Commun. math. Phys. 33, 145—164 (1973)

The $(\phi^4)_2$ Field Theory as a Classical Ising Model

Barry Simon*
C.N.R.S., Marseille, France

Robert B. Griffiths**
Chemistry Dept., Cornell University, Ithaca, New York

Received May 16, 1973

Abstract. We approximate a (spatially cutoff) $(\phi^4)_2$ Euclidean field theory by an ensemble of spin 1/2 Ising spins with ferromagnetic pair interactions. This approximation is used to prove a Lee-Yang theorem and GHS type correlation inequalities for the $(\phi^4)_2$ theory. Application of these results are presented.

1. Introduction

Rigorous tools in the theory of the Ising model fall roughly into two classes. Certain results have only been proven directly for what we will call "classical Ising models" by which we mean ferromagnets with spin 1/2 (i.e. s_i can have the values ± 1) spins and pair interactions. Others hold for "general Ising models" by which we mean that arbitrary (ferromagnetic) many body interactions are allowed and that individual spins can take an arbitrary set of values (including continuous values) with fairly arbitrary uncoupled single spin probability distributions. The first class includes the zero theorem of Lee and Yang [15, 1] and the correlation inequalities of the Griffiths-Hurst-Sherman (GHS) type [8]. The second class includes the correlation inequalities of Griffiths, Kelly, and Sherman (GKS) [6, 13] and of Fortuin, Kasteleyn, and Ginibre (FKG) [3]. (GKS inequalities were originally proven for classical models [6] but were eventually proven with many body interactions [13], higher spin [7] and arbitrary even spin distributions [4].)

Recently, Guerra, Rosen and Simon [12] have shown how the $P(\phi)_2$ Euclidean field theory [16, 12] can be approximated by general Ising models. As a consequence of this approximation (which they called the "lattice approximation") they were able to prove GKS and FKG inequalities. Our goal in this paper is to investigate the possibility of approximating such field theories by classical Ising models.

* Alfred P. Sloan Foundation Fellow. Permanent address: Departments of Mathematics and Physics, Princeton University, Princeton, N. J. 08540, USA.

** J. S. Guggenheim Memorial Foundation Fellow. Permanent address: Department of Physics, Carnegie-Mellon University, Pittsburgh, Pa. 15213, USA.

The idea of the approximation we will use is based on a method used by Griffiths [7] to prove various results for ferromagnets where each spin can take the values $-n, -n+2, \ldots, n-2, n$ with equal weight. What was done in that case was to find a ferromagnet of n coupled spin 1/2 spins so that the probability distribution for the total spin gave equal weight to each possibility. Knowing this, if one is given a system, S, of m spin $n/2$ spins, $s_1, \ldots, s_m$, ferromagnetically coupled, one easily constructs an "analog system" of mn spin 1/2 spins $\{\sigma_{ij}\}$, $1 \leqq i \leqq m$, $1 \leqq j \leqq n$, so that the σ_{ij} are ferromagnetically coupled and so that the probability distribution for the random variables $s_i = \sum_{j=1}^{n} \sigma_{ij}$ in the analog system is the same as in the system S. Inequalities and a Lee-Yang theorem for the system S follow from those for the analog system.

Now, to approximate a $P(\phi)_2$ Euclidean field theory with a classical Ising model, we first approximate it with the lattice approximation of [12]. Thus, it is approximated by a family of continuous spins with ferromagnetic pair interactions where each single spin has an (uncoupled) distribution proportional to $\exp(-Q(s))\,ds$. Here Q is even if P is even, and is a polynomial with $\deg Q = \deg P$. The next step is to realize each single spin as a limit of classical Ising systems in the sense of the last paragraph. In § 2, we will succeed in doing this if $Q(s) = as^4 + bs^2 (a > 0,$ b real). The basis of our approximation is the DeMoivre-Laplace limit theorem [2, pp. 179–182] which says that a suitably scaled binomial distribution approaches a Gaussian. By using a ferromagnetic pair interaction, we can cancel the Gaussian and rescale to obtain $\exp(-s^4)$. The ferromagnetic pair interaction (see the proof of Theorem 1) is of the form in which every spin interacts equally with every other spin. This leads, in a suitable thermodynamic limit, to the "mean-field" model of ferromagnetism [24], and an $\exp(-s^4)$ distribution is to be expected at the critical point of such a model. (The exponent of s, 4, is $\delta + 1$ in the usual notation for critical point indices in the theory of phase transitions [26].)

For general polynomials, Q, we only have the negative result that not all $\exp(-as^6 - bs^4 - cs^2)$ are limits of classical Ising models. In fact both the Lee-Yang theorem and the GHS inequalities fail for some ferromagnets with single spin distributions from this sixth degree class. Of course, we only know that these theorems do not hold for all $(\phi^6)_2$ theories in the lattice approximation. They could be true for all (continuous) $P(\phi)_2$ theories and only be provable by some different approximation method (although this does seem unlikely).

Given our work in § 2 (which is the technical core of the paper), we are able to prove GHS inequalities and a Lee-Yang theorem for $(a\phi^4 + b\phi^2)_2$ theories (§ 3). The applications we give of these results are closely patterned after those in classical statistical mechanics. Our most interesting application of the GHS inequalities is patterned after a theorem of Lebowitz [14]. Typically, it also requires the correlation inequalities of Guerra, Rosen and Simon [12] – in this case those of FKG type. We show (§8) that if $(a\phi^4 + b\phi^2)_2$ has a mass gap in the infinite volume limit, then $\langle\phi(0)\rangle$ for the $(a\phi^4 + b\phi^2 - \mu\phi)_2$ theory is continuous at $\mu = 0$. This is closely connected to "Bogoliubov's criterion" for spontaneous symmetry breaking. Our application of Lee-Yang is classical: We obtain analyticity of the "pressure", i.e. the energy per unit volume [9, 10], in the coefficient of the linear term in P (§ 6).

We close this introduction by emphasizing that in a certain sense, we expect the Lee-Yang theorem to be a more useful tool in quantum field theory than in classical statistical mechanics. The Lee-Yang theorem is known to be a powerful tool in the study of classical Ising models at non-zero field, but in statistical mechanics the main region of interest is the zero field region, especially near the critical temperature. In field theory, there is a different situation. While the region of dynamical instability is of undoubted interest (if it exists!), "normal" field theories are of great direct interest and we hope the Lee-Yang theorem will be a powerful tool in their study.

2. Approximating Continuous Spins by Spin 1/2 Ensembles

This section is the technical heart of the paper. We will show that a continuous probability density proportional to $\exp(-as^4 - bs^2)$ can be well-approximated by the probability density for the total magnetization of ensembles of suitably scaled spin 1/2 Ising spins with ferromagnetic pair interactions.

Let $s_1, \ldots, s_N$ be random variables which can take the values ± 1. A joint probability distribution for them (on $\{-1, 1\}^N$) will be called a *ferromagnetic pair Gibbs measure* if

$$P(s_1, \ldots, s_N) = Z^{-1} \exp(-H(s))$$

where

$$Z = \sum_{s_1 = \pm 1, \ldots, s_N = \pm 1} \exp(-H(s)),$$

$$H(s) = -\Sigma a_{ij} s_i s_j,$$

and where each $a_{ij} \geqq 0$. The associated probability distribution, w, for

$$\mu \equiv \sum_{i=1}^{N} s_i$$

will be called a *ferromagnetic pair magnetization*, i.e. w is a measure on $-N, -N+2, \ldots, N-2, N$ given by

$$w(\mu) = \sum_{s_1, \ldots, s_N \mid \Sigma s_i = \mu} P(s_1, \ldots, s_N).$$

A function, F, on $\mathbb{R}$ is called a *ferromagnetic pair distribution* if there exist constants δ and c, an integer N and a ferromagnetic pair magnetization, $w(\mu)$, $\mu = -N, -N+2, \ldots, N$, so that

$$F(s) = \begin{cases} c\,w(\mu) & \text{if} \quad \delta(\mu-1) \leqq s < \delta(\mu+1) \\ 0 & \text{if} \quad s \geqq \delta(N+1) \quad \text{or} \quad s < (-N-1)\,\delta\,; \end{cases}$$

δ will be called the *mesh* of F. Given F, $\mu_F(s)$ is the unique function with the properties that $\mu_F(s) - N$ is always an even integer and

$$s/\delta - 1 < \mu_F(s) \leqq s/\delta + 1\,.$$

Consequently we have

$$F(s) = c\,w(\mu_F(s))\,.$$

Finally, a function F will be called a *strong limit of ferromagnetic pair distributions* or, for short, a *ferromagnetic limit distribution* (f.l.d.) if and only if there exists a sequence of ferromagnetic pair distributions, $F_n(x)$, with meshes, δ_n, so that:

(1) $\delta_n \to 0$.

(2) $F_n(x) \to F(x)$ for each x.

(3) For each k, there exists a constant D_k with

$$|F_n(x)| \leqq D_k \exp(-k|x|^2)$$

for all n and x.

Remark. Condition (3) rules out the possibility that $F(x)$ be a Gaussian. It is possible to weaken (3) if we require a positive definiteness condition on our eventual continuous spin coupling (defined before Theorem 2). For simplicity of exposition, we avoid this, although as a result, we must add a step to the proofs in § 3.

The main theorem of this section is the following.

Theorem 1. *Let $a > 0$ and let b be real. Then* $\exp(-as^4 - bs^2)$ *is a ferromagnetic limit distribution.*

The idea behind the proof is very simple. We will take

$$H_N = -d_N \left(\sum_{i=1}^{N} s_i \right)^2$$

for suitable d_N so that

$$w(\mu) = \binom{N}{(N+\mu)/2} \exp(d_N \mu^2)$$

where $\binom{a}{b}$ is a binomial coefficient. We will then choose δ_N in such a way that for s fixed, $\mu_N(s) \to \infty$ but $\mu/N \to 0$. The d_N's will be choosen to cancel the quadratic term in μ/N in an expansion of the logarithm of the binomial coefficient as $N \to \infty$ leaving a quartic term.

We first note that since $F(s/\alpha)$ is clearly a f.l.d. if $F(s)$ is, we need only consider the case $a = 1/12$.

Lemma 1. *Let*

$$f(n) = n \log n + 1/2 \log(n+1) - n + 1/2 \log(2\pi)$$

(where $0 \log 0 \equiv 0$*). Then*

(a) $|\log n! - f(n)|$ *is bounded as n runs through* $0, 1, 2, \ldots$

(b) $\lim_{n \to \infty} [\log n! - f(n)] = 0.$

Proof. This is Stirling's formula, with the exception that we have written $1/2 \log(n+1)$ in place of $1/2 \log n$ to allow (a) be be true if $n=0$. See, e.g. [2, pp. 52–54]. □

Lemma 2. *Let* N, μ *be integers whose sum is even and with* $N \geqq |\mu|$, *and let* $x = \mu/N$. *Let*

$$G(N, \mu) = D_N - N h(x) - 1/2 \log(1 - x + 2/N) - 1/2 \log(1 + x + 2/N)$$

where

$$D_N = (N+1) \log 2 + 1/2 \log[(N+1)/N^2] - 1/2 \log 2\pi\,, \tag{1}$$

$$h(x) = 1/2[(1+x)\log(1+x) + (1-x)\log(1-x)]\,. \tag{2}$$

Then

(a) $\left|\log\binom{N}{(N+\mu)/2} - G(N, \mu)\right|$ *is bounded independently of N and* μ.

(b) *For each fixed* $\varepsilon > 0$:

$$\lim_{\substack{N \to \infty \\ |\mu/N| \leqq 1 - \varepsilon}} \left[\log\binom{N}{(N+\mu)/2} - G(N, \mu)\right] = 0\,.$$

Proof. By Lemma 1, it is sufficient to prove that

$$G(N, \mu) = f(N) - f((N+\mu)/2) - f((N-\mu)/2)\,.$$

This follows by straight-forward manipulations. □

Lemma 3. *Let* $h(x)$ *be given by* (2). *Then*

(a) $h(x) \geqq 1/2\, x^2 + 1/12\, x^4$ *for all* x *in* $[-1, 1]$.

(b) $\lim_{x \to 0} x^{-5}[h(x) - 1/2\, x^2 - 1/12\, x^4] = 0$.

Proof. The Taylor series for $h(x)$ is

$$\sum_{n=1}^{\infty} \frac{1}{2n(2n-1)} x^{2n}\,.$$

Since the series converges for $|x| \leqq 1$, (a) and (b) follow. □

Proof of Theorem 1. Let

$$H_N(s_1, \ldots, s_N) = (bN^{-3/2} - N^{-1}/2)\left(\sum_{i=1}^{N} s_i\right)^2$$

$$c_N = \exp(-D_N) \qquad (D_N \text{ given by (1)})$$

$$\delta_N = N^{-3/4}$$

and let $F_N(s)$ be the corresponding pair distributions. For b fixed, $H_N(s)$ is ferromagnetic for N large ($N > 4b^2$) so we need only show that

$$F_N(s) \to \exp(-bs^2 - s^4/12)$$

for each fixed s and that for each k there is a constant R_k with

$$|f_N(s)| \leqq R_k \exp(-k|s|^2) \tag{3}$$

for all s and all N.

If we fix s, then

$$|\mu_N(s) - N^{3/4} s| \leqq 2$$

so that

$$\mu_N(s)/N^{3/4} \to s\,.$$

Now

$$F(s) = c_N \binom{N}{(N + \mu_N(s))/2} \exp\left(\frac{\mu_N(s)^2}{2N} - \frac{b\mu_N(s)^2}{N^{3/2}}\right).$$

Letting

$$x_N(s) = \mu_N(s)/N \to 0\,,$$

we see applying Lemmas 2 and 3, that

$$\begin{aligned}\log F_N(s) &\sim -Nh(x_N(s)) - 1/2 \log\left(1 - x_N(s) + \frac{2}{N}\right)\\ &\quad - 1/2 \log\left(1 + x_N(s) + \frac{2}{N}\right) + \frac{\mu_N(s)^2}{2N} - bs^2\\ &\sim -N\left(\frac{\mu_N(s)^2}{2N^2} + \frac{\mu_N(s)^4}{12N^4} + O\left(\frac{\mu_N(s)^6}{N^6}\right)\right) + \frac{\mu_N(s)^2}{2N} - bs^2\\ &\sim -bs^2 - s^4/12\end{aligned}$$

so

$$F_N(s) \to \exp(-bs^2 - s^4/12)$$

pointwise. Moreover, since

$$-h(x) \leqq -x^2/2 - x^4/12$$

it follows that:

$$\log F(s) \leqq \frac{-b\mu_N(s)^2}{N^{3/2}} - \frac{1}{12}\,\frac{\mu_N(s)^4}{N^3} - K_N(s)$$

if $|s| \leqq (N+1)/N^{3/4}$, and $-\infty$ otherwise. Here,

$$K_N(s) = -1/2 \log(1 - x_N(s) + 2/N) - 1/2 \log(1 + x_N(s) + 2/N)\,.$$

Let

$$K(s) \equiv \sup_{\{N \,|\, |s| \leqq (N+1)/N^{3/4}\}} K_N(s)\,.$$

If N is such that $|x_N(s)| \leqq 1/2$, e.g. if $N \geqq (4s)^4 + 8$, then $K_N(s) \leqq 1/2 \log 2$. On the other hand, if $N \leqq (4s)^4 + 8$, then

$$K_N(s) \leqq -1/2 \log(2/N) \leqq 1/2 \log(128 s^4 + 4).$$

Thus

$$K(s) \leqq 1/2 \log 2 + 1/2 \log(128 s^4 + 4).$$

We conclude that

$$\begin{aligned} F(s) &\leqq (256 s^4 + 8)^{1/2} \exp(|b|\, s^2) \quad \text{for} \quad |s| \leqq 2 \\ &\leqq (256 s^4 + 8)^{1/2} \exp(|b|\, s^2 - (|s| - 2)^4/12), \qquad |s| \geqq 2, \end{aligned}$$

so that (3) follows. □

Now suppose that $F_1(s_1) \ldots F_n(s_n)$ are f.l.d.'s and let a_{ij} be a matrix of positive elements; let $h_i \in \mathbb{R}$. We call the measure

$$dv = G(s)\, d^n s / \int G(s)\, d^n s$$

where

$$G(s) = F_1(s_1) \ldots F_n(s_n) \exp\Big(\sum_{i \neq j} a_{ij} s_i s_j + \sum_i h_i s_i \Big) \tag{4}$$

a *continuous classical ferromagnet* (c.c.f.). If $h_1 = \cdots = h_n = 0$, we say dv is a *zero field c.c.f.*; if $h_1 \geqq 0, \ldots, h_n \geqq 0$, we say dv is a *positive field c.c.f.*

Theorem 2. *Let $\langle \cdot \rangle$ be the expectation value for a positive field c.c.f. Then for any i, j, k:*

$$\begin{aligned} &\langle s_i s_j s_k \rangle + 2 \langle s_i \rangle \langle s_j \rangle \langle s_k \rangle - \langle s_i \rangle \langle s_j s_k \rangle \\ &\quad - \langle s_i s_k \rangle \langle s_j \rangle - \langle s_i s_j \rangle \langle s_k \rangle \leqq 0 . \end{aligned} \tag{5}$$

Proof. Approximate $F_1(\cdot), \ldots, F_n(\cdot)$ with ferromagnetic pair distributions $F_1^{(m)}(\cdot), \ldots, F_n^{(m)}(\cdot)$, let $\mu_i^{(m)}(s)$ be the corresponding μ's and let $\langle\ \rangle_m$ be the expectation value obtained when the F_i in (4) are replaced with $F_i^{(m)}$. Write

$$\mu_i^{(m)} = \sigma_{i1}^{(m)} + \cdots + \sigma_{ik_m}^{(m)} .$$

By the GHS inequalities for spin 1/2 ferromagnets [8],

$$\begin{aligned} &\langle \sigma_\alpha \sigma_\beta \sigma_\gamma \rangle_m + 2 \langle \sigma_\alpha \rangle_m \langle \sigma_\beta \rangle_m \langle \sigma_\gamma \rangle_m \\ &\leqq \langle \sigma_\alpha \rangle_m \langle \sigma_\beta \sigma_\gamma \rangle_m + \langle \sigma_\alpha \sigma_\gamma \rangle_m \langle \sigma_\beta \rangle_m + \langle \sigma_\alpha \sigma_\beta \rangle_m \langle \sigma_\gamma \rangle_m . \end{aligned}$$

Because this inequality is multilinear it remains true if σ_α is replaced by $\mu_i^{(m)}$, etc. Since $\mu_i^{(m)}(s_i)\, \delta_i^{(m)} \to s_i$ as $m \to \infty$ and

$$|\mu_i^{(m)} \delta_i^{(m)}| \leqq (\max \delta_i^{(m)}) (|s_i| + 2),$$

(5) follows from the dominated convergence theorem. □

Theorem 3. *Let $\langle \cdot \rangle$ be the expectation value for a zero field c.c.f. For any complex $h_1, \ldots, h_n$ define*

$$Z(h_1, \ldots, h_n) = \langle \exp(h_1 s_1 + \cdots + h_n s_n) \rangle .$$

Then $Z \neq 0$ provided each h_i is either 0 or has strictly positive real part.

Proof. As in Theorem 2, approximate $\langle\ \rangle$ with $\langle\ \rangle_m$ and Z with

$$Z_m(h) \equiv \langle \exp(h_1 \mu_1^{(m)}(s_1) + \cdots + h_n \mu_n^{(m)}(s_m))\rangle .$$

Then by the dominated convergence theorem $Z_m(h)$ converges to $Z(h)$ uniformly on compacts of $\mathbb{C}^n$. By relabeling indices suppose $\mathrm{Re}\,h_1, \ldots, \mathrm{Re}\,h_k > 0$, $h_{k+1} = \cdots = h_n = 0$. Then $Z(h_1, \ldots, h_k, 0, \ldots, 0)$ is not identically 0 (since $Z(0, \ldots, 0) = 1$) and by the classical Lee-Yang theorem [15],

$$Z_m(h_1, \ldots, h_k, 0, \ldots, 0) \neq 0 \quad \text{if} \quad \mathrm{Re}\,h_1 > 0, \ldots, \mathrm{Re}\,h_k > 0 .$$

The theorem now follows from the lemma below. □

The following lemma follows from the argument principle (see e.g. [21]):

Lemma 4. *Let $f_1, \ldots, f_n, \ldots$ be functions analytic on a connected open set $D \subset \mathbb{C}^k$. Suppose that $f_n \to f$ uniformly on compact subsets of D and that for some open $D' \subset D$, f_n is non-vanishing on D' for all n. Then either $f \equiv 0$ or f is non-vanishing on D!*

We are now able to prove

Theorem 4. *Not every distribution* $\exp(-P(s))$ *for* $P(s)$ *an even semibounded sixth degree polynomial is a* f.l.d.

Remarks. 1. We will actually prove more; namely we will show that the Lee-Yang theorem fails for some sixth degree polynomials, and the same is true of the GHS inequality.

2. By using the method of Theorem 1, but adding a $(\Sigma s_i)^4$ term, it is easy to prove that any $\exp(-as^6 - bs^4 - cs^2)$ distribution, $a > 0$, can be approximated by ferromagnets with two and four body interactions. Hence we also see that the Lee-Yang theorem cannot hold for arbitrary ferromagnets with four body interactions, and the same is true for the GHS inequalities (neither result is new).

Proof. Let $P(s) = q[s^2(s-1)^2(s+1)^2] + 2\varepsilon[s^2 - s^4/2] - 1/2 \log q$. Then $P(s)$ has minima at $s = 0, \pm 1$ with

$$\begin{aligned} P'(0) &= 0, & P''(0) &= 2[q + 2\varepsilon] \\ P'(\pm 1) &= 0, & P''(\pm 1) &= 8[q - \varepsilon] \\ P(\pm 1) - P(0) &= \varepsilon . \end{aligned}$$

In the Gaussian approximation,

$$e^{-P(s)}\, ds \to \sqrt{\pi}\, \{\delta(s) + e^{-\varepsilon}[\delta(s-1) + \delta(s+1)]/2\}$$

and, in fact, it is easy to prove that

$$\lim_{q\to\infty} \int_{-\infty}^{\infty} f(s)\, e^{-P(s)}\, ds = \sqrt{\pi}\, \{f(0) + e^{-\varepsilon}[f(1) + f(-1)]/2\}$$

for any f continuous near 0, 1 and -1 and obeying

$$|f(s)| \leqq \text{const} \exp(s^6)\,.$$

Thus by following Theorem 2 (resp. 3) we could conclude a GHS inequality (resp. Lee-Yang theorem) for single spin distributions

$$dv = \delta(s+1) + \delta(s-1) + \alpha\delta(s)$$

if a GHS inequality (resp. Lee-Yang theorem) held for all

$$\exp(-as^6 - bs^4 - cs^2)\,.$$

a) *Lee-Yang theorem.* If $\alpha = 3$,

$$G(h) = \int \exp(hs)\, dv = x + x^{-1} + 3$$

(if $x = e^{-h}$) has its zeros when x is real, not when $|x| = 1$. So the Lee-Yang theorem fails for some sixth degree polynomial self-couplings.

b) *GHS inequalities.* If

$$M = \int s \exp(sh)\, dv / \int \exp(hs)\, dv$$

then explicit computation shows that

$$\frac{\partial^2 M}{\partial h^2} = \frac{x(1-x)}{[1+\alpha x + x^2]^3} \{\alpha + (8 + \alpha - \alpha^2)(x + x^2) + \alpha x^3\}$$

where $x = e^{-h}$. If GHS held, we would have $\partial^2 M/\partial h^2 \leqq 0$ for all $x > 1$. Because of the $-\alpha^2$ term, this evidently fails for α large. □

This theorem *suggests* that the Lee-Yang theorem *may* not hold for $P(\phi)_2$ theories with $\deg P \geqq 6$.

3. Theorems for the $(\phi^4)_2$ Euclidean Field Theory

In [12], a spatially cutoff $P(\phi)_2$ Euclidean field theory was approximated by a continuous Ising ferromagnet. The Ising spins were coupled by a ferromagnetic pair interaction and the individual spins have an unperturbed distribution of the form $\exp(-Q(s))$ where

$$Q(s) = \sum_{n=0}^{2N} a_n s^n \quad \text{if} \quad P(X) = \sum_{n=0}^{2N} b_n X^n$$

and, in general, the highest odd powers of P and Q agree. Thus, by Theorem 1, if P is of the special form $P(X) = aX^4 + bX^2 - \mu X$, then the approximation of [12] is by c.c.f.'s. As a result, it is easy to prove:

Theorem 5. *(GHS Inequalities for $(\phi^4)_2$.) Let $\langle\ \rangle$ be the expectation value for a spatially cutoff $P(X)$ Euclidean field theory, where*

$$P(X) = aX^4 + bX^2 - \mu X$$

with $a > 0$, $\mu \geqq 0$. Then for any positive C^∞ functions f, g, h of compact support:

$$\begin{aligned}&\langle\phi(f)\,\phi(g)\,\phi(h)\rangle + 2\langle\phi(f)\rangle\,\langle\phi(g)\rangle\,\langle\phi(h)\rangle - \langle\phi(f)\rangle\,\langle\phi(g)\,\phi(h)\rangle\\&\quad - \langle\phi(f)\,\phi(h)\rangle\,\langle\phi(g)\rangle - \langle\phi(f)\,\phi(g)\rangle\,\langle\phi(h)\rangle \leqq 0\,.\end{aligned}$$

Theorem 6. *(Lee-Yang theorem for $(\phi^4)_2$.) Let $\langle\ \rangle_0$ denote the free Euclidean field theory expectation value. Let $a > 0$, b real and let $g, h_1, \ldots, h_n$ be positive C^∞ functions on $\mathbb{R}^2$ with compact support. Let*

$$F(\lambda_1, \ldots, \lambda_n) = \Big\langle \exp\Big(-\int g(x) :a\phi^4(x): + b:\phi^2(x): + \sum_{i=1}^{n} \lambda_i \int h_i(x)\,\phi(x)\Big)\Big\rangle_0$$

for $(\lambda_1, \ldots, \lambda_n) \in \mathbb{C}^n$. Then $F(\lambda) \neq 0$ if $\mathrm{Re}\,\lambda_1, \ldots, \mathrm{Re}\,\lambda_n$ are all strictly positive.

Using the lattice approximation, these theorems follow from Theorems 2 and 3 in just the way that Theorems 2 and 3 follow from the classical Ising model theorems.

Remarks. 1. Technically, we have only proved Theorems 2 and 3 for $\exp(-Q(s))$; $Q(s) = as^4 + bs^2$ if $a > 0$ and not if $a = 0$, $b > 0$. Thus, in proving Theorems 5 and 6 we should pick some $h \in \mathcal{S}$ with h strictly positive, consider first $g + \varepsilon h$ cutoffs and then take ε to 0.

2. Once these theorems are proven with smooth cutoffs and smearing functions they extend to more general cutoffs by a limiting argument.

3. In Theorem 6, $\langle\ \rangle_0$ can be replaced with a Dirichlet or half-Dirichlet expectation value [12] without changing the proof. A similar change is possible in Theorem 5.

4. Since the mass renormalization in $(\phi^4)_3$ and the mass and coupling constant renormalizations in $(\phi^4)_4$ do not change the degree, these theorems should be provable in higher dimensions once one controls local ultraviolet divergences.

Important partial control on the ultraviolet divergences in $(\phi^4)_3$ has been recently obtained by Glimm-Jaffe [27]. Since these results are in terms of removing a conventional momentum cutoff rather than a lattice cutoff and since they are boundedness theorems rather than convergence theorems, we are not yet in possession of classical Ising theorems for $(\phi^4)_3$ but we regard [27] as a hopeful indication.

4. First Application of GHS: Concavity of the "Magnetization"

Our first application of the GHS inequality mimics the original application of Griffiths, Hurst, and Sherman [8]. By an extension [12] of a result of Nelson [18], if $P(X)=aX^4+bX^2-\mu X$, then the Dirichlet boundary condition states for $P(\phi)_2$ have an infinite volume limit. (See [12] for a discussion of Dirichlet and the related half-Dirichlet theory.) We denote this state as $\langle\ \rangle_p$. It is a translationally invariant state, so $\langle\phi(x)\rangle_p$ is a number $M(P)$. For $P(X)=aX^4+bX^2-\mu X$ $(a>0)$, we write $M(a,b,\mu)$ for $M(P)$. This M is the analogue of the magnetization in an Ising magnet. The properties of M are summarized in:

Theorem 7. *Fix $a>b$, b real. In the region $\mu>0$, $M(a,b,\mu)$ is*

(1) *Strictly positive.*

(2) *Strictly monotone.*

(3) *Concave.*

(4) *Continuous.*

Remark. From $M(a,b,-\mu)=-M(a,b,\mu)$, one can read off properties in the region $\mu<0$.

Proof. That M is positive and monotone is a consequence of the Griffiths inequalities (see [12]); (4) follows from (3) and given (3), the strict monotonicity and strict positivity follow from the fact that M is not bounded, which we prove in a lemma below. This leaves the proof of (3). Fix $f\in C_0^\infty(\mathbb{R}^2)$, positive. Then, letting $\langle\ \rangle_{P,\Lambda}$ denote the Dirichlet state in volume Λ, we see that

$$\left(\int f\, d^2x\right) M(P)=\lim_{|\Lambda|\to\infty}\langle\phi(f)\rangle_{P,\Lambda}$$

by Nelson's result [18]. Thus it is sufficient to prove that

$$M_\Lambda(a,b,\mu)\equiv\langle\phi(f)\rangle_{aX^4+bX^2-\mu X,\Lambda}$$

is concave in μ for each fixed Λ with $\Lambda\supset\operatorname{supp} f$. Letting $Q(X)=aX^4+bX^2$, we obtain:

$$M_\Lambda(a,b,\mu)=\frac{\langle\phi(f)\exp(\mu\phi(\chi_\Lambda))\rangle_{Q,\Lambda}}{\langle\exp(\mu\phi(\chi_\Lambda))\rangle_{Q,\Lambda}}$$

where, here and below, χ_Λ is the characteristic function of the set Λ. Thus $M_\Lambda(a,b,\mu)$ is analytic in μ and

$$\begin{aligned} d^2M_\Lambda(a,b,\mu)/d\mu^2=&\langle\phi(f)\,\phi(\chi_\Lambda)^2\rangle+2\langle\phi(f)\rangle\langle\phi(\chi_\Lambda)\rangle^2\\ &-\langle\phi(f)\rangle\langle\phi(\chi_\Lambda)^2\rangle-2\langle\phi(f)\,\phi(\chi_\Lambda)\rangle\langle\phi(\chi_\Lambda)\rangle\end{aligned}$$

where

$$\langle\ \rangle\equiv\langle\ \rangle_{Q-\mu X,\Lambda}\,.$$

Thus by Theorem 5, $d^2M_\Lambda/d\mu^2\leqq 0$ and M_Λ is concave in μ. □

Lemma 5. *$M(a, b, \mu)$ is not bounded from above.*

Proof. Suppose that $M(a, b, \mu) \leqq M_0$ for all μ. Let $Q(X) = aX^4 + bX^2$. Then

$$\langle \phi(f) \rangle_{Q-\mu X} \leqq M_0 \int f \, d^2 x$$

for all positive f. Since

$$\langle \phi(f) \rangle_{Q-\mu X, \Lambda} \leqq \langle \phi(f) \rangle_{Q-\mu X}$$

[18], we see that

$$\langle \phi(f) \rangle_{Q-\mu X, \Lambda} \leqq M_0 \int f \, d^2 x$$

for all $\mu \geqq 0$ and all Λ with $\Lambda \supset \operatorname{supp} f$. Suppose that $\|f\|_\infty \leqq 1$. By the second Griffiths inequality,

$$G(\alpha) = \frac{\langle \phi(f) \exp\{\mu[\phi(f) + \alpha \phi(\chi_\Lambda - f)]\} \rangle}{\langle \exp\{\mu[\phi(f) + \alpha \phi(\chi_\Lambda - f)]\} \rangle}$$

is monotone increasing as α increases. Thus:

$$\frac{\langle \phi(f) \, e^{\mu \phi(f)} \rangle_{Q, \Lambda}}{\langle e^{\mu \phi(f)} \rangle_{Q, \Lambda}} \leqq M_0 \int f \, d^2 x$$

for all $\mu > 0$. Letting ν be the probability distribution for $\phi(f)$ in $\langle \ \rangle_{Q, \Lambda}$, we see that

$$\int x e^{\mu x} \, d\nu \leqq \left(M_0 \int f \, d^2 x\right) \int e^{\mu x} \, d\nu$$

for all $\mu > 0$. It follows that ν has support in $[-\infty, M_0 \int f \, d^2 x]$. Since ν is symmetric,

$$\langle \phi(f)^2 \rangle_{Q, \Lambda} \leqq \left(M_0 \int f \, d^2 x\right)^2 .$$

Since Λ is arbitrary, we conclude that for all $f \in C_0^\infty$ and $0 \leqq f \leqq 1$,

$$\langle \phi(f)^2 \rangle_Q \leqq \left(M_0 \int f \, d^2 x\right)^2 . \tag{6}$$

Since $\langle \ \rangle_Q$ is a Euclidean invariant state obeying positive definiteness (in the sense of Symanzik [20] and Nelson [17]), and Nelson's reflection axiom [17], there is a positive measure, $\varkappa$, on $(0, \infty)$ with

$$\langle \phi(f)^2 \rangle_Q \geqq \int d\kappa(m^2) \int f(x) \, f(y) \, S_m(x-y) \, dx \, dy \tag{7}$$

where $S_m(x-y)$ is the Euclidean propagator for the free field of mass m. Moreover $\int d\kappa(m^2) = 1$ since the convergence of the Dirichlet Schwinger functions implies the convergence of the Dirichlet Wightman functions [5, 12] which in turn implies that

$$\langle \phi(x) \, \dot{\phi}(y) \rangle - \langle \dot{\phi}(y) \, \phi(x) \rangle = i\delta(x-y) .$$

Because of the logarithmic singularity of $S_m(x)$ at $x = 0$, (6) and (7) are inconsistent. □

5. Second Application of GHS: Monotonicity of the Mass Gap

This application is modelled after one of Lebowitz [14] in his study of Ising magnets:

Theorem 8. *Fix $a>0$ and b real. Let $\langle\ \rangle_{\mu,g}$ denote the expectation value for the $:a\phi^4+b\phi^2-\mu\phi:$ Euclidean field theory with cutoff $g\in C_0^\infty(\mathbb{R}^2)$. Let f, h be positive test functions. Then*

$$\langle\phi(f)\,\phi(h)\rangle_{\mu,g}-\langle\phi(f)\rangle_{\mu,g}\langle\phi(h)\rangle_{\mu,g}$$

is a monotone decreasing function of μ in the region $\mu\geqq 0$.

Remark. The same result holds if $\langle\ \rangle_{\mu,g}$ is replaced with a Dirichlet state.

Proof. The given function is analytic in μ and its derivative is negative by Theorem 5. □

It was proven in [19] that the rate of falloff of the truncated two point function

$$\langle\phi(f)\,\phi(h)\rangle_T\equiv\langle\phi(f)\,\phi(h)\rangle-\langle\phi(f)\rangle\,\langle\phi(h)\rangle$$

as $\mathrm{dist}(\mathrm{supp} f, \mathrm{supp}\, h)\to\infty$ determines the mass gap. Thus Theorem 8 implies:

Theorem 9. *Fix $a>0$ and b real. Let $m(\mu)$ denote the (Hamiltonian theory) mass gap for $(a\phi^4+b\phi^2-\mu\phi)_2$ in one of the following situations:*

(a) *With fixed Fock spatial cutoff $g\in L^{1+\varepsilon}+L^2$* [11].

(b) *With Dirichlet boundary conditions in an interval* $[0, l]$ [12].

(c) *Dirichlet or half-Dirichlet states in the infinite volume limit* [18, 12].

(d) *Free states in the infinite volume limit when a, b and μ are all small* [5].

Then $m(\mu)$ is monotone increasing in the region $\mu\geqq 0$ (in case (d), *for μ small and $\mu\geqq 0$).*

Proof. Let us prove (c). The others are the same. Let $\mu_1>\mu_2>0$. Then going to the $|\Lambda|\to\infty$ limit in Theorem 8,

$$\langle\phi(f)\,\phi(h)\rangle_{T,\mu_1}\leqq\langle\phi(f)\,\phi(h)\rangle_{T,\mu_2}$$

if f and h have disjoint compact supports. Fix f with support in $\{(x,t)\,|\,t<0\}$, let $\tilde{f}$ be the reflection of f in the $f=0$ axis and let $\tilde{f}_t$ be the translate of $\tilde{f}$. Then

$$\langle\phi(\tilde{f}_t)\,\phi(f)\rangle_{T,\mu_2}\leqq C\exp(-m(\mu_2)\,t)$$

for all t. Since this therefore holds for $\langle\ \rangle_{T,\mu_1}$, the theorems in [19] imply $-m(\mu_1)\leqq -m(\mu_2)$. □

6. Analyticity of the Energy per Unit Volume

This application is patterned after the original application by Yang and Lee [25] of their theorem. We need the following lemma which does not seem to be readily available in just this form in the literature:

Lemma 6. *Let f_n be a sequence of functions analytic in a region $R \subset \mathbb{C}$. Suppose that*

(a) *$f_n(z)$ converges for z in D a determing subset of R ($D \subset R$ is called determining if every function analytic in R vanishing on D is 0).*

(b) *For every compact $K \subset R$,*

$$\sup_{z \in K; n} \operatorname{Re} f_n(z) < \infty .$$

Then there exists a function f analytic in R so that f_n converges to f uniformly on compacts of R.

Proof. Let $g_n = \exp(f_n)$. Then, by (a), $g_n(z)$ converges to a non-zero number for each $z \in D$. By (b),

$$\sup_{n; z \in K} |g_n(z)| < \infty ,$$

so by the Vitali convergence theorem [21], $g_n \to g$ uniformly on compacts. Since g is non-zero on D, g is non-zero on all of R by Lemma 4. For any closed curve C in R,

$$\oint_C \frac{g_n'(z)}{g_n(z)} dz = 0$$

so this remains true if g replaces g_n. Thus $f = \log g$ is a single-valued analytic function in R and $f_n \to f$. □

Theorem 10. *Fix $a > 0$, b. Let $\alpha_\infty^{(D)}(\mu)$ be the energy per unit volume [9, 10, 12] for the $(a\phi^4 - b\phi^2 - \mu\phi)_2$ field theory. Then $\alpha_\infty^{(D)}(\mu)$ is real analytic on $(0, \infty)$ and has an analytic continuation to the half plane $\{\mu \mid \operatorname{Re}\mu > 0\}$. Moreover for any μ in the half plane*

$$\alpha_\infty^{(D)}(\mu) = \lim_{|\Lambda| \to \infty} \frac{1}{|\Lambda|} \log \left\langle \exp\left(- \int_\Lambda (a : \phi^4(x): + b : \phi^2(x): - \mu\phi(x))\, dx\right) \right\rangle_{\Lambda, d}$$

where $\langle \cdot \rangle_{\Lambda, d}$ is the Dirichlet boundary condition free field in Λ.

Remarks. 1. The limit can be taken as $|\Lambda| \to \infty$ in the Fisher sense.

2. This theorem holds with free boundary condition pressure [9, 10, 12], i.e. $\langle\ \rangle_0$ replacing $\langle\ \rangle_{\Lambda, d}$. For use in the next section we emphasize Dirichlet states.

Proof. Consider the functions

$$\alpha_\Lambda^{(D)}(\mu) = \frac{1}{|\Lambda|} \log \langle \cdots \rangle_{\Lambda, d} \,.$$

By the zero theorem, $\alpha_\Lambda(\mu)$ is analytic in the half plane where $\mathrm{Re}\,\mu > 0$. For any μ,

$$\left|\left\langle \exp\left(-\int_\Lambda a:\phi^4(x)+b\phi^2(x): -\mu\phi(x)\right)\right\rangle_{\Lambda,d}\right|$$
$$\leqq \left\langle \exp\left(-\int_\Lambda a:\phi^4(x)+b\phi^2(x): -\mathrm{Re}\,\mu\phi(x)\right)\right\rangle_{\Lambda,d}$$

so

$$\mathrm{Re}\,\alpha_\Lambda^{(D)}(\mu) \leqq \alpha_\Lambda^{(D)}(\mathrm{Re}\,\mu) \leqq \alpha_\infty^{(D)}(\mathrm{Re}\,\mu)$$

where we have used the superadditivity of the Dirichlet pressure in the last step [12]. Thus $\mathrm{Re}\,\alpha_\Lambda^{(D)}(\mu)$ is uniformly bounded on compacts of the right half plane. Moreover, by [12], $\alpha_\Lambda^{(D)}(\mu)$ converges to $\alpha_\infty^{(D)}(\mu)$ for μ real. The theorem follows from Lemma 6. □

Henceforth we will drop the superscript from $\alpha_\infty^{(D)}$ and $\alpha_\Lambda^{(D)}$.

7. Magnetization is the Derivative of the Pressure

Formally, the magnetization is the derivative of the pressure. We show this for the $\mu \neq 0$ region:

Theorem 11. *Fix $a>0$, b. Let $\alpha_\infty(\mu)$ be the Dirichlet pressure for $(a\phi^4+b\phi^2-\mu\phi)_2$. Then*

$$M(a,b,\mu) = d\alpha_\infty(\mu)/d\mu$$

for all $\mu \neq 0$.

Proof. By symmetry, we need only consider the $\mu>0$ region. Since $\alpha_\Lambda(\mu) \to \alpha_\infty(\mu)$ uniformly on compacts of the right half μ plane, $d\alpha_\Lambda/d\mu$ converges to $d\alpha_\infty/d\mu$ for $\mu>0$. Thus, we need only prove that for fixed $\mu>0$,

$$M(a,b,\mu) = \lim_{n\to\infty} \frac{1}{|\Lambda_n|} \langle \phi(\chi_{\Lambda_n})\rangle_{\Lambda_n,d} \tag{8}$$

for some sequence Λ_n going to ∞ (Fisher). On the one hand, by the monotonicity of $\langle\phi(f)\rangle_{\Lambda,d}$ in Λ [18],

$$\langle\phi(\chi_\Lambda)\rangle_{\Lambda,d} \leqq \langle\phi(\chi_\Lambda)\rangle_{\infty,d} \equiv |\Lambda|\, M(a,b,\mu)\,.$$

On the other hand, let $\varepsilon>0$ and let R_0 denote the unit square centered at the origin. Then, for all large Λ, say for Λ containing the square of side s centered at the origin

$$\langle\phi(\chi_{R_0})\rangle_{\Lambda,d} \geqq (M(a,b,\mu)-\varepsilon)\,.$$

By translation covariance, for any unit square R_i

$$\langle\phi(\chi_{R_i})\rangle_{\Lambda,d} \geqq (M(a,b,\mu)-\varepsilon)$$

so long as the square of side s with the same center as R_i is contained in Λ. Let Λ_n be the square of side $(s-1)+n$ centered at the origin. Then n^2

non-overlapping unit squares of the type just discussed can fit inside Λ_n. By the first Griffiths inequality,

$$\langle \phi(\chi_{\Lambda_n}) \rangle \geqq \langle \phi(\chi_{n^2 R_i}) \rangle$$

so

$$\langle \phi(\chi_{\Lambda_n}) \rangle \geqq n^2 (M(a, b, \mu) - \varepsilon).$$

Since $n^2/(n+s-1)^2 \to 1$, (8) follows. □

Corollary 1. *Fix $a > 0$, b real. Then $M(a, b, \mu)$ is real analytic in the region $\mu > 0$ and has a continuation to the entire right half plane.*

Corollary 2. *The pressure $\alpha_\infty(\mu)$ is strictly monotone and strictly convex in the region $\mu > 0$.*

8. The Definition of a Phase Transition in $(\phi^4)_2$

The last corollary of Theorem 11 says:

Theorem 12. *Fix $a > 0, b$. Then $\alpha_\infty(\mu)$ is right differentiable at $\mu = 0$ and its right derivative is the limit as μ decreases to 0 of $M(a, b, \mu)$. In particular, the following are equivalent:*

(1) *$\alpha_\infty(\mu)$ is differentiable at $\mu = 0$.*

(2) $\lim_{\mu \downarrow 0} M(a, b, \mu) = 0$.

(3) *$M(a, b, \mu)$ is continuous at $\mu = 0$.*

Proof. By the second Griffiths inequality [12], the limit as μ decreases to 0 of $M(a, b, \mu)$ exists. Call it M_0. Since

$$\alpha_\infty(\mu) - \alpha_\infty(0) = \int_0^\mu \alpha'_\infty(\lambda)\, d\lambda,$$

Theorem 11 implies that

$$M_0 \mu \leqq \alpha_\infty(\mu) - \alpha_\infty(0) \leqq \mu M(a, b, \mu).$$

Thus the limit as μ decreases to zero of $[\alpha_\infty(\mu) - \alpha_\infty(0)]/\mu$ exists and equals M_0. Given this, (1) and (2) are clearly equivalent. That (2) and (3) are equivalent follows from $M(a, b, -\mu) = -M(a, b, \mu)$. □

Theorem 12 shows that two possible definitions of "phase transition" agree. The following result, patterned after a study of Lebowitz [14], shows a relation to a third common definition.

Theorem 13. *If the infinite volume (Dirichlet) $:a\phi^4 + b\phi^2:$ field theory has a mass gap, then $M(a, b, \mu)$ is continuous in μ at $\mu = 0$.*

Proof. Let $\alpha \in \mathbb{Z}^2$, let Λ_α be the unit square centered at α and let χ_α be its characteristic function. By the technique of the transfer matrix

[12, 16], if $j > 1$

$$\langle \phi(\chi_{0,0})\, \phi(\chi_{j,k}) \rangle \leqq \exp[-(j-1)\, m]\, \langle \phi(\chi_{0,0})\, \phi(\chi_{1,0}) \rangle$$

where $\langle\ \rangle$ is the infinite volume (Dirichlet) state for $:a\phi^4 + b\phi^2:$ and m is the mass gap. By Euclidean symmetry and monotonicity of $\langle\ \rangle_{\Lambda,d}$ in Λ, we conclude that

$$\langle \phi(\chi_{0,0})\, \phi(\chi_{j,k}) \rangle_{\Lambda,T} \leqq e^{-m[\min(|j|,|k|)-1]}\, \langle \phi(\chi_{0,0})\, \phi(\chi_{0,1}) \rangle$$

where $\langle\ \rangle_{\Lambda,T}$ is the truncated two point function and $(j,k) \neq (0,0)$. By using Theorem 8,

$$\langle \phi(\chi_{0,0})\, \phi(\chi_{j,k}) \rangle_{\mu,\Lambda,T} \leqq C e^{-m[\min(|j|,|k|)-1]}\,. \tag{9}$$

In (9), there is no restriction on j, k for we take

$$C = \max(\langle \phi(\chi_{0,0})\, \phi(\chi_{0,0}) \rangle, \langle \phi(\chi_{0,0})\, \phi(\chi_{0,1}) \rangle)\,.$$

C is finite because the two point function can be written in a Kallen-Lehmann representation with spectral integral on $[m, \infty)$ and with total weight 1 (see the proof of Lemma 5), so that the coincident point singularity is only logarithmic.

Now, let Λ be a finite union of Λ_α including $\Lambda_{0,0}$, say $\Lambda = \bigcup_{\alpha \in I} \Lambda_\alpha$. Then for any $\mu > 0$,

$$\begin{aligned} \frac{d}{d\mu} \langle \phi(\chi_{0,0}) \rangle_{\Lambda,\mu} &= \langle \phi(\chi_{0,0}) \sum_{\alpha \in I} \phi(\chi_\Lambda) \rangle_{T,\Lambda,\mu} \\ &\leqq C \sum_{(j,k) \in \mathbb{Z}^2} e^{-m[\min(|j|,|k|)-1]} = D < \infty \end{aligned}$$

by (9). Thus for all such Λ and any $\mu > 0$

$$|\langle \phi(\chi_{0,0}) \rangle_{\Lambda,\mu}| \leqq D\mu\,.$$

Taking Λ to ∞, we find that

$$|M(a,b,\mu)| \leqq D\mu$$

for $\mu > 0$. Thus M is continuous at $\mu = 0$. □

Remark. It may be possible by further mimicking Lebowitz to prove that M is C^∞ at $\mu = 0$ if there is a mass gap.

9. How Good is the Conventional Wisdom?

It is perhaps useful to summarize what we know about the $a\phi^4 + b\phi^2 - \mu\phi$ theory for fixed a as far as mass gap and magnetization is concerned. This summary uses results from [5, 12] as well as from this paper. We suppose the bare mass is large. Then one of the following holds:

(a) There is a mass gap for all b, μ. Also, $M(a, b, \mu)$ is continuous in μ for each fixed b,

or

(b) There is a critical value of b, call it b_1. There is a mass gap if $b > b_1$ for all μ and not one if $\mu = 0$ and $b < b_1$. Again, $M(a, b, \mu)$ is continuous in μ for each fixed b,

or

(c) There are critical values of b, call them b_1 and b_2 with $b_2 \leqq b_1$. There is a mass gap if $b > b_1$ for all μ and not one if $\mu = 0$ and $b < b_1$. $M(a, b, \mu)$ is continuous in μ if $b > b_2$ and discontinuous at $\mu = 0$ if $b < b_2$.

Moreover, in any event,

(i) $M(a, b, \mu)$ is real analytic in μ away from $\mu = 0$.

(ii) $M(a, b, \mu)$ is monotone non-decreasing as μ increases or, for $\mu > 0$, as b decreases.

(iii) $m(b, \mu)$ is monotone non-decreasing as $|\mu|$ is increased with b fixed or as b is decreased with $\mu = 0$.

Of course, we expect that (c) occurs and that $b_1 = b_2$.

It is useful to compare what can be shown rigorously with what Wightman calls "the conventional wisdom" [22, 23]. Let us describe this picture, often associated with Goldstone. Given a polynomial, P, we define

$$\tilde{P}(X) = P(X) + \tfrac{1}{2} m_0 X^2$$

where m_0 is the bare mass. Then the conventional wisdom asserts:

(α) The $P(\phi)_2$ theory with bare mass m_0 leads to a theory with a unique vacuum (and unique "equilibrium state") if and only if $\tilde{P}(X)$ has a unique minimum at a point X_0.

(β) If $\tilde{P}(X)$ has a unique minimum, X_0 is $\langle \phi(x) \rangle$ in the infinite volume theory.

(γ) In case $\tilde{P}(X)$ has a unique minimum, the physical mass is given by $m_{\mathrm{Gold}} \equiv P''(X_0) + m_0$.

(δ) If $\tilde{P}(X)$ has a non-unique minimum, say at $X_1, \ldots, X_k$, then the infinite volume theory has a k-fold degenerate vacuum and the k pure states obtained by a central decomposition have $\langle \phi(x) \rangle$ and the physical masses given by X_i and $P''(X_i) + m_0$.

Aside from its crudeness, this picture has the defect of being invariant under the formal change $\langle P, m_0 \rangle \to \langle P - \tfrac{1}{2} \delta m_0 X^2, m_0 + \delta m_0 \rangle$ while the $:P(\phi)_2:$ theories are not because of the difference in Wick ordering. Nevertheless, we can think of the conventional wisdom as a qualitative picture and see how it compares with the qualitative results of this paper. Fix $Q(X) = aX^4 + bX^2$ and consider the family of polynomials $P_\mu(X) = Q(X) - \mu X$. Then, in the conventional wisdom picture:

(1) $\tilde{P}_\mu$ has a unique minimum at a point $X_0(\mu) > 0$ if $\mu > 0$. (If $\tilde{P}_\mu$ has minima at a and b it is of the form $c_1 + c_2(X-a)^2(X-b)^2$. The vanishing of the coefficient of X^3 implies that μ, the coefficient of X, also vanishes.)

(2) Let X_+ be the largest minimum for $\tilde{Q}(X)$. Notice that $X_+ = 0$ if and only if $\tilde{Q}(X)$ has a unique minimum, in which case $X_0(\mu)$ is real analytic at $\mu = 0$. In any case, $X_+ = \lim_{\mu \downarrow 0} X_0(\mu)$.

(3) Differentiating $P'_\mu(X_0(\mu)) = 0$ with respect to μ, we find the striking equation:

$$m_{\text{Gold}}(\mu)\,(\partial X_0/\partial\mu) = 1 \tag{10}$$

which implies monotonicity of X_0.

(4) By (10),

$$\frac{\partial m_G}{\partial\mu}\,\frac{\partial X_0}{\partial\mu} = -\,m_G(\mu)\,\frac{\partial^2 X}{\partial\mu^2},$$

so concavity of X_0 is equivalent to monotonicity of m_{Gold}.

(5) The quantity

$$m_{\text{Gold}}(\mu) = 12aX_0^2(\mu) + 2b + m_0$$

is monotone increasing in μ and [by (4)], $X_0(\mu)$ is concave.

At the risk of belaboring the obvious, we point out that (1) agrees with Corollary 1 to Theorem 11, (2) with Theorem 12, (4) and (5) with Theorems 7 and 9.

Acknowledgements. We would like to thank Professors J. Lebowitz and E. Lieb for valuable discussions. In addition one of us (B.S.) would like to thank Professor N. Kuiper for the hospitality shown him while a visitor at the I.H.E.S. and Professor A. Visconti for the hospitality of the C.N.R.S. The other (R.B.G.) is grateful for the hospitality of the Cornell Chemistry Department.

References

1. Asano, T.: Phys. Rev. Lett. **24**, 1409 (1970); — J. Phys. Soc. Jap. **29**, 350 (1970)
2. Feller, W.: An introduction to probability theory and its applications, Vol. I. Third Edition. New York: Wiley 1968
3. Fortuin, C. M., Kasteleyn, P. W., Ginibre, J.: Commun. math. Phys. **22**, 89 (1971)
4. Ginibre, J.: Commun. math. Phys. **16**, 310 (1970)
5. Glimm, J., Spencer, T.: N.Y.U. Preprint
6. Griffiths, R. B.: J. Math. Phys. **8**, 478, 484 (1967)
7. Griffiths, R. B.: J. Math. Phys. **10**, 1559 (1969)
8. Griffiths, R. B., Hurst, C. A., Sherman, S.: J. Math. Phys. **11**, 790 (1970)
9. Guerra, F.: Phys. Rev. Lett. **28**, 1213 (1972)
10. Guerra, F., Rosen, L., Simon, B.: Commun. math. Phys. **27**, 10 (1972)
11. Guerra, F., Rosen, L., Simon, B.: Commun. math. Phys. **29**, 233 (1973)
12. Guerra, F., Rosen, L., Simon, B.: Ann. Math., to appear

13. Kelly, D. G., Sherman, S.: J. Math. Phys. **9**, 466 (1968)
14. Lebowitz, J.: Commun. math. Phys. **28**, 313 (1972)
15. Lee, T. D., Yang, C. N.: Phys. Rev. **87**, 410 (1952)
16. Nelson, E.: Proc. A.M.S. 1971 Summer Symposium at Berkeley
17. Nelson, E.: J. Func. Anal. **12**, 97 (1973)
18. Nelson, E.: Proc. 1973 Erice Summer School in Mathematical Physics, to appear
19. Simon, B.: Commun. math. Phys. **31**, 127—136 (1973)
20. Symanzik, K.: In: Jost, R. (Ed.): Local Quantum Theory. New York: Academic Press 1969
21. Titchmarsh, E. C.: The theory of functions, 2nd ed. London: Oxford University Press 1939
22. Wightman, A. S.: Phys. Today **22**, No. 9, 53 (1969)
23. Wightman, A. S.: In: Proc. Coral Gables Conference 1972, to appear
24. Bitter, F.: Introduction to ferromagnetism, p. 153. New York: McGraw-Hill Book Co. 1937
25. Yang, C. N., Lee, T. D.: Phys. Rev. **87**, 404 (1952)
26. Fisher, M. E.: Repts. Progr. Phys. **30**, 615 (1967)
27. Glimm, J., Jaffe, A.: Fort. für Phys., to appear

Barry Simon
Department of Mathematics and Physics
Princeton University
Princeton, N.J. 08540, USA