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THE PRESSURE IS INDEPENDENT OF THE BOUNDARY CONDITIONS FOR $P(\phi)_2$ FIELD THEORIES

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This research announcement represents a continuation of our program [6] of applying statistical mechanical methods to study the $P(\phi)_2$ Euclidean quantum field theory [11], [18], [15]. Our main results are:

(a) The pressures and the ground state energies for different boundary conditions converge to the same infinite volume limit.

(b) The Gibbs variational equality for the entropy density is satisfied.

(c) For interactions of the type $P(x) = \lambda x^4 - \mu x$ with $\lambda > 0$ and $\mu \neq 0$, the infinite volume Dirichlet theory has a mass gap.

The free field of mass $m > 0$ (in two dimensions) is the Gaussian process ϕ indexed by $S(\mathbf{R}^2)$ with variance

$$\int \phi(f)^2 \, d\mu_0 = \langle f, (-\Delta + m^2)^{-1} f \rangle, \tag{1}$$

where $\langle \cdot , \cdot \rangle$ denotes the $L^2(\mathbf{R}^2)$ inner product and Δ is a two-dimensional Laplacian. It is convenient to realize $d\mu_0$ as a Borel measure on $S'(\mathbf{R}^2)$ [15]. Given a bounded rectangle Λ in $\mathbf{R}^2$, we introduce three additional Gaussian processes, indexed by $C_0^\infty(\Lambda)$, with variance given by the analogue of (1) but with Δ replaced by the Laplacian on $L^2(\Lambda)$ with Dirichlet, Neumann or periodic boundary conditions. We denote the corresponding measures by $d\mu_{0,\Lambda}^X$, $X = D, N$, or P. (In the cases of Dirichlet and Neumann boundary conditions, Λ may be taken to be any bounded open region.)

Let P be a polynomial bounded below as a function on $\mathbf{R}$ and normalized by $P(0) = 0$. For $\Lambda \subset \mathbf{R}^2$ a bounded rectangle, define the inter-

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action in volume Λ by

$$U_\Lambda = \int_\Lambda :P(\phi(x)): d^2x, \tag{2}$$

where : : denotes that the Wick subtractions are made with respect to $d\mu_0$ (for the definition see, for instance, [3], [13], or [15]). U_Λ^X denotes the analogue of (2) with Wick subtractions made with respect to $d\mu_{0,\Lambda}^X$ instead of $d\mu_0$.

According to the basic correspondence with statistical mechanics, the free pressure (i.e. free boundary conditions) in the region Λ is defined by

$$\alpha_\Lambda = |\Lambda|^{-1} \log\left(\int \exp(-U_\Lambda)\, d\mu_0\right), \tag{3}$$

where $|\Lambda|$ is the area of Λ. Similarly the X-pressure α_Λ^X is defined as in (3) but with $d\mu_0$ replaced by $d\mu_{0,\Lambda}^X$ *and* U_Λ by U_Λ^X. The half-X pressure α_Λ^{HX} is defined by replacing $d\mu_0$ by $d\mu_{0,\Lambda}^X$ but leaving U_Λ unchanged.

The Schwinger functions are defined as the moments of the above measures. Thus

$$S_\Lambda^X(x_1, \cdots, x_n) = \frac{\int \phi(x_1)\phi(x_2)\cdots\phi(x_n)\exp(-U_\Lambda^X)\, d\mu_\Lambda^X}{\int \exp(-U_\Lambda^X)\, d\mu_\Lambda^X}, \tag{4}$$

and similarly for the half-X Schwinger functions S_Λ^{HX} where U_Λ^X in (4) is replaced by U_Λ.

In [4], [5] it was established that $\alpha_\infty = \lim_{\Lambda\to\infty}\alpha_\Lambda$ exists. The main technical result we wish to announce here is

THEOREM 1. *For any semibounded polynomial P, the limits $\alpha_\infty^X = \lim_{\Lambda\to\infty}\alpha_\Lambda^X$ and $\alpha_\infty^{HX} = \lim_{\Lambda\to\infty}\alpha_\Lambda^{HX}$ ($X = D, P$ or N) all exist and equal α_∞.*

REMARKS. 1. For the analogous problem in ordinary statistical mechanics, there is extensive literature on the independence of pressure on boundary conditions; see e.g. [1], [2], [12].

2. The existence of α_∞^D and the inequality $\alpha_\infty^D \leqslant \alpha_\infty$ are to be found in our paper [6]. Spencer [17] has proved the convergence of α_Λ^P in the case of "large external field".

3. Detailed proofs of the assertions in Theorem 1 will be presented elsewhere [7]. A preliminary and somewhat more complex version of our

proof of the equality $\alpha_\infty = \alpha_\infty^D$ has appeared in [15].

4. Rather than being precise here on the sense in which the regions Λ go to infinity, we note that any sequence of rectangles going to infinity in both directions will do. More general conditions on $\Lambda \longrightarrow \infty$ will be specified in [7].

While Theorem 1 is of some interest in its own right, it is primarily useful as a technical tool. We conclude with a brief description of several applications:

1. There are a variety of "translations" of Theorem 1 into statements about the Fock space energy per unit volume. For instance (see [7] for a proof and for related results):

THEOREM 2. *Let E_V denote the ground state energy of the periodic Hamiltonian on an interval of length V as defined by Glimm and Jaffe* [3]. *Then*

$$\lim_{V\to\infty} - E_V/V = \alpha_\infty .$$

2. Theorem 1 plays a role in the proof of the Gibbs variational *equality* (for notation see [6]).

THEOREM 3. *For any semibounded polynomial P,*

$$\alpha_\infty(P) = \sup_f\, [s(f) - \rho(f, P)]$$

where the supremum is taken over all tempered, translation invariant states, $s(f)$ is the entropy density, and $\rho(f, P)$ is the mean interaction.

Details appear in [7]. A sketch of the main idea has been presented in Guerra's contribution to [18].

3. The main use of Theorem 1 is to provide flexibility in the choice of boundary conditions. For example, the equality $\alpha_\infty^P = \alpha_\infty^D$ is an ingredient in Spencer's proof [17] that, when $P(x) = \lambda x^4 - \mu x$, α_∞ is jointly real analytic in λ for $\lambda > 0$ and in complex μ with $\operatorname{Re} \mu \neq 0$.

4. It is an interesting question whether the infinite volume Schwinger functions $S^X = \lim_{\Lambda\to\infty} S_\Lambda^X$ exist and are independent of X. When $P(x) = \lambda x^4 - \mu x$, it is known that S^D exists [6], [15] and that S^P exists [17] for sufficiently large μ, i.e. $|\mu| \geqslant c(\lambda)$ for some $c(\lambda)$. Theorem 1, the GKS inequalities [6], the Simon-Griffiths results [16], and the fact that the Schwinger functions are related to derivatives of α_∞ allow one to prove [7]:

THEOREM 4. *Let* $P(x) = \lambda x^4 - \mu x$ *and suppose that the infinite volume Dirichlet theory with* $\mu = 0$ *has a mass gap. Then for* $|\mu| \geqslant c(\lambda)$, $S^D = S^P$.

5. By using Theorem 1, Spencer's results, [17] and a new technique of Lebowitz and Penrose [10], we can improve the result of [14] as follows [8]:

THEOREM 5. *For* $P(x) = \lambda x^4 - \mu x$, $\mu \neq 0$, *the infinite volume Dirichlet theory has a mass gap.*

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