b,\mu)=0.$$

It is our hope and expectation that theorems 1 and 2 will become as powerful a tool in the study of field theories with $\mu \neq 0$ as they are in the theory of the Ising model[20] at nonzero magnetic field. In particular, partly motivated by Ref. 20, one of us has proven[21] the following:

Theorem 7.—The infinite-volume[17] $:a\varphi^4 + b\varphi^2 - \mu\varphi:$ field theory ($a>0$, $\mu \neq 0$) possesses a unique vacuum.

Combined with results from Ref. 6, this concludes the proof of the Wightman axioms for a class of strongly coupled theories.

Finally, let us say a word about the limitations to two dimensions. In the lattice approximation, theorems 1 and 2 hold in any number of dimensions. In two (space-time) dimensions we can take the lattice spacing δ to zero without any renormalizations. In three or four dimensions, nontrivial ultraviolet divergences occur and so renormalizations are needed, and we do not yet know how to control these theories as $\delta \to 0$. However, if perturbation theory is an accurate guide for an $:a\varphi^4 + b\varphi^2 - \mu\varphi:$ theory, the counter terms will only be quartic, and so we expect that our theorems will remain valid.

We should like to thank Professor J. Lebowitz and Professor E. Lieb for valuable discussions. In addition, one of us (B.S.) would like to thank Professor N. Kuiper for the hospitality shown him while a visitor at the Institut des Haute Etudes Scientifique, and Professor A. Visconti for the hospitality of the Centre National de la Recherche Scientifique. The other (R.B.G.) is grateful for the hospitality of the Cornell Chemistry Department.

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