Correlation inequalities and the mass gap in $P(\phi)_2$. II. Uniqueness of the vacuum for a class of strongly coupled theories

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Abstract: We prove uniqueness of the vacuum in the infinite volume (half-Dirichlet) $P(\phi)_2$ theory when $P(X) = aX^4 + bX^2 - \mu X$; $a > 0$, $\mu \neq 0$. This completes the proof of the Wightman axioms for such theories.

This paper is a contribution to the $P(\phi)_2$ quantum field theory whose foundations have been presented by Glimm-Jaffe [3] and others (see [4], [5], [15] for reviews of results and a guide to the literature).

It is a development in the program proposed and initiated by Guerra, Rosen and Simon [7] of using statistical mechanical methods in constructive field theory. This program, in turn, is an offshoot of the introduction of Euclidean techniques and probability ideas in field theory by Symanzik [19] and Nelson [10].

In order to describe our main result, we recall the definition of half-Dirichlet states [7]. Throughout this paper, we only deal with half-Dirichlet states and fields, so we do not add a superscript "D" as was done in [7]. We also fix a bare mass, m, throughout. Given a bounded open region $\Lambda \subset \mathbf{R}^2$, we write $-\Delta_\Lambda$ for the Dirichlet boundary condition Laplacian on $L^2(\Lambda, d^2x)$, i.e. the Friedrichs extension of $-\Delta$ on $C_0^\infty(\Lambda)$. $H_{-1}(\Lambda)$ denotes the completion of $L^2(\Lambda)$ in norm $\langle \cdot, (-\Delta_\Lambda + m^2)^{-1} \cdot \rangle^{1/2}$. The free Dirichlet theory in region Λ is the Gaussian random field over $H_{-1}(\Lambda)$ with mean zero and covariance matrix:

$$\langle \phi(f)\phi(g) \rangle_{0,\Lambda} = \big(f, (-\Delta_\Lambda + m^2)^{-1} g\big) .$$

We will denote the expectation value with respect to this Gaussian measure by $\langle \cdot \rangle_{0,\Lambda}$. If P is a semi-bounded polynomial, the expectation value

$$\langle \cdot \rangle_{P,\Lambda} = \frac{\langle \cdot \exp\big(-U(\chi_\Lambda)\big) \rangle_{0,\Lambda}}{\langle \exp\big(-U(\chi_\Lambda)\big) \rangle_{0,\Lambda}} ,$$

where $U(g) = \int g(x) \colon P(\phi(x)) \colon dx$ and where χ_Λ is the characteristic function

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of Λ, defines the interaction field in region Λ with Dirichlet boundary conditions.

Here : : denotes μ_0 (free) Wick ordering; hence the term half-Dirichlet. If the polynomial P intended is clear, we write $\langle\ \rangle_\Lambda$.

The importance of half-Dirichlet states was pointed out by Nelson [12] who noted that monotonicity of the Schwinger functions resulted. Bounds allowing the convergence of the Schwinger functions are due to Guerra et al. [7] and Fröhlich [2]. The existence of a half-Dirichlet transfer matrix [17] then implies convergence of the Wightman functions. One therefore has (see e.g. [17] for details):

THEOREM. *Let $P(X) = Q(X) - \mu X$ where Q is an even semibounded polynomial. If $f_1, \cdots, f_n$ are functions in $C_0^\infty(\mathbf{R}^2)$ with disjoint supports, then $\lim_{\Lambda\to\infty} \langle\phi(f_1)\cdots\phi(f_n)\rangle_{P,\Lambda}$ exists and is the (smeared) Schwinger function of a theory obeying all the Wightman axioms except possibly uniqueness of the vacuum.*

The basic inequality of Nelson is the following consequence of the second Griffiths inequality [7]:

(1) $$\langle\phi(f_1)\cdots\phi(f_n)\rangle_\Lambda \leqq \langle\phi(f_1)\cdots\phi(f_n)\rangle_{\Lambda'}$$

if

$$\Lambda\subset\Lambda'\ ,\ f_1\geqq 0, \cdots, f_n\geqq 0\ ,\ \mu\geqq 0\ .$$

One handles the case $\mu<0$ by using

(2) $$\langle\phi(f_1)\cdots\phi(f_n)\rangle_{Q+\mu X,\Lambda} = (-1)^n < \phi(f_1)\cdots\phi(f_n)\rangle_{Q-\mu X,\Lambda}\ .$$

We write $\langle\ \rangle_{P,\infty}$ for the infinite volume expectation values.

Our goal in this paper is to prove:

THEOREM. *Let $P(X) = aX^4 + bX^2 - \mu X$ where $a>0$ and $\mu\neq 0$. Then the Wightman theory defined by $\langle\ \rangle_{P,\infty}$ possesses a unique vacuum.*

Remarks 1. This completes the proof of the Wightman axioms for a class of $P(\phi)_2$ theories without any restriction on the magnitude of the overall coupling constant. For small coupling constant (high temperature in the statistical mechanical picture) and arbitrary semibounded polynomial, the uniqueness of the vacuum has already been proved by Glimm-Jaffe-Spencer [6].

2. Our theorem is motivated by a result of Lebowitz and Penrose [9] who prove exponential falloff of truncated correlation functions for finite-range, pair-coupled Ising ferromagnets at non-zero magnetic field. We do not see how to mimic their proof which depends on a Mayer expansion since this expansion relies on the boundedness and discreteness of Ising spins. The use of the Lee-Yang zero theorem to control truncated vacuum expectation

values was suggested by another argument in [9].

3. We expect that $\langle\ \rangle_{P,\infty}$ possesses a mass gap under the hypotheses of the theorem.*

4. This theorem is "predicted" by the "conventional wisdom" model of [20], [18].

5. By equation (2) we need only consider the case $\mu > 0$.

6. Without *a priori* knowledge of uniqueness of the vacuum in $\langle\ \rangle_{P,\infty}$, one could construct theories with unique vacuum by a decomposition procedure applied to $\langle\ \rangle_{P,\infty}$. See [1].

The proof of this theorem uses many of the techniques recently developed in the statistical mechanical approach to $P(\phi)_2$ [7], [12], [16], [18]. The basic idea is very simple.

On the one hand, we will show, using FKG inequalities (indirectly) and the "transfer matrix", that if $\langle\cdot\rangle_\infty$ does not have a unique vacuum, then

$$\langle\phi(\chi_\Lambda)\phi(\chi_\Lambda)\rangle_{T,\infty} \geqq C\,|\,\Lambda\,|^2;\ C > 0 \tag{3}$$

where

$$\langle\phi(f)\phi(g)\rangle_{T,\cdots} \equiv \langle\phi(f)\phi(g)\rangle_{\cdots} - \langle\phi(f)\rangle_{\cdots}\langle\phi(g)\rangle_{\cdots}$$

and $|\,\Lambda\,|$ = volume of Λ. On the other hand using the Lee-Yang theorem, and the second Griffiths inequality, we will prove for squares Λ, that

$$\langle\phi(\chi_\Lambda)\phi(\chi_\Lambda)\rangle_{T,\infty} \leqq d\,|\,\Lambda\,|;\ d < \infty\ . \tag{4}$$

LEMMA 1. *$\langle\phi(x)\phi(y)\rangle_{T,\infty}$ is a function, f, of $|x-y|$. $f(\cdot)$ is a real analytic, monotone decreasing, positive function on* $(0, \infty)$.

Remarks 1. This result holds in any scalar theory obeying the Osterwalder-Schrader axioms [13] without necessarily having unique vacuum.

2. A proof can also be based on the Källén-Lehmann representation [14].

Proof. That the truncated two-point function only depends on $|x-y|$ is a consequence of Euclidean invariance. Real analyticity is a consequence of the fact that the non-coincident points in the Euclidean region lie in the permuted extended forward tube [13]. Pick $g \in C_0^\infty(\mathbf{R}^2)$ with support in $\{\langle a, t\rangle \mid t < 0\}$. Let $g_s(a, t) = g(a, s-t)$. Then, there is a vector, ψ, in the physical Hilbert space so that

$$\langle\phi(g)\phi(\tilde{g}_s)\rangle_{T,\infty} = (\psi, e^{-sH}\psi) \tag{5}$$

for all $s > 0$ where H is the Hamiltonian on the physical Hilbert space. For example, $\psi = E_0(\phi(g)) - (\Omega_0, \phi(g))\Omega_0$ in the Nelson language [11] and $\psi = V(\theta g) - (\Omega_0, \phi(g)\Omega_0)\Omega_0$ in the Osterwalder-Schrader language [13]. By (5) and

* This has been proven by Guerra, Rosen, Simon (submitted to Commun. Math. Phys.).

the spectral theorem, $\langle\phi(g)\phi(g_s)\rangle_{T,\infty}$ is monotone decreasing and positive. Letting $g \to \delta$, the Dirac measure at (0, 0), we complete the proof. □

LEMMA 2. *$\langle\cdot\rangle_\infty$ has a unique vacuum if and only if*

$$\lim_{|x-y|\to\infty}\langle\phi(x)\phi(y)\rangle_{T,\infty} = 0 .$$

In particular, if there is not a unique vacuum, then there is a $C > 0$ with

$$\langle\phi(x)\phi(y)\rangle_{T,\infty} \geqq C$$

for all x, y.

Proof. If $\langle\cdot\rangle_\infty$ has a unique vacuum, then by equation (5), $\langle\phi(g)\phi(\tilde{g}_s)\rangle_{T,\infty}\to 0$ as $s \to \infty$. Taking $g \to \delta$ and using the fact that the resulting convergence is uniform for $s \in [1, \infty)$, we see that the limit is 0. Conversely, if the limit is 0, then $\lim_{s\to\infty}\langle\phi(g)\phi(\tilde{g}_s)\rangle_{T,\infty} = 0$ for any $g \in C_0^\infty$ with support in $\{\langle a, t\rangle \mid t<0\}$. It follows from the FGK inequalities [7] that $\langle\ \rangle_\infty$ has a unique vacuum (this is the main theorem, Theorem 6, of [16]). The second statement follows from the first and Lemma 1. □

This proves (3).

The basic estimate going into the proof of equation (4) is:

LEMMA 3. *For all squares, Λ, of side bigger than* 1

(6) $$\langle\phi(\chi_\Lambda)\phi(\chi_\Lambda)\rangle_{T,\Lambda} \leqq d\,|\Lambda|$$

for some $d < \infty$.

Remark. Once we have Lemma 4, the condition that Λ have side bigger than 1 can be dropped.

Proof. Let

$$\alpha_\Lambda(\mu) = \frac{1}{|\Lambda|}\ln\left\langle\exp\left(-\int_\Lambda(a\colon\phi^4(x)\colon + b\colon\phi^2(x)\colon - \mu\phi(x))dx\right)\right\rangle_{0,\Lambda} .$$

The functions $\alpha_\Lambda(\mu)$ are entire in μ (a, b fixed with $a > 0$ and b real). By a result in [8], they converge when μ is real, as $|\Lambda| \to \infty$, suitably to α_∞. By the Lee-Yang theorem [18] (see especially Theorem 10 of [18]), $\alpha_\Lambda(\mu)$ converges to a function $\alpha_\infty(\mu)$ uniformly on compacts of the right half plane. In particular, by the Cauchy integral formula, $d^2\alpha_\Lambda/d\mu^2$ converges for any $\mu > 0$. Thus

$$\mathrm{Sup}_{\{\Lambda\mid\Lambda\text{ is a square, }|\Lambda|\geqq 1\}}\, d^2\alpha_\Lambda/d\mu^2 \equiv d < \infty .$$

Since

$$\frac{d^2}{d\mu^2}\alpha_\Lambda(\mu) = \frac{1}{|\Lambda|}\langle\phi(\chi_\Lambda)\phi(\chi_\Lambda)\rangle_{T,\,aX^4+bX^2-\mu X} ,$$

(6) follows. □

LEMMA 4. *Let d be given by* (6). *Then for any square* Λ,

$$\langle\phi(\chi_\Lambda)\phi(\chi_\Lambda)\rangle_{T,\infty} \leqq d\,|\Lambda|\ .$$

Proof. Suppose not. Then for some $\varepsilon > 0$ and some square Λ_0 centered at the origin, $\langle\phi(\chi_{\Lambda_0})\phi(\chi_{\Lambda_0})\rangle_{T,\infty} \geqq (d + 2\varepsilon)\,|\Lambda_0|$. By equation (1), the limit $\Lambda \to \infty$ is in the sense of the direction given by inclusion. (We make use of the fact that even though, *a priori*, $\langle\phi(\chi_{\Lambda_0})\phi(\chi_{\Lambda_0})\rangle_\infty$ may be ∞, $\langle\phi(\chi_{\Lambda_0})\rangle_\infty^2$ is finite.) Thus, there is some square Λ' centered at the origin so that $\Lambda' \subset \Lambda$ implies that $\langle\phi(\chi_{\Lambda_0})\phi(\chi_{\Lambda_0})\rangle_{T,\Lambda} \geqq (d + \varepsilon)\,|\Lambda_0|$. Let $l_0 = |\Lambda_0|^{1/2}$ and $|\Lambda'|^{1/2} = l_0 + 2a$. By translation covariance, we conclude that if Λ_α is any square of side l_0 so that the square with the same center and side $l_0 + 2a$ is contained in Λ, then

$$\langle\phi(\chi_{\Lambda_\alpha})\phi(\chi_{\Lambda_\alpha})\rangle_{T,\Lambda} \geqq (d + \varepsilon)l_0^2\ . \tag{7}$$

Let $\Lambda^{(k)}$ be the square of side $kl_0 + 2a$ centered at the origin. $\Lambda^{(k)}$ can be decomposed into a "corridor" of width a about its boundary and k^2 squares Λ_α, each with side l_0, mutually disjoint, and so that each obeys the geometric condition required for (7) to hold with $\Lambda = \Lambda^{(k)}$. By the second Griffiths inequality (6):

$$\begin{aligned}\langle\phi(\chi_{\Lambda^{(k)}})\phi(\chi_{\Lambda^{(k)}})\rangle_{T,\Lambda^{(k)}} &\geqq \textstyle\sum_{\alpha=1}^{k^2} \langle\phi(\chi_{\Lambda_\alpha}),\ \phi(\chi_{\Lambda_\alpha})\rangle_{T,\Lambda^{(k)}}\\ &\geqq (d + \varepsilon)(kl_0)^2\ .\end{aligned}$$

Choosing k so that $(d + \varepsilon)(kl_0)^2 > d(kl_0 + 2a)^2$, we obtain a contradiction to Lemma 3. □

Our theorem follows directly from Lemmas 2 and 4.

Acknowledgment: I should like to thank Prof. J. M. Combes for the hospitality of Centre Universitaire, Toulon.

Appendix A: Uniformity of convergence of the one-point function

Geometric considerations of the type used in the proof of Lemma 4 have led us to an improvement of equation (8) of [18] which we feel may be of some use:

PROPOSITION 1. *Let* $M = \langle\phi(x)\rangle_{P,\infty}$ *where* $P(X) = Q(X) - \mu X$, *Q an even polynomial,* $\mu \geqq 0$. *Then for any* $\varepsilon > 0$, *there is a* δ *so that*

$$M - \varepsilon \leqq \langle\phi(x)\rangle_{P,\Lambda} \leqq M \tag{7}$$

for any region Λ *and any* $x \in \Lambda$ *with* dist $(x, \partial\Lambda) > \delta$.

Remarks 1. (7) is interpreted in a "distributional" sense; i.e., it means that there is a neighborhood of x so that for any non-negative function, f, with support in the neighborhood

$$(M - \varepsilon)\left(\int f d^2x\right) \leqq \langle\phi(f)\rangle_{P,\Lambda} \leqq M\left(\int f d^2x\right).$$

2. It must be true that $x \longrightarrow \langle\phi(x)\rangle_{P,\Lambda}$ is continuous, in which case (7) holds pointwise and our proof below can be made less wordy.

3. By equation (2), if $\mu < 0$, the proposition holds if $\langle\phi(x)\rangle_{P,\Lambda}$ is replaced by $\langle -\phi(x)\rangle_{P,\Lambda}$ in equation (7) and in the definition of M.

Proof. Let S be the unit square centered at the origin. Let Λ_0 be a square centered at the origin of side $r > 1$ so that

$$\langle\phi(\chi_S)\rangle_{\Lambda_0} \geqq (M - \varepsilon) \equiv \langle\phi(\chi_S)\rangle_\infty - \varepsilon .$$

Let n be a positive integer. Since S is the disjoint union of n^2 squares of side $1/n$, we can find some square, S_n, of side $1/n$ so that

$$\langle\phi(X_{S_n})\rangle_{\Lambda_0} \geqq (M - \varepsilon)n^{-2} .$$

Let Λ_1 be the square of side $r + 2$ centered at the origin. For any square, M_n, of side $1/n$, with $M_n \subset S$, we claim that

$$\langle\phi(X_{M_n})\rangle_{\Lambda_1} \geqq (M - \varepsilon)n^{-2} . \tag{8}$$

(8) comes from translation covariance, the monotonicity of equation (1) and the fact that we can find $\Lambda_0' \subset \Lambda_1$ so that the geometric relation of M_n inside Λ_0' is identical to that of S_n inside Λ_0. By approximating with Riemann sums,

$$\langle\phi(f)\rangle_{\Lambda_1} \geqq (M - \varepsilon)\int f d^2x \tag{9}$$

if $\operatorname{supp} f \subset S$, and $f \geqq 0$. Pick $\delta = (r + 2\sqrt{2})$. Given any x and Λ with $x \in \Lambda$, $\operatorname{dist}(x, \partial\Lambda) > \delta$, we can fit the square Λ_1' of side $(r + 2)$ and center x inside Λ. (7) follows from equations (1) and (9). □

Appendix B: Non-unique vacuums and spontaneous magnetization

As a typical application of our main theorem (together with correlation inequalities), we have:

PROPOSITION 2. *Let $Q(X) = aX^4 + bX^2$ with $a > 0$. If $\langle\ \rangle_{Q,\infty}$ does not have a unique vacuum, then*

$$\lim_{\mu\downarrow 0} \langle\phi(0)\rangle_{Q-\mu X,\infty} > 0 = \langle\phi(0)\rangle_{Q,\infty} .$$

Remarks 1. The existence of the limit on the left is proved in [7].

2. Consider the following four meanings of "dynamical instability" in the $Q(\phi)_2$ theory:

(a) $\langle\ \rangle_{Q,\infty}$ does not have a unique vacuum.

(b) $\langle\ \rangle_{Q,\infty}$ does not have a mass gap.

(c) $\lim_{\mu \downarrow 0} \langle \phi(0) \rangle_{Q-\mu X,\infty} > 0$.

(d) The pressure $\alpha_\infty(\mu)$ is not differentiable at $\mu = 0$.

Combining Proposition 2 with results of [18], we have

$$(a) \Longrightarrow (c) \Longleftrightarrow (d) \Longrightarrow (b) .$$

Proof: On account of Lemma 2, if $\langle \ \rangle_{Q,\infty}$ does not have a unique vacuum, then $\lim_{x\to\infty} \langle \phi(x)\phi(0) \rangle_{Q,\infty} \equiv C^2 > 0$. But by the Griffiths inequalities (see [7]) $\langle \phi(x)\phi(0) \rangle_{Q-\mu X}$ is monotone increasing in μ. Thus $\lim_{x\to\infty} \langle \phi(x)\phi(0) \rangle_{Q-\mu X} \geqq C^2$ if $\mu > 0$. By the theorem

$$\lim_{x\to\infty} \langle \phi(x)\phi(0) \rangle_{Q-\mu X} = \left(\langle \phi(0) \rangle_{Q-\mu X}\right)^2 \text{ if } \mu > 0 .$$

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(Received April 19, 1973)
(Revised October 30, 1973)