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Positivity of the Hamiltonian Semigroup and the Construction of Euclidean Region Fields

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Abstract. We present conditions under which a Wightman field theory can be 'continued' to a *field theory* in the Euclidean region. A basic role is played by the condition that the semigroup e^{-tH} be positivity preserving in the realization of the physical Hilbert space diagonalizing the time zero fields. Finally we present a model in one-dimensional space time obeying all the Osterwalder–Schrader axioms without an associated Euclidean region field.

1. Introduction

Much of the recent progress in constructive quantum field theory is founded on the use of Euclidean techniques. Since the pioneering work of Wightman [16] and Hall and Wightman [2], we have known how to analytically continue Minkowski expectation values to Euclidean region expectation values. It is only recently that we have learned to go in the opposite direction. The first and germinal result in this line is the reconstruction theorem of Nelson [7]. Following ideas of Symanzik [15], Nelson started with Euclidean region fields and showed how, if the Euclidean fields obeyed suitable axioms, one could construct an associated Minkowski region field theory obeying all the axioms of Gårding and Wightman [4, 14, 17].

Since the distinction between fields and expectation values will concern us in this paper, we emphasize that there is an asymmetry in the above results. This was partially rectified by Osterwalder and Schrader [9] in a beautiful paper.[2]) They presented a set of axioms for Euclidean region expectation values which were equivalent to the Wightman axioms for Minkowski expectation values (which are in turn known to be equivalent to the Gårding–Wightman axioms for fields [16]). In this situation, two questions naturally arise:

I) Are the Nelson axioms strictly stronger (other than for technical reasons) than the Osterwalder–Schrader (equivalently Gårding–Wightman) axioms?

II) (Assuming the answer to I) is yes.) What must be added to the Gårding–Wightman axioms to assure us that a Euclidean region *field* theory exists?

One expects that the answer to I) is yes because so much of Nelson's structure, especially the Markov property,[3]) is absent from the Osterwalder–Schrader axioms.

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[2]) There is a technical error in Ref. [9] but modified axioms equivalent to the Wightman axioms exist; see Refs. [20], [21].

[3]) One can have Euclidean region fields without the Markov property. See Ref. [22].

In fact, we will show in Section 5 that the 'Wightman field theory' in one dimension (generalized anharmonic oscillator) associated with the Hamiltonian $p^2 + q^2 + p^4$ yields a Euclidean region theory obeying all the OS-axioms but not the N-axioms.

II) is of especial interest because so much of the progress in the $P(\varphi)_2$ model has depended on the Euclidean *field* structure of the free field and it is of interest to know on what this depends. The answer we give to II) is not definitive: Basically we wish to present the thesis that, except for technicalities, Nelson's axioms are equivalent to the Wightman axioms together with the condition that e^{-tH} be positively preserving. Unfortunately, the 'technicalities' enter in many places and, in particular, at the very beginning: in order to have a notion of 'positivity preserving', we will need time zero fields (as operators) and thereby we restrict ourselves essentially to theories with finite field strength renormalization, Z^{-1}.

The structure of this paper is as follows: In Sections 2 and 3 we show how to construct Euclidean region fields once it is known that e^{-tH} is positivity preserving. This is just an expression of the standard idea from probability literature that the essential ingredient needed to construct a Markov process is a 'probability semigroup'. In fact, our construction is a special case of a general result in the probability literature [18]. We provide a detailed proof for the reader's convenience. In Section 4 we discuss the extent to which Nelson's axioms can be recovered for these Euclidean region fields. In Section 5 we give the details of the $p^2 + g^2 + p^4$ counter-example. In Section 6 we discuss parity invariance in Nelson theories. Finally, in an appendix, we present a new proof that $\Gamma(A)$ is positivity preserving when A is a contraction; this proof is based on the connection between positivity and Markov processes exploited in the rest of this paper.

2. Construction of a Euclidean Q-space

One of the weak points of the current paper is that we require the existence of time zero field operators; explicitly we require, in addition to the Gårding–Wightman axioms, that (d = dimension of space-time)

A) For each $f \in \mathscr{S}(\mathbb{R}^{d-1})$, there is a self-adjoint operator $\varphi_0(f)$ so that

i) $|\varphi_0(f)| \leqslant c(H+1)^n$ for c, n suitable.
ii) $\varphi_0(f)$ is essentially self-adjoint on $C^\infty(H)$.
iii) For any $g \in \mathscr{S}(\mathbb{R}^d)$

$$\theta(g) = \int_{-\infty}^{\infty} e^{iHt} \varphi_0(g_t) e^{-iHt} dt$$

where θ is the Wightman field and $g_t(x) = g(x,t)$.

Remark: Except for ii), these hold in any Wightman theory coming from a Nelson theory over $\mathscr{H}_{-1}(\mathbb{R}^d)$. ii) holds, for example, if each $\varphi(f)$ ($f \in \mathscr{H}_{-1}$) is L^2 (which in turn holds under assumptions E), F) below; see Section 3).

From local commutativity, we conclude that formally $\varphi_0(f)$ and $\varphi_0(g)$ commute but this does not automatically yield the stronger:

B) For any $f, g \in \mathscr{S}(\mathbb{R}^{d-1})$, $\varphi_0(f)$ and $\varphi_0(g)$ commute in the sense that their resolvents commute.

In addition, we require

C) $\{\varphi_0(f) | f \in \mathscr{S}(\mathbb{R}^{d-1})\}$ have Ω as cyclic vector.

Remark: B) and C) hold in any theory obeying Nelson's axioms.

The critical hypothesis for purposes of constructing a Euclidean Q-space is:

D) e^{-tH} is positivity preserving, explicitly:

$$\langle F(\varphi_0(f_1), \ldots, \varphi_0(f_n))\,\Omega, e^{-tH}\, G(\varphi_0(g_1), \ldots, \varphi_0(g_m))\,\Omega\rangle \geqslant 0$$

for any $f_1, \ldots, f_n, g_1, \ldots, g_m \in \mathscr{S}(\mathbb{R}^{d-1})$ and F, G, bounded, *positive* measurable functions on $\mathbb{R}^n$ (resp. $\mathbb{R}^m$).

Remark: Since the vacuum is unique, one can show that the inner product in question is strictly positive so long as $t > 0$, $\langle F\Omega, F\Omega\rangle > 0$, $\langle G\Omega, G\Omega\rangle > 0$. See Ref. [12].

That D) holds in any Nelson theory over $\mathscr{H}_{-1}$ is sufficiently important that we state it as a theorem:

Theorem 1: D) holds for the Wightman theory associated to any Nelson theory over $\mathscr{H}_{-1}(\mathbb{R}^d)$.

Proof:

$$\langle F(\varphi_0(f_1)\ldots)\,\Omega, e^{-tH}\, G(\varphi_0(g_1), \ldots)\,\Omega\rangle = \int d\mu\, F(\varphi(f_1 \otimes \delta_0), \ldots)\, G(\varphi(g_1 \otimes \delta_t), \ldots)$$

is clearly non-negative. ■

The main result of this section is the following:

Theorem 2: Let $\varphi_0(\cdot)$ be the time zero field associated to a scalar, Hermitean Wightman field theory obeying A)–D). Then there exists a probability measure space $(Q, \sum, \nu)$ and for each $t \in \mathbb{R}$ a random field $\varphi(\cdot, t)$ over $\mathscr{S}(\mathbb{R}^{d-1})$ so that, for any $t_1, \ldots, t_n \geqslant 0$; $\{f_{ij}\}_{i=1;\, j=1}^{k_i;\, n}$ and F_i bounded measurable functions on $\mathbb{R}^{k_i}$:

$$\langle \Omega, F_0(\varphi_0(f_{0j}))\, e^{-t_1 H} \ldots e^{-t_n H}\, F_n(\varphi_0(f_{nj}))\,\Omega\rangle = \int d\nu\, F_0(\varphi(f_{0j}, s_0)) \ldots F_n(\varphi(f_{nj}, s))$$

where s_0 is arbitrary and $t_i = s_i - s_{i-1}$.

Proof: We use an idea of Nelson [5] to reduce the proof of the countable additivity of the measure to the Reisz–Markov representation theorem. Let $\mathscr{m}$ be the Von Neumann algebra generated by $\{e^{i\varphi_0(f)} | f \in \mathscr{S}(\mathbb{R}^{d-1})\}$. By B), C), $\mathscr{m}$ is a maximal abelian von-Neumann algebra. For each t, let Q_t be a copy of spec($\mathscr{m}$), the Gel'fand spectrum of $\mathscr{m}$. We will take

$$Q = \mathop{X}_{t \in \mathbb{R}} Q_t$$

and Σ, the Baire sets on this compact set. In order to construct ν we need:

Lemma: Let $Y_1, \ldots, Y_n$ be compact Hausdorff spaces. Let μ_n be a Baire measure on Y_n and for each $i = 1, \ldots, n-1$, let A_i a bounded map from $C(Y_i)$ to $C(Y_{i+1})$ be given so that $A_i 1 = 1$ and $A_i f \geqslant 0$ if $f \geqslant 0$. Then there exists a unique measure ν on $Y = Y_1 \times Y_2 \times \ldots \times Y_n$ so that for any $f_1, \ldots, f_n$ in $C(Y_1), \ldots, C(Y_n)$ (respectively),

$$\int f_1(y_1) \ldots f_n(y_n)\, d\nu = \int f_n A_{n-1}[f_{n-1} \cdot A_{n-2}[f_{n-2} \ldots f_2 \cdot A_1 f_1] \ldots]\, d\mu_n.$$

Proof: Since $\{f_1 \otimes \ldots \otimes f_n\}$ are total in $C(Y)$ (by Stone–Weierstrass), uniqueness is clear. Thus we need only prove existence. Consider first the case $n = 2$. For each $y_2 \in Y_2$ let $\nu_{y_2} = A_1^*(\delta_{y_2})$ where δ_{y_2} is the Dirac measure at y_2 and A_1^* is the adjoint of A_1. Since A_1 is positivity preserving ν_{y_2} is a positive measure and since $A_1 1 = 1$ it is normalized. Moreover $y_2 \mapsto \nu_{y_2}$ is clearly weak-$*$ continuous. Let $f \in C(Y_1 \times Y_2)$. Then it is easy to see that

$$y_2 \mapsto \int_{Y_1} f(y_1, y_2)\, d\nu_{y_2}(y_1) \equiv G_{y_2}(f)$$

is continuous. Let

$$\int f\, d\nu = \int_{Y_2} G_{y_2}(f)\, d\mu_2(y_2).$$

Then clearly $d\nu$ defines a positive (normalized) measure. Moreover

$$G_{y_2}(f_1 f_2) = f_2(y_2) \int f_1\, d(A_1^* \delta_{y_2}) = f_2(y_2)\,(A_1 f_1)\,(y_2)$$

so

$$\int f_1 f_2\, d\nu = \int f_2 (A_1 f_1)\, d\mu_2.$$

This is the proof for $n = 2$.

Now, by induction, we can handle the case of general n. Supposing the theorem true for $n - 1$, let $Y_1, \ldots, Y_n$ be given together with $A_1, \ldots, A_{n-1}, d\mu_n$. By the inductive hypothesis, find $d\tilde{\nu}$ on $Z_2 = Y_2 \times \ldots \times Y_n$ with

$$\int f_2 \ldots f_n\, d\tilde{\nu} = \int f_n A_{n-1}[\ldots f_2 \cdot A_2 f_2]\, d\mu_n.$$

Let $Z_1 = Y_1$. By the case $n = 2$ above, we find $d\nu$ on $Z_1 \times Z_2$ with

$$\int d\nu f_1 g_2 = \int g_2 (B_1 f_2)\, d\tilde{\nu}$$

where

$$(B_1 f_1)\,(y_2, \ldots, y_n) = (A_1 f)\,(y_2).$$

It is easily verified that ν has the required property. ■

Now we return to the proof of Theorem 2. $\langle\Omega, \cdot\Omega\rangle$ defines a measure $d\mu$ on spec($\mathscr{m}$) in such a way that the Wightman Hilbert space, $\mathscr{H}$, is L^2 (spec($\mathscr{m}$), $d\mu$) and $\Omega = 1$. Since spec($\mathscr{m}$) is a Stone space, $\mathscr{m} \cong L^\infty(\text{spec}(\mathscr{m}), d\mu) \cong C(\text{spec}(\mathscr{m}))$. Now e^{-tH} is a contraction on L^2 taking 1 into 1 and thus by a standard argument (see e.g. Ref. [13]) taking $L^\infty \equiv C(\text{spec}(\mathscr{m}))$ into itself. Let A_t denote this map on $L^\infty \equiv C(\text{spec}(\mathscr{m}))$. By D), A_t is positivity preserving.

We now pass this structure to the Q_t (copies of spec($\mathscr{m}$)). For each t, $d\mu$ passes to a measure $d\mu_t$ on Q_t and for each $t, s \in \mathbb{R}$, a map $A_{t,s}$ is defined from $C(Q_s)$ to $C(Q_t)$ by pulling over $A_{|t-s|}$.

Given $t_1 < t_2 < \ldots < t_n$ define $d\nu_{t_1,\ldots,t_n}$ on $Q_{t_1} \times \ldots \times Q_{t_n}$ by applying the lemma with $d\mu_n = d\mu_{n_t}$ and $A_j = A_{t_{j+1},t_j}$. Given a function F in $C(Q)$ which only depends on $q_{t_1}, \ldots, q_{t_n}$ let

$$\int F\,d\nu = \int F(q_{t_1}, \ldots, q_{t_n})\,d\nu_{t_1,\ldots,t_n}$$

It is easily seen that this defines a positive linear functional on $\bigcup_n (\bigcup_{t_1,\ldots,t_n} C(Q_{t_1} \times \ldots \times Q_{t_n}))$ which thus extends uniquely to $C(Q)$ (since the latter functions are dense by Stone–Weïerstrass). By construction $d\nu$ obeys the basic property asserted by the theorem. ■

3. Construction of Euclidean Region Fields over $\mathscr{H}_{-1}(\mathbb{R}^d)$

While it is not absolutely necessary, it is technically simplifying if we suppose

E) There is a mass gap above 0 in the energy momentum spectrum

F) Z is not zero; i.e. $\int_0^\infty d\rho(m^2) < \infty$ where ρ is the Källén–Lehmann measure for $\langle\Omega_0, \varphi(x)\varphi(y)\Omega_0\rangle - \langle\Omega_0, \varphi(x)\Omega_0\rangle^2$.

Given that we are already supposing that time zero fields exist, these hypotheses are not especially strong. We can now prove:

Theorem 3: Suppose that the hypotheses of Theorem 2 hold and that, in addition, E) and F) hold. Then there exists a random field, φ_N, over the space (Q, Σ, ν) indexed by $\mathscr{H}_{-1}(\mathbb{R}^d)$ so that

1) For each $f \in \mathscr{H}_{-1}$, $\varphi_N(f) \in L^2(Q, d\nu)$ and $f \mapsto \varphi_N(f)$ is a continuous map from $\mathscr{H}_{-1}$ to L^2.

2) For each $t \in \mathbb{R}$ and $f \in \mathscr{S}(\mathbb{R}^{d-1})$

$$\varphi_N(f \otimes \delta_t) = \varphi(f, t) - \langle\Omega, \varphi(x)\Omega\rangle \int f(x)\,d^{d-1}x.$$

Remark: Of course, if $\langle\Omega, \varphi(x)\Omega\rangle = 0$, the last correction term is not needed.

Proof: We will show below that

$$\sum_{n=1}^N f_n \otimes \delta_{t_n} \; (N < \infty; f_n \in C_0^\infty(\mathbb{R}^{d-1}))$$

is dense in $\mathscr{H}_{-1}(\mathbb{R}^d)$ so that uniqueness is obvious. Let $d\rho$ be the Källén–Lehmann measure and let

$$\langle f, g\rangle_\rho = \int_{m_{\text{gap}}}^\infty d\rho(m^2) \int \frac{d^d k}{k^2 + m^2} \hat{f}(k)^* \hat{g}(k).$$

Clearly, $\langle\,,\,\rangle_\rho$ is finite for $f, g \in \mathscr{H}_{-1}$, and

$$\langle f, g\rangle_\rho \leqslant \|f\|_{\mathscr{H}_{-1}} \|g\|_{\mathscr{H}_{-1}} c$$

where $c = \int d\rho$ if $m_{\text{gap}} \geqslant 1$ and $c = m_{\text{gap}}^{-2} \int d\rho$ if $m_{\text{gap}} \leqslant 1$. Let

$$\tilde{\varphi}(f, t) = \varphi(f, t) - \langle\Omega, \varphi(x)\Omega\rangle \int f\,d^{d-1}x.$$

Then by direct computation:

$$\int_Q \tilde{\varphi}(f, t)\tilde{\varphi}(g, \delta)\,d\nu = \langle f \otimes \delta_t, g \otimes \delta_s\rangle_\rho.$$

Thus, letting

$$\varphi_N(\sum f_n \otimes \delta_{t_n}) = \sum_n \tilde{\varphi}(f_n, t_n)$$

we see that φ_N is continuous from a dense set of $\mathscr{H}_{-1}$ to $L^2(Q_N, d\nu)$ and so extends to $\mathscr{H}_{-1}$. ■

In the above we used:

Lemma: $\{f \otimes \delta_t | f \in C^\infty(\mathbb{R}^{d-1})\}$ are total in $\mathscr{H}_{-1}(\mathbb{R}^d)$.

Proof: Consider first the case $d = 1$. Let $g \in C_0^\infty(\mathbb{R})$ with supp$g \subset [-N, N]$ (N a positive integer). Let

$$g_n = \sum_{m=-Nn}^{Nn} \frac{1}{n} g\left(\frac{m}{n}\right) \delta_{m/n}.$$

Then

$$\hat{g}_n(k) = (2\pi)^{-\frac{1}{2}} \sum \frac{1}{n} g\left(\frac{m}{n}\right) e^{-ikm/n}$$

from which it follows that $\hat{g}_n \to \hat{g}$ pointwise. Clearly

$$\|\hat{g}_n\|_\infty \leqslant (2\pi)^{-\frac{1}{2}}(2N)\|g\|_\infty$$

so

$$|\hat{g}_n(k)|^2/(1 + k^2) \leqslant c/1 + k^2 \in L^1.$$

It thus follows from the dominated convergence theorem that $g_n \to g$ in $\mathscr{H}_{-1}(\mathbb{R}^1)$. Similarly if $g \in C_0^\infty(\mathbb{R}^d)$ is of the form $g = f(x)\,h(t)$ then $f(x)\,h_n$ converges to g in the norm $\int |\hat{f}|^2 d^d k / 1 + k_n^2$ so *a fortiori* in $\mathscr{H}_{-1}$ norm. Since these functions are easily seen to be total in $\mathscr{H}_{-1}$ the result is proven. ∎

4. Obstructions to Nelson's Axioms

It is clear that the random field φ_N we have constructed on (Q, Σ, ν) obeys most of Nelson's axioms by direct construction. The only axioms not immediately evident are:

i) Rotation invariance of $d\nu$ and the related covariance of the fields.

ii) The Markov property. In fact, we do not think it is possible to prove these without adding extremely strong additional hypotheses. However, it is easy to show that weak forms of i) and ii) hold, namely:

i') The non-coincident 'Schwinger functions', $\int d\nu\, \phi_N(f_1) \ldots \phi_N(f_n)$ (suppf_i and suppf_j disjoint for all $i \neq j$) are rotation invariant.

ii') Let $\Sigma_{(0,b)}$ be the σ-algebra generated by the $\varphi(f)$ for those f with support in $\{(x,t) \mid 0 \leqslant t \leqslant b\}$. Then, if F is $\Sigma_{[a,\infty)}$ measurable

$$E(F \mid \Sigma_{(-\infty,a]}) = E(F \mid \Sigma_{(a)}).$$

Only i') and ii') are used in proving the Nelson reconstruction theorem. Thus, we could (by suitably modifying (E), (F) and translating to Euclidean space) obtain Nelson-like axioms (but with i') and ii') replacing i) and ii)) equivalent to a set of Wightman-like axioms (plus technical assumptions plus positivity). We have not bothered to explicitly do this since we find i') and ii') so unnatural.

We note that the obstruction to proving i') from i) is not only the non-coincidence, but more strongly the non-uniqueness of the moment problem for suitable moments. (This is connected to the old Borchers–Zimmerman proposal for self-adjointness and its counter-example, see Hepp [3].) To extend ii') even to general half-planes, we will need i)!

5. A Counter-Example in One-Dimensional Space-Time

The only models of scalar Wightman fields in dimension 2 or more that we now have are:

a) Free fields, or more generally, generalized free fields.

b) Wick polynomials (and in two dimensions, Wick entire functions) in fields of class a).

c) Certain $P(\varphi)_2$ models.

These all yield theories which can be proven to have associated Euclidean fields. To see this for theories in class a), we note:

Theorem 4: In any theory obeying the Osterwalder–Schrader (equivalently Wightman) axioms, the two-point (Schwinger) function obeys the Nelson–Symanzik positivity condition:

$$\int f(x) f(y)\, S_2(x-y)\, dx\, dy \geqslant 0$$

for all real valued $f \in \mathscr{S}(\mathbb{R}^d)$.

Proof: By the Källén–Lehmann representation, S_2 is a 'super-position' of free field S_2's for which the inequality has been noted by Symanzik [15]. ∎

For generalized free fields, NS positivity of S_2 is all that is needed to produce a probability measure space, for one uses the Gaussian random process indexed by the Hilbert space obtained by completing $\mathscr{S}$ in the inner product $\int f(S_2 * f)$. For theories of type b), we can use the measure space of a) and just take the suitable Wick polynomial in the Euclidean generalized free field as the new Euclidean field.

Finally, the whole point of recent developments and of Nelson's work [6] in particular is that $P(\varphi)_2$ is expected to obey Nelson's axioms.

We thus pass to one-dimensional space-time where we see that certain 'Wightman' theories violate the conclusion of Theorem 1 (and so cannot be Nelson theories since they violate the general NS positivity condition):

Theorem 5: Let $H = p^4 + p^2 + q^2$ on $L^2(\mathbb{R}, dq)$ where $p = i^{-1} d/dx \cdot H$ has a unique ground state. With 'field' q and Hamiltonian H, we have a theory obeying all the Wightman axioms but for which condition D) fails.

Proof: Since H is unitarily equivalent to $p^2 + q^4 + q^2$, it has a unique ground state, ψ, which is positive definitive as a function of q. In particular, ψ is strictly positive near 0 and continuous. Let $F = q^4 \psi$. Then $e^{-tH} F = F_t$ is strictly negative near 0 for t small. For $F(t,q) \equiv F_t(q)$ obeys

$$\frac{\partial}{\partial t} F(t,q) = -\frac{\partial^4}{\partial q^4} F(t,q) + \frac{\partial^2}{\partial q^2} F(t,q) + q^2 F(t,q).$$

Thus $(\partial/\partial t)\, F(t,0)|_{t=0} < 0$ and $F(0,0) = 0$ so $F(t,0)$ is negative for t small. Since $F(t,q)$ is continuous in t and q, we obtain the result that for some t_0 and some δ, $F(t_0, q) < 0$ for $|q| < \delta$. By shrinking δ, we can suppose that ψ is non-negative in $(-\delta, \delta)$. Pick G non-negative with support in $(-\delta, \delta)$. Then

$$(G\psi, e^{-tH} q^4 \psi) < 0. \quad ∎$$

6. Parity Invariance

In this section we wish to explain the following: Let us suppose that space-time has even dimension. Nelson's axioms imply invariance of the Schwinger functions under the full Euclidean group and thus separate P and T symmetry (C is trivial for Hermitean fields). We were thus prepared to explicitly assume this in order to build up a Nelson theory. This seems to be unnecessary. In which hypothesis is separate P and T invariance buried? There is a related question: namely as used by Nelson, T invariance seems to be related to self-adjointness of H – surely H is self-adjoint in theories without T. The following result in 'classical' axiomatic field theory explains what is going on:

Theorem 6: In a scalar Wightman theory P (equivalently T) invariance is equivalent to reality of the non-coincident Schwinger functions.

Proof: Let W_n denote the Wightman functions in the forward tube. Making space and time separately explicit, P is equivalent to

$$W_n(x_1, t_1; \ldots; x_n, t_n) = W_n(-x_1, t_1; \ldots; -x_n q t_n).$$

Thus, the theorem is true if we can prove

$$S_n(-x_1, t_1; \ldots; -x_n, t_n) = \overline{S_n(x_1, t_1; \ldots; x_n, t_n)}$$

in the Euclidean region. But we see directly that

$$S_n(x_n, -t_n; \ldots; x_1, -t_1) = \overline{S_n(x_1, t_1; \ldots; x_n, t_n)}$$

in the Euclidean region from which the desired equality follows by the facts that $PT \in$ complex $(\mathscr{L}_+^\uparrow)$, and total symmetry of S. ■

Thus, in some sense, by supposing the Schwinger functions real, Nelson is already imposing P-invariance once he can reconstruct a Wightman theory. And our hypothesis D) implies P-invariance since it implies reality of S_n in the Euclidean region.

Note added in Typescript

By using an idea of Fröhlich [19] one can actually prove the measure ν of Theorem 2 is Euclidean invariant if A) i) is replaced with the stronger

$$|\varphi_0(f)| \leqslant C(H+1)$$

This is discussed in detail in Ref. [20], Chaps. IV, VI.

APPENDIX

A New Proof That $\Gamma(A)$ is Positivity Preserving

Let $\mathscr{H}$ be a real Hilbert space and $\Gamma(\mathscr{H})$ the L^2-space associated with the Gaussian random process indexed by $\mathscr{H}$. If A is a contraction, then $\Gamma(A)$ is defined on $\Gamma(\mathscr{H})$.

It is known to be positivity preserving – this fact has a variety of proofs [**1**, **8**, **10**, **11**, **13**]. We wish to present another proof which helps 'explain' why the result is true. This proof is based on the fact that to say that B is positivity preserving is essentially equivalent to the ability to construct a Markov chain with B as transition matrix. Since $\Gamma(A)$ is positivity preserving it must be the transition matrix of a chain. Thus to prove that $\Gamma(A)$ is positivity preserving, we need only construct the Markov chain *a priori*.

Given $A: \mathscr{H} \to \mathscr{H}$, we define an operator valued function on $\mathbb{Z}$ by

$$\begin{aligned} A(n) &= A^n \qquad n \geqslant 0 \\ &= (A^*)^{-n} \qquad n \leqslant 0. \end{aligned}$$

Lemma: Let $f(n)$ be an $\mathscr{H}$-valued function on $\mathbb{Z}$ which is non-zero for only finitely many integers. Then, for any contraction A:

$$\sum_{n,m} (f(n), A(n-m) f(m)) \geqslant 0.$$

Proof: Suppose $\|A\| < 1$. Let

$$\hat{f}(\theta) = (2\pi)^{-\frac{1}{2}} \sum_n e^{in\theta} f(n)$$

and

$$\hat{A}(\theta) = \sum_n e^{in\theta} A(n).$$

Then

$$\sum_{n,m} (f(n), A(n-m) f(m)) = \int_0^{2\pi} (\hat{f}(\theta), \hat{A}(\theta)\hat{f}(\theta))\, d\theta$$

so we need only show that $\hat{A}(\theta)$ is positive. But by explicit computation

$$\hat{A}(\theta) = (1 - A^* e^{-i\theta})^{-1} (1 - A^* A)(1 - e^{i\theta} A) \geqslant 0.$$

A simple limiting argument handles the case $\|A\| = 1$. ■

Theorem: If A is a contraction, $\Gamma(A)$ is positivity preserving.

Proof: Let $\mathscr{K}$ be the Hilbert space obtained by completing $\{f(n)\}$ of the lemma in the norm

$$\|f\|_{\mathscr{K}}^2 = \sum (f(n), A(n-m) f(m))$$

For each n, let $\mathscr{H}_n$ be the set of functions in $\mathscr{K}$ with support on $\{n\}$ – $\mathscr{H}_n$ is naturally isomorphic to $\mathscr{H}$. Thus for each n, there is a natural imbedding α_n of $\Gamma(\mathscr{H})$ into $\Gamma(\mathscr{K})$, α_n is a homomorphism and so, in particular, is positivity preserving. Moreover, it is easy to see that

$$(\alpha_1(F), \alpha_0(G))_{\Gamma(\mathscr{K})} = (F, \Gamma(A) G)_{\Gamma(\mathscr{H})}$$

so we see that $\Gamma(A)$ is positivity preserving. ■

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