PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 45, Number 3, September 1974

POINTWISE BOUNDS ON EIGENFUNCTIONS AND WAVE PACKETS IN N-BODY QUANTUM SYSTEMS. II

BARRY SIMON[1]

ABSTRACT. We provide a simple proof (and mild improvement) of Schnol's result that L^2 eigenfunctions of $-\Delta + V$ are $O(\exp(-ar))$ for any $a > 0$ whenever $V \to \infty$ as $r \to \infty$.

Despite a rather large literature (reviewed in [6]) on the exponential falloff of Schrödinger operators, $-\Delta + V$, one of the strongest results is one of the first, that of Schnol [9], who asserts that L^2 solutions of $(-\Delta + V)\psi = E\psi$ obey pointwise bounds of the form

$$|\psi(r)| \le C_a \exp(-ar) \tag{1}$$

if V is continuous and bounded below and E is in the discrete spectrum of $-\Delta + V$. The constant a in Schnol's result can be taken arbitrarily obeying $a < f(d(E))$, where f is a universal function depending on E and V only through the lower bound on V and $d(E)$ is the distance of E from the essential spectrum of $-\Delta + V$.

For the general multiparticle quantum system, Schnol's results have two obvious weaknesses: V is not bounded below in atomic and other systems of interest, and secondly Schnol's function $f(E)$ behaves as $\ln E$ as $E \to \infty$ instead of as $\sqrt{E}$ which is suggested by spherically symmetric examples and the theory of ordinary differential equations [8]. Much of the literature on the subject deals with these weaknesses. Due to the recent work of O'Connor [6], Combes-Thomas [2] and Simon [10], we now have nearly maximally good results for the case $V(r) = \sum V_{ij}(r_i - r_j)$ with $V_{ij}(x) \to 0$ as $x \to \infty$.

There is another case which is better handled by Schnol's result, namely where $V \to \infty$ at ∞ with V bounded below (generalized harmonic oscillator) and our goal in this note is to use the methods of the recent papers just quoted to obtain Schnol's results for this case.

Received by the editors December 3, 1973.

AMS (MOS) subject classifications (1970). Primary 35B40, 81A81; Secondary 35D10, 26A16, 42A12.

[1] A Sloan fellow.

First, we can exploit the Combes-Thomas idea:

Theorem 1. *Let* $V = V_1 + V_2$ *where* $V_2 \geq 0$, $V_2 \in L^1_{loc}(\mathbf{R}^n)$ *and where* V_1 *is a form bounded perturbation of* $-\Delta$ *with form bound, a, less than* 1 *(e.g.* $V_1 \in L^{n/2}(\mathbf{R}^n)$ *will do if* $n \geq 3$*; see e.g.* [7]*). Let* $H = -\Delta + V$ *defined as a sum of quadratic forms* [4], [11]*. Suppose that* $-(1-a)\Delta + V_2$ *has compact resolvent (e.g. if* $\inf_{|r|>R} V_2(r) \to \infty$ *as* $R \to \infty$*). Then any* L^2 *eigenfunction of* $-\Delta + V$ *lies in the domain of* $\exp(cr)$ *for any* $c > 0$.

Proof. Let $H(\vec{b})$ be defined as $(i\vec{\nabla} - \vec{b})^2 + V$ for any $\vec{b} \in \mathbf{C}^n$. It is easy to see that $H(\vec{b})$ is an entire analytic family of type (B) in the sense of Kato [4] with invariant form domain $D(-\Delta^{1/2}) \cap Q(V_2)$. Moreover, since $H(0)$ has compact resolvent by hypothesis and $H(\vec{b})$ is unitarily equivalent to $H(0)$ if $\vec{b} \in \mathbf{R}^n$, $H(\vec{b})$ has compact resolvent for all $\vec{b}$. By mimicking the Combes-Thomas arguments, one easily sees that eigenvectors of H are entire vectors [5] for the group $\exp(i\vec{b} \cdot \vec{r})$ which implies our result. □

To obtain pointwise bounds on eigenfunctions we are *not* able to mimic Simon [10], who combines L^2 exponential bounds with L^∞ bounds of Kato [3] for vectors in $C^\infty(H)$, because Kato's methods only work for potentials going to zero at infinity [3], [10]. Instead we combine the L^2 bounds with L^∞ bounds of Davies [1] for analytic vectors of H. Use of Davies' ideas restricts us to V's which are bounded below, thereby recovering Schnol's result:

Theorem 2. *Let* $V \geq 0$, $V \in L^1_{loc}(\mathbf{R}^n)$, $\inf_{|r|>R} V(r) \to \infty$ *as* $R \to \infty$. *Let* $\psi \in L^2(\mathbf{R}^n)$ *lie in* $Q(-\Delta) \cap Q(V)$ *and obey* $-\Delta\psi + V\psi = E\psi$. *Then for any* $a > 0$, *there is a* C *with* $|\psi(r)| \leq C \exp(-a|r|)$.

Proof. It is obviously sufficient to show that for each $b \in \mathbf{R}^n$, $\phi \equiv \exp(b \cdot r)\psi \in L^\infty$. But ϕ obeys $H(ib)\phi = E\phi$ with $\phi \in L^2$ by Theorem 1. Now, write $H(ib) = H_0(ib) + V$ and consider the semigroup $\exp(-tH_0(ib))$. It is easy to write down an explicit kernel for it and see that:

(a) $\exp(-tH_0(ib))$ is positivity preserving;

(b) $\exp(-tH_0(ib))$: $L^2(\mathbf{R}^n) \to L^\infty(\mathbf{R}^n)$ for $t > 0$.

Write $V_n = \min(V, n)$. Then $H_0(ib) + V_n$ converges to $H(ib)$ in strong resolvent sense as $n \to \infty$ so that

$$\exp(-tH(ib))\phi = \operatorname*{s-lim}_{n\to\infty} \left(\operatorname*{s-lim}_{m\to\infty} (\exp(-tH_0(ib)/m)\exp(-tV_n/m))^m \right)\phi$$

and thus since $e^{-tV_n/m} \leq 1$ $(V \geq 0!)$ and (a)

$$|\exp(-tH(ib))\phi| \leq \exp(-tH_0(ib))|\phi|$$

pointwise. Thus, by (b), $\exp(-tH(ib))\phi = \exp(-tE)\phi$ is in L^∞ so that ϕ is in L^∞. □

Remark. Depending on how fast V goes to infinity we expect ψ to obey $\exp(-x^\alpha)$ bounds for $\alpha > 1$. We hope to return to this question in a future publication.

REFERENCES

1. E. B. Davies, *Properties of the Green's functions of operators*, J. London Math. Soc. (to appear).

2. J. M. Combes and L. Thomas, *Asymptotic behavior of eigenfunctions for multiparticle Schrödinger operators*, Comm. Math. Phys. 34 (1974), 251–270.

3. T. Kato, *On the eigenfunctions of many-particle systems in quantum mechanics*, Comm. Pure Appl. Math. 10 (1957), 151–177. MR 19, 501.

4. ———, *Perturbation theory for linear operators*, Die Grundlehren der math. Wissenschaften, Band 132, Springer-Verlag, New York, 1966. MR 34 #3324.

5. E. Nelson, *Analytic vectors*, Ann. of Math. (2) 70 (1959), 572–615. MR 21 #5901.

6. A. O'Connor, *Exponential decay of bound state wave functions*, Comm. Math. Phys. 32 (1973), 319.

7. M. Reed and B. Simon, *Methods of modern mathematical physics*. Vol. II, Academic Press, New York (to appear).

8. T. Regge and V. de Alfaro, *Potential scattering*, North-Holland, Amsterdam; Interscience, New York, 1965. MR 32 #8724.

9. J. E. Šnol′, *On the behavior of the eigenfunctions of Schrödinger's equation*, Mat. Sb. 46 (88) (1957), 273–286; erratum, 259. (Russian) MR 23 #A2618.

10. B. Simon, *Pointwise bounds on eigenfunctions and wave packets in N-body quantum systems*. I, Proc. Amer. Math. Soc. 42 (1974), 395–401.

11. ———, *Quantum mechanics for Hamiltonians defined on quadratic forms*, Princeton Univ. Press, Princeton, N. J., 1971.

DEPARTMENT OF MATHEMATICS AND PHYSICS, PRINCETON UNIVERSITY, PRINCETON, NEW JERSEY 08540