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Correlation Inequalities and the Mass Gap in $P(\phi)_2$

III. Mass Gap for a Class of Strongly Coupled Theories with Nonzero External Field

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Abstract. We consider the infinite volume Dirichlet (or half-Dirichlet) $P(\phi)_2$ quantum field theory with $P(X)=aX^4+bX^4+bX^2-\mu X\,(a>0)$. If $\mu \neq 0$ there is a positive mass gap in the energy spectrum. If the gap vanishes as $\mu \to 0$, it goes to zero no faster than linearly yielding a bound on a critical exponent.

§ 1. Introduction

In this paper, we discuss various aspects of the $P(\phi)_2$ Euclidean field theory [28, 23]. In the statistical mechanical approach to these theories which we have advocated elsewhere [10] (see also our contributions to [28]), one of the subprograms concerns the use of Ising model techniques. These techniques are especially useful in the study of the $:a\phi^4+b\phi^2-\mu\phi:_2$ theory where both the lattice approximation [10] and classical Ising approximation [24] are available. In fact, in II of this series [21], we used these techniques to complete the proof of the Wightman axioms for these theories when $\mu \neq 0$. In essence, the result of that note was that 0 was a simple eigenvalue of the Hamiltonian in the infinite volume Dirichlet theory. Using very different techniques, based in part on the cluster expansion of [7, 8], Spencer [25] proved that the theories with $|\mu|$ large (and periodic B.C.) have a mass gap, i.e. that 0 is a simple, *isolated*, eigenvalue of the Hamiltonian. Our goal in this note is to extend this result to any $\mu \neq 0$.

As before, our proof is modelled on a result in the theory of Ising models, namely the recent work of Lebowitz and Penrose [14, 15] on clustering. They, in turn, rely on subharmonicity ideas first introduced by Penrose and Elvey [16]. In the present context, this basic idea of "superharmonic continuation" is very simple and beautiful: Let $m_l(\mu)$ be the mass gap for the (periodic) Hamiltonian on $[-l/2, l/2]$ with interaction polynomial $P(X)=aX^4=bX^2-\mu X$. We show that $m_l(\mu)$ has a continuation to a *nonnegative superharmonic* function $M_l(\mu)$ in the region $\mathrm{Re}\,\mu>0$ where the Lee-Yang theorem of the classical Ising approximation applies [24]. Now for large real μ, Spencer [25] assures us that $M_l(\mu)$ is bounded

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away from zero independently of l. It follows from the theory of superharmonic functions that $M_l(\mu)$ is strictly positive for all $\mathrm{Re}\,\mu > 0$ (independently of l). (For the reader's convenience, we summarize the facts about superharmonic functions that we shall use in an appendix.)

Were it not for the technicalities of boundary conditions, our proof would be quite brief; for, as we shall see, with the proper incantations about compact operators, the Lebowitz-Penrose argument easily extends from the Ising case where the cutoff transfer matrix is a discrete semigroup of finite matrices to the field theory case where the cutoff (periodic) transfer matrix is a continuous semigroup of compact operators (see § 2). The difficulty with boundary conditions is the following: Spencer's results are proven with periodic B.C. while it is only for Dirichlet and Half-Dirichlet B.C., that we know the Schwinger functions are convergent for all μ. By a somewhat elaborate sequence of arguments (§ 4), we are able to circumvent this difficulty with a minimum of new proofs but only by appealing to our theorem [11] on the independence of pressure on B.C. and by using ϕ-bounds [5] and Frohlich bounds [3] for periodic states (§ 3).

While we succeed in proving the existence of a mass gap without too many technical estimates, it is at a high cost since we leave open two questions which we expect could be settled if we systematically extended the cluster expansion of Glimm-Jaffe-Spencer [8]. First, we do not establish the infinite volume convergence of the periodic states if $\mu \neq 0$, even though in principle this should be a consequence of the uniform mass gap (see [8]). Secondly, while we prove the existence of a mass gap in the infinite volume Dirichlet theories, we do not prove that an l-independent mass gap exists in the finite l theories. In applications, this would be a considerable technical advantage. While Spencer's proof is much simplified by using B.C. invariant under translation of the fields it might be possible to carry it through with Dirichlet B.C. in which case our methods in § 2 would give the result of an l-independent gap. We remark that by an argument in [11], if one solves both of the above questions affirmatively, then the Dirichlet and periodic states agree.

§ 2. Superharmonicity of the Mass Gap

We first note the following theorem generalizing a result of Lebowitz-Penrose in the finite matrix case:

Theorem 2.1[1]**.** *Let X be a Banach space and let $A(\mu)$ be an operator-valued analytic function on $\Omega \subset \mathbb{C}$. Let* $\mathrm{spr}(A(\mu))$ *denote the spectral radius of $A(\mu)$. Then* $\ln \mathrm{spr}(A(\mu))$ *is subharmonic on Ω.*

Proof. Consider first the function $N(\mu) = \ln \|A(\mu)\|$. As a function with values in $\mathbb{R} \cup \{-\infty\}$, N is clearly continuous and so upper semicontinuous. For each $\chi \in X$, $l \in X^*$ let $N_{\chi,l}(\mu) = \ln |l(A(\mu)\chi)|$. Then $N_{\chi,l}$ is subharmonic by Theorem A.3. Since $N(\mu) = \sup\{N_{\chi,l}(\mu) \mid \|\chi\| = \|l\| = 1\}$, N is subharmonic by Theorem A.4. Similarly $N_n(\mu) \equiv 1/2^n \ln \|A(\mu)^{2^n}\|$ is subharmonic. Clearly $N_n(\mu)$ is monotone non-increasing so $\lim_{n\to\infty} N_n(\mu) = \ln \mathrm{spr}(A(\mu))$ is subharmonic by Theorem A.5. □

Now we want to discuss spatially cutoff $(\phi^4)_2$ Hamiltonians with periodic B.C. We use freely results from [11].

[1] Added in proof: A similar result appear in [30].

Lemma 2.2. *Let* $P(X)=aX^4+bX^2; a>0$. *Fix* $l>0$. *Let* $H(0)$ *denote the periodic Hamiltonian (transfer matrix) in interval* $(-l/2, l/2)$ *with interaction* P *and let*

$$H(\mu)=H(0)-\mu \int_{-l/2}^{l/2} \phi(x)\,dx .$$

Then:

(a) $H(\mu)$ *is an entire analytic family of generators of holomorphic semigroups and in particular* $B(\mu)=e^{-H(\mu)}$ *is a bounded operator-valued analytic function.*

(b) $B(\mu)$ *is compact for all* $\mu \in \mathbb{C}$.

(c) *For all* μ *with* $\operatorname{Re}\mu>0$, *all* n *and all positive test functions* f, g *on* $(-l/2, l/2)$

$$Z(\mu)\equiv (e^{\phi(f)}\Omega_0, B(\mu)^n e^{\phi(g)}\Omega_0) \neq 0 .$$

Proof. (a) and (b) follow easily from the facts that $H(0)$ is bounded below with compact resolvent [11] and that $\int_{-l/2}^{l/2} \phi(x)\,dx$ is an $H(0)$-form bounded perturbation with relative bound 0 (see e.g. [13; p. 498] or [18; II; § X.2]).

In terms of Euclidean fields ϕ_E,

$$Z(\mu)=\int \exp\left[-\int_{-n/2}^{n/2} dt \int_{-l/2}^{l/2} dx(:P(\phi_E(x,t)):\right.$$
$$\left. -(\mu+f(x)\,\delta(t+\tfrac{1}{2}n)+g(x)\,\delta(t-\tfrac{1}{2}n))\,\phi_E(x,t)\,d\mu_l^P\right.$$

where $d\mu_l^P$ is the Gaussian measure with covariance $(-\Delta+m^2)^{-1}$ with periodic B.C. on the sides of the strip $[-l/2, l/2]\times\mathbb{R}$. By passing to the lattice approximation [10] and then further to the classical Ising approximation [24] we can approximate $Z(\mu)$ by (analytic) functions $Z_j(\mu)$ to which the Lee-Yang Theorem applies, i.e. $Z_j(\mu)\neq 0$ provided $\operatorname{Re}\mu\neq 0$. It is easy to check that this approximation is uniform for μ in compact subsets of $\mathbb{C}$. It follows from Hurwitz' Theorem that $Z(\mu)$ is identically zero or nowhere zero in $\operatorname{Re}\mu>0$. But for real μ, $B(\mu)$ is positivity improving so that the inner product is positive [22]. □

Now, for any $f, g \geqq 0$ on $(-l/2, l/2)$, let

$$F_n(\mu; f, g)=n^{-1}\ln(e^{\phi(f)}\Omega_0, B(\mu)^n e^{\phi(g)}\Omega_0)$$

where we use (c) above to define a continuous logarithm in $\operatorname{Re}\mu>0$ choosing the unique value of the logarithm which assures us that F_n is real if $\mu>0$.

Lemma 2.3. (a) *For all* μ *with* $\operatorname{Re}\mu>0$, $\lim_{n\to\infty} F_n(\mu; f, g)$ *exists, is analytic in* μ *and independent of* f *and* g. *Let* $\alpha(\mu)$ *denote the limit.*

(b) $|e^{\alpha(\mu)}|=\operatorname{spr}(B(\mu))$.

(c) $e(\mu)\equiv e^{\alpha(\mu)}$ *is the unique eigenvalue of* $B(\mu)$ *whose magnitude is* $\operatorname{spr}(B(\mu))$ *and it is a simple isolated eigenvalue of* $B(\mu)$.

(d) *The spectral projection* $P(\mu)$ *corresponding to* $e(\mu)$ *is analytic in* $\operatorname{Re}\mu>0$.

Proof. (a) Let $\Sigma(\mu)=\operatorname{spr}B(\mu)$, $\psi(f)=e^{\phi(f)}\Omega_0$ and Ω_μ the (unique positive) vacuum vector for $H(\mu)$. For any $f, g\geqq 0$, the standard Lee-Yang argument ([19]; see also [24]) together with the convergence of $F_n(\mu; f, g)$ when μ is real implies that $F_n(\mu; f, g)\to F(\mu; f, g)$ as $n\to\infty$ and that the limit is analytic (see [24, Theorem 10]). But for μ real, $F(\mu; f, g)=-\inf\sigma(H(\mu))$ since $\langle\psi(f), \Omega_\mu\rangle>0$

(both are strictly positive vectors). Since $F(\mu; f, g)$ is independent of f, g for μ real, it is independent of f, g for all μ with $\mathrm{Re}\,\mu > 0$ by analytic continuation.

(b) Clearly

$$|e^{F_n}| \leqq \|\psi(f)\|^{1/n} \|\psi(g)\|^{1/n} \|B(\mu)^n\|^{1/n}$$

so that, by the spectral radius formula

$$\Sigma(\mu) = \lim_n \|B(\mu)^n\|^{1/n},$$

we have $|e^{\alpha(\mu)}| \leqq \Sigma(\mu)$ (so that, in particular $\Sigma(\mu) > 0$).

We now prove the reverse inequality. Fix $\varepsilon > 0$. Then for n sufficiently large (in a way that may depend on f and g)

$$|\langle \psi(f), B^n \psi(g) \rangle| = e^{n \mathrm{Re} F_n} \leqq e^{n\,(\mathrm{Re}\alpha + \varepsilon)}$$

so that

$$|\langle \psi(f), B^n \psi(g) \rangle| \leqq C_{f,g}\, e^{n(\mathrm{Re}\alpha + \varepsilon)}.$$

Since the $\{\psi(f)\}$ are total, there is a dense set of vectors η_i with

$$|\langle \eta_i, B^n \eta_j \rangle| \leqq C_{ij} \exp(n(\mathrm{Re}\alpha + \varepsilon)). \tag{1}$$

Now since $B(\mu)$ is compact there are finitely many eigenvalues $\lambda_1, \ldots, \lambda_k$ with $|\lambda_i| = \Sigma(\mu)$ (see e.g. [18, I, § VI.5]), associated finite dimensional eigenprojections P_i and eigenilpotents N_i with (see e.g. [13]) $P_i P_j = 0$ if $i \neq j$ and $N_i P_i = P_i N_i = N_i$ such that

$$C(\mu) = B(\mu) - \sum_{i=1}^{k} (\lambda_i P_i + K_i) \equiv B(\mu) - A(\mu)$$

satisfies $\mathrm{spr}\, C(\mu) < \Sigma(\mu)$. Define m_i by $N_i^{m_i} \neq 0$, $N_i^{m_i+1} = 0$ and by renumbering, if necessary, suppose $m_1 \geqq m_2 \geqq \cdots \geqq m_k$. Since the η's are dense, we can choose η_1, η_2 with

$$\langle \eta_1, N_1^{m_1} \eta_2 \rangle \geqq 1$$
$$\langle \eta_1, N_i^{m_i} \eta_2 \rangle \leqq 1/2k \qquad i = 2, \ldots, k.$$

Then:

$$\langle \eta_1, B(\mu)^n \eta_2 \rangle = \sum_{i=1}^{k} \left(\sum_{j=0}^{m_i} \lambda_i^{n-j} \binom{n}{j} \langle \eta_1, N_i^j \eta_2 \rangle \right) + \langle \eta_1, C(\mu)^n \eta_2 \rangle$$

Since $\mathrm{spr}\, C(\mu) < \Sigma(\mu) = |\lambda_1|$, the dominant term in this sum is $\binom{n}{m_1} \lambda_1^{n-m_1} \times \langle \eta_1, N_1^{m_1} \eta_2 \rangle$ so that $\lim_{n \to \infty} |\langle \eta_1, B(\mu)^n \eta_2 \rangle|^{1/n} = |\lambda_1|$. Thus, by (1), $\Sigma(\mu) = |\lambda_1| \leqq e^{(\mathrm{Re}\alpha + \varepsilon)}$. Since ε is arbitrary, we have completed the proof that $\Sigma(\mu) = |e^{\alpha(\mu)}|$.

(c) Our proof will use the fact that since $|e(\mu)| = \mathrm{spr}\, B(\mu)$ by part (b), $|\lambda(\mu)/e(\mu)| \leqq 1$ if $\lambda(\mu)$ is any eigenvalue of $B(\mu)$. Let $W = \{\mu | e(\mu)$ obeys part (c)$\}$. Since $(0, \infty) \subset W$, W is clearly non-empty. Next suppose $\mu_0 \in W$. By standard eigenvalue perturbation theory [2, 13, 18], there is a neighborhood N of μ_0 so that for $\mu \in N$, there is a unique eigenvalue $f(\mu)$ with $|f(\mu)| = \mathrm{spr\,rad}\, B(\mu)$ and it is simple. By (b), $|f(\mu)/e(\mu)| \leqq 1$ so since $f(\mu)/e(\mu)$ is analytic near μ_0 and equal to 1 at μ_0, $f(\mu) = e(\mu)$ near μ_0 by the maximum modulus principle. W is open. Next let $\mu_n \in W$ and suppose $\mu_n \to \mu_\infty$. Since $B(\mu_n) \to B(\mu_\infty)$, the permanency of spectrum implies that $e(\mu_\infty) \in \mathrm{spec}(B(\mu_\infty))$ and so is an eigenvalue. Let $\alpha_1 = e(\mu_\infty), \ldots, \alpha_m$ be all the eigenvalues of $B(\mu_\infty)$ with $|\alpha_i| = \Sigma(\mu_\infty)$ counting multiplicity. Then for μ near μ_∞, there are $n \leqq m$ functions, f_i, analytic near μ_∞ with at worst algebraic singularities at μ_∞ so that all the branches of f_i are eigenvalues of $B(\mu)$ and these eigenvalues coalesce to $\alpha_1, \ldots, \alpha_m$. As above, $|f_i(\mu)/e(\mu)| \leqq 1$ near μ_∞ and $|f_i(\mu_\infty)/e(\mu_\infty)| = 1$ so that $f_i(\mu)/e(\mu) = \text{const.}$ by the maximum modulus principle on the Reimann surface for $(\mu - \mu_\infty)^{1/k}$. Thus $\alpha_i e(\mu_n)/\alpha_1$ is an eigenvalue of $B(\mu_n)$ for n large. Since $\mu_n \in W$, $m = 1$ so that $\mu_\infty \in W$. Thus W is closed and so W is the whole right half plane.

(d) follows easily from the standard formula

$$P(\mu) = (-2\pi i)^{-1} \int_{|\lambda - e(\mu)| = \varepsilon} (B(\mu) - \lambda)^{-1} \, d\lambda$$

where $\varepsilon > 0$ is sufficiently small that no other points of $\operatorname{spec}(B(\mu))$ lie in or on the circle. □

Theorem 2.4. *Fix l. For μ real, let $m_l(\mu)$ denote the mass gap for the Hamiltonian $H(\mu)$ of Lemma 2.2. Then there exists a function $M_l(\mu)$ in $\{\mu | \operatorname{Re}\mu > 0\}$ so that:*

(1) *$M_l(\mu)$ is superharmonic,*

(2) *$M_l(\mu) \geqq 0$,*

(3) *$M_l(\mu) = m_l(\mu)$ for μ real and positive.*

Proof. Let $A(\mu) = B(\mu) - e(\mu) P(\mu)$ where $B(\mu)$, $e(\mu)$, $P(\mu)$ are given by Lemmas 2.2 and 2.3. Then $A(\mu)$ is analytic and $e(\mu)$ is non-vanishing and analytic by Lemma 2.3. Thus, by Theorem 2.1,

$$M_l(\mu) \equiv -\ln \operatorname{spr}(A(\mu)) + \ln |e(\mu)|$$

is superharmonic. Since, clearly $\operatorname{spr}(A(\mu)) \leqq \operatorname{spr}(B(\mu)) = |e(\mu)|$, $M_l(\mu) \geqq 0$. Finally for μ real and positive, it is clear that $e(\mu) = \exp(-\inf \sigma(H))$ and $\operatorname{spr}(A(\mu)) = \exp(-\inf \sigma(H) - m_l(\mu))$, so $M_l(\mu) = m_l(\mu)$. □

From Theorem 2.4, Spencer's result [25] and Theorem A.6, we obtain:

Theorem 2.5. *Let $m_l(\mu)$ be as in Theorem 2.4. Then there exists a strictly positive function $M_\infty(\mu)$ on $(0, \infty)$ obeying $M_\infty(\mu) \geqq c\mu$ for $\mu \in (0,1)$ $(c > 0)$ so that*

$$m_l(\mu) \geqq M_\infty(\mu)$$

for all $\mu \in (0, \infty)$, and all $l > 1$.

Proof. By Spencer's result [25], $m_l(\mu) \geqq d$, some positive constant, for all $l > 1$, $\mu \geqq \mu_0$ sufficiently large. The bound now follows by Theorem A.6. □

§ 3. ϕ-Bounds for Periodic States

For technical purposes, we require the ϕ bounds of Glimm-Jaffe [5] in the case of periodic Hamiltonians. The original method [5] covers this case (see Theorem 1.1_v of [5]) but we provide here a proof along the lines of [9]. We first consider the half-periodic Hamiltonian H_l, i.e. the sum of the *periodic* free Hamiltonian, $H_{0,l}$, in box $(-l/2, l/2)$ and the free B.C. Wick ordered interaction $\int_{-l/2}^{l/2} :a\phi^4 + b\phi^2 - \mu\phi: dx$; let E_l be the vacuum energy for H_l. Since we will only consider half-periodic and periodic B.C. in this section, we denote the objects

H_l^{HP}, E_l^{HP}, etc.; H_l^P, E_l^P, etc. of [11] by H_l, E_l, etc.; $\tilde{H}_l$, $\tilde{E}_l$, etc. We also note that H_l differs from the H_V of Glimm-Jaffe [4, 5] with $V=l$ but in such a simple manner that it is easy to obtain one set of ϕ bounds from the other (see [11] for a discussion of this distinction); basically $H_V = H_l \otimes 1 + 1 \otimes B$ with B a $d\Gamma$ operator.

Theorem 3.1. *For each compact subset* C *of* $(0, \infty) \times R \times R$ *there is a norm* $|||\cdot|||$ *on* $\mathscr{S}$ *so that*

$$\pm \phi(f) \leqq |||f|||\,(H_l - E_l + 1)$$

for all $l \geqq 2$ *with* $\operatorname{supp} f \subset (-l/2, l/2)$ *and all* $(a, b, \mu) \in C$.

Proof. We will prove that for all f with $\|f\|_\infty \leqq 1$ and $\operatorname{supp} f$ in $(-1/2, 1/2)$ and for $l \geqq 2$

$$\pm \phi(f) \leqq (H_l - E_l) + d \tag{2}$$

where d is a constant only depending on the subset C. Using the translation covariance of H_l it is then easy to establish the result with $|||f||| = \|(1+x^2) f\|_\infty$. We will also suppose that f is the characteristic function of the interval $(-1/2, 1/2)$. The general f is handled similarly as in [9]. We will also deal with a fixed (a, b, μ) noting now that all constants obtained below are uniformly bounded on compacts C.

Let f be the characteristic function of $(-1/2, 1/2)$ and define

$$A_t = \frac{\operatorname{Tr}[\exp(-t(H_l \pm \phi(f)))]}{\operatorname{Tr}[\exp(-tH_l)]}.$$

The proof of (2) reduces to showing that

$$A_t = O(e^{dt}) \tag{3}$$

where the constant d depends only on the set C. For, by the monotone convergence theorem

$$\lim_{t\to\infty} t^{-1} \ln A_t = E_l - E(H_l \pm \phi(f)),$$

where $E(H)$ denotes the inf of the spectrum of the semibounded operator H. Hence by (3)

$$E_l - E(H_l \pm \phi(f)) \leqq d$$

and the desired estimate (2) follows by the following argument of Glimm and Jaffe [5]:

$$\begin{aligned} H_l \mp \phi(f) &\geqq E(H_l \mp \phi(f)) \\ &\geqq E_l - d. \end{aligned}$$

Now for periodic states Nelson's symmetry takes the somewhat subtle form [11]:

$$\frac{\operatorname{Tr}(e^{-tH_l})}{\operatorname{Tr}(e^{-tH_{0,l}})} = \frac{\operatorname{Tr}(e^{-lH_t})}{\operatorname{Tr}(e^{-lH_{0,t}})}.$$

Applying a slight generalization of this to A_t we obtain

$$A_t = \frac{\operatorname{Tr}[\exp(-(l-1)H_t)\exp(-H_t^{\pm})]}{\operatorname{Tr}[\exp(-lH_t)]}$$

where $H_t^{\pm} = H_t \pm \int_{-t/2}^{t/2} \phi(x)\,dx$. Since for positive operators A, B, $\mathrm{Tr}(AB) \leqq \|B\|\,\mathrm{Tr}(A)$ (see e.g. [18, I; § V1.6]),

$$A_t \leqq \|e^{-H_t^{\pm}}\| \frac{\mathrm{Tr}(e^{-(l-1)H_t})}{\mathrm{Tr}(e^{-lH_t})} \leqq e^{ct} \frac{\mathrm{Tr}(e^{-(l-1)H_t})}{\mathrm{Tr}(e^{-lH_t})}$$

by the linear lower bound for $H_t^{\pm}$ [11].

To control the ratio of traces we note that by Holder's inequality

$$f(l) = \ln \mathrm{Tr}(e^{-lH_t})$$

is a convex function of l.

By the Lemma below we deduce that

$$\frac{\mathrm{Tr}(e^{-(l-1)H_t})}{\mathrm{Tr}(e^{-lH_t})} \leqq \frac{\mathrm{Tr}(e^{-H_t})}{\mathrm{Tr}(e^{-2H_t})}.$$

Thus by Nelson's symmetry

$$\begin{aligned} A_t &\leqq e^{ct}\,\mathrm{Tr}(e^{-H_t})/\mathrm{Tr}(e^{-2H_t}) \\ &= e^{ct} \frac{\mathrm{Tr}(e^{-tH_1})\,\mathrm{Tr}(e^{-H_{0,t}})\,\mathrm{Tr}(e^{-tH_{0,2}})}{\mathrm{Tr}(e^{-tH_2})\,\mathrm{Tr}(e^{-tH_{0,1}})\,\mathrm{Tr}(e^{-2H_{0,t}})}. \end{aligned} \tag{4}$$

By explicit computation [11], for $x > 0$

$$\lim_{t\to\infty} t^{-1} \ln \mathrm{Tr}(e^{-xH_{0,t}}) = -\frac{1}{2\pi} \int \ln(1 - e^{-x\mu(k)})\,dk$$

where $\mu(k) = (k^2 + m^2)^{1/2}$. The desired inequality (3) thus follows from (4). □

Lemma 3.2. *If $f(l)$ is convex, then $f(l+1) - f(l)$ is monotone non-decreasing in l.*

Proof. Let $t_0 \leqq t_1$. Then one can find θ with

$$t_0 + 1 = \theta t_0 + (1-\theta)(t_1 + 1).$$

It follows that

$$t_1 = (1-\theta)\,t_0 + \theta(t_1 + 1)$$

so by convexity

$$f(t_0 + 1) + f(t_1) \leqq f(t_0) + f(t_1 + 1)$$

which is the result we wanted. □

Theorem 3.3. *Theorem 3.1 continues to hold if the Half-Periodic Hamiltonian, H_l, is replaced by $\tilde{H}_l$, the periodic Hamiltonian.*

Proof. By the standard formula for a change of Wick ordering [10, Lemma V.27], $:\phi^2(x):_{P,l} = :\phi^2(x): - c_l$ and

$$:\phi^4:_{P,l} = :\phi^4: - 6c_l :\phi^2: + 3c_l^2$$

where the *constant* c_l can be explicitly computed [11] and shown to vanish exponentially as $l \to \infty$. Therefore $\tilde{H}_l$ and H_l differ only by quadratic and constant terms:

$$\tilde{H}_l(a, b, \mu) = H_l(a, b - 6ac_l, \mu) + (3ac_l^2 - bc_l)\, l\,,$$

or

$$(\tilde{H}_l - \tilde{E}_l)(a, b, \mu) = (H_l - E_l)(a, b - 6ac_l, \mu)\,.$$

Clearly, as (a, b, μ) runs through a compact set C, $(a, b - 6ac_l, \mu)$ also runs through a compact set of $(0, \infty) \times \mathbb{R} \times \mathbb{R}$ for $l \geqq 2$, so that the ϕ-bounds for $\tilde{H}_l$ follow from those for H_l. □

In our application of the ϕ-bounds below, we need them in Frohlich's form [3]:

Theorem 3.4. *Fix a, b, μ. Let v_l be the measure for the periodic B.C. Euclidean theory for $P(X) = aX^4 + bX^2 - \mu X$ in the strip $(-l/2, l/2) \times (-\infty, \infty)$. Then for any $f \in C_0^\infty(R^2)$, these are constants d and α, so that for all l with* $\operatorname{supp} f \subset (-l/2, l/2) \times R$:

$$\int \exp(\pm \alpha \phi_E(f))\, dv_l \leqq d \tag{5}$$

and in particular

$$\int \phi_E(f)^2\, dv_l \leqq 2d/\alpha^2\,. \tag{6}$$

Proof. Let Ω_l be the vacuum vector for H_l. Then by a standard argument using the FKN formula (see, for example, [10, Lemma II.13]),

$$\int e^{\pm\alpha\phi_E(f)}\, dv_l \leqq \left(\Omega_l, \exp\left[-\int_{-\infty}^{\infty} E(\tilde{H}_l - E_l \pm \alpha\phi(f_t))\, dt\right]\Omega_l\right)$$

where $f_t(x) = f(x, t)$. But by Theorem 3.3, if $\alpha |||f_t||| < 1$,

$$-E(\tilde{H}_l - E_l \pm \alpha\phi(f_t)) \leqq C\,,$$

where C is independent of l. Choosing α suffiently small that $\sup |||f_t||| < \alpha^{-1}$ we deduce (5). The bound (6) follows from (5) and the estimate $x^2 \leqq e^x + e^{-x}$. □

Remark. It is fairly easy as in [3, 23] to strengthen Theorem 3.1 and 3.3 so that one can take $\alpha = 1$ in Theorem 3.4.

§ 4. Mass Gap for the Infinite Volume Dirichlet States

In this section we will prove our main result:

Theorem 4.1. *The infinite volume Dirichlet $(a\phi^4 + b\phi^2 - \mu\phi)_2$ theory* [10, 23] *has a mass gap for any $\mu \neq 0$ (and $a > 0$).*

Remarks. 1. By mass gap, we mean that the Hamiltonian has 0 as an isolated point of its spectrum and that 0 is a simple eigenvalue.

2. This result generalizes that of II of this series [21] where it was proven that 0 is a simple eigenvalue.

3. On account of the FKG inequalities [10], it is sufficient to prove [20] that (the fields ϕ in this section are Euclidean fields):

$$\langle \phi(f_t)\, \phi(f) \rangle_{D,\infty} - \langle \phi(f) \rangle^2{}_{D,\infty} \leqq c(f)\, e^{-mt}$$

for all non-negative $f \in C_0^\infty(R)^2$, where $f_t(x,s)=f(x,s+t)$, $c(f)$ is an f dependent constant, m is a strictly positive f-independent constant and $\langle\ \rangle_{D,\infty}$ is the infinite volume Dirichlet state.

4. By a similar method, one can prove the same result for the infinite volume Half-Dirichlet states.

Proof. We let $\langle\ \rangle_{P,l}$ and $\langle\ \rangle_{D,l}$ represent the periodic and Dirichlet states for the strip $(-l/2, l/2)\times(-\infty,\infty)$ and $\langle\ \rangle_{P,l,t}$ the periodic states in $(-l/2,l/2)\times(-t/2, t/2)$. We first note that by definition $\langle\phi(f)\rangle_{D,\infty}=\lim_{l\to\infty}\langle\phi(f)\rangle_{D,l}$. Moreover:

Lemma 4.2. $\lim_{l\to\infty}\langle\phi(f)\rangle_{P,l}=\langle\phi(f)\rangle_{D,\infty}$.

Proof. By a simple argument [24, 21] employing the Lee-Yang theorem and Nelson's monotonity theorem,

$$\langle\phi(f)\rangle_{D,\infty}=\frac{d\alpha_\infty^D}{d\mu}\int f(x)\,d^2x\,.$$

An argument similar to a piece of the above employing the Lee-Yang theorem [24] then shows that

$$\lim_{l\to\infty}\lim_{t\to\infty}\frac{1}{lt}\langle\phi(\chi_{l,t})\rangle_{P,l,t}=\frac{d\alpha_\infty^P}{d\mu}$$

where $\chi_{l,t}$ is the characteristic function of $(-l/2,l/2)\times(-t/2,t/2)$. Thus, although we have not proven convergence of the periodic states, we can prove convergence of the one point function. But $\langle\ \rangle_{P,l,t}$ is translation invariant so

$$\langle\phi(f)\rangle_{P,l}=\lim_{t\to\infty}\langle\phi(f)\rangle_{P,l,t}=\int f(x)\,d^2x\lim_{t\to\infty}\frac{1}{lt}\langle\phi(\chi_{l,t})\rangle_{P,l,t}\,.$$

Hence

$$\lim_{l\to\infty}\langle\phi(f)\rangle_{P,l}=\frac{d\alpha_\infty^P}{d\mu}\int f(x)\,d^2x\,.$$

The lemma now follows from the equality of α_∞^P and α_∞^D [11]. □

Secondly, by comparing periodic and Dirichlet B.C. in the lattice approximation, we have the following Griffith's inequalities:

Lemma 4.3 [11]. *For any* $f, g \geqq 0$

$$\langle\phi(f)\,\phi(g)\rangle_{D,l}\leqq\langle\phi(f)\,\phi(g)\rangle_{P,l}\,.$$

Returning to the proof of Theorem 4.1, we note that in any field theory with time translation invariance, if F is measurable w.r.t. the fields in $\mathbb{R}\times(-\infty,0]$ and G w.r.t. the fields in $\mathbb{R}\times[t_0,\infty)$ with $t_0>0$ and if the transfer matrix H has a gap M then

$$\langle FG\rangle-\langle F\rangle\langle G\rangle\leqq e^{-Mt_0}\langle F^2\rangle^{1/2}\langle G^2\rangle^{1/2}\,. \tag{7}$$

For, letting Ω be the vacuum for H and P_t the conditional expectation onto the fields at time t translated to time 0 fields (J_t^* in the language of [10] for the free field).

$$\begin{aligned}\langle FG\rangle-\langle F\rangle\langle G\rangle&=\langle P_0F, e^{-t_0H}P_tG\rangle-\langle P_0F,\Omega\rangle\langle\Omega,P_tG\rangle\\&\leqq e^{-t_0M}\langle P_0F,P_0F\rangle^{1/2}\langle P_tG,P_tG\rangle^{1/2}\,.\end{aligned}$$

(7) follows by noting that the conditional expectation is a contraction on each L^p so that $\|P_t F\|_2 \leqq \|F\|_2$. Thus by Theorem 2.5 ($m = M_\infty(\mu)$):

$$\langle \phi(f)\,\phi(f_t)\rangle_{P,l} - \langle \phi(f)\rangle_{P,l}^2 \leqq c\,(f)\,e^{-mt}.$$

By Lemma 4.3:

$$\langle \phi(f)\,\phi(f_t)\rangle_{D,l} - \langle \phi(f)\rangle_{P,l}^2 \leqq c(f)\,e^{-mt}.$$

If we take $l \to \infty$ and use Lemma 4.2:

$$\langle \phi(f)\,\phi(f_t)\rangle_{D,\infty} - \langle \phi(f)\rangle_{D,\infty}^2 \leqq c(f)\,e^{-mt}$$

which implies there is a mass gap of size at least m. □

§ 5. A Bound on a Critical Exponent

Glimm and Jaffe [6] have raised the question of obtaining bounds on critical exponents. There is a natural critical exponent associated with the divergence of the correlation length $m(\mu)^{-1}$ as $\mu \to 0$ at the critical value of b (or a or m_0). We define ν_H by:

$$m(\mu)^{-1} \sim \mu^{-\nu_H}$$

at critical point. Interestingly enough the analogous critical exponent in magnetic systems does not seem to have even been given a name in the standard sources [1, 26]! From the bound $M_\infty(\mu) \geqq c\mu$, we have:

Theorem 5.1. $\nu_H \leqq 1$.

For comparison, we compute the classical (i.e. Goldstone) value. For $P(X) = X^4 - \mu X$, the minimum for $\mu > 0$ occurs at $X = (\mu/4)^{1/3}$ where the curvature is $P''((\mu/4)^{1/3}) = 3(2\mu)^{2/3}$, i.e.

$$\nu_H^{\text{classical}} = 1/3\,.$$

Thus, our bounds, unlike those obtained by Glimm-Jaffe [6] for the critical exponents they consider, are not by classical values.

Remark. If scaling holds, then $\nu_H = \nu/\Delta$ in terms of the usual indices [1] so one has [1]: $\nu_H = 8/15$ in two dimensional Ising, $\nu_H = 0.401$ in three dimensional Ising and $\nu_H = 0.408$ in three dimensional spin 1/2 Heisenberg.

It is, of course, no coincidence that $\nu_H \leqq 1$ also holds in the above cases since the subharmonicity arguments also work for spin systems [15].

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Appendix

Subharmonic Functions

In this appendix, we provide for the reader's convenience a review of those aspects of the theory of subharmonic functions that we use. For the general theory, see Heins [12] or Radó [17].

Definition. A function f from an open set $\Omega \subset \mathbb{C}$ to $\mathbb{R} \cup \{\infty\}$ is called *subharmonic* if and only if:

(i) f is upper semicontinuous on Ω.

(ii) For any $a \in \Omega$ and $r > 0$ with $\{z \,|\, |z-a| \leqq r\} \subset \Omega$, we have

$$f(a) \leqq \frac{1}{2\pi} \int_0^{2\pi} f(a + r e^{i\theta})\, d\theta .$$

Definition. A function is called *superharmonic* if it is the negative of a subharmonic function and *harmonic* if it is both sub- and superharmonic.

Remarks. 1. We recall that f upper semicontinuous means that $f(x) \geqq \lim f(x_n)$ whenever $x_n \to x$ or equivalently $\{x \,|\, f(x) < a\}$ is open for any a. In particular, by this last, f is measurable and its restriction to any circle $a = r e^{i\theta}$ is measurable. Moreover upper semicontinuous functions are bounded from above on compacts so the integral in (ii) is always convergent or divergent to $-\infty$.

2. Upper semicontinuous functions are functions which take their maximum value on compacts; so by using (ii) there will be a maximum principle for subharmonic functions. This is the point of requiring upper semicontinuity.

3. Harmonic functions are thus finite valued, continuous functions obeying a mean value equality.

4. Among the basic properties of subharmonic functions which we do not develop (or require) are the facts that subharmonicity is a local property (i.e. (ii) need only hold for small r) the equivalence of (i), (ii) to a definition by comparison with harmonic functions, and the connection with the distributional inequality $\Delta f \geqq 0$.

First we construct critical examples of subharmonic functions (Theorem A.3):

Lemma A.1. *If g is an analytic function in Ω, then* Reg *and* Img *are harmonic in Ω.*

Proof. Continuity is obvious. The Cauchy integral theorem $g(a) = (2\pi i)^{-1} \int_{|z|=r} g(a+z) \frac{dz}{z}$ and the fact that $d(r e^{i\theta}) = i r d\theta$ immediately imply $g(a) = (2\pi)^{-1} \int_0^{2\pi} g(a + r e^{i\theta})\, d\theta$ completing the proof. □

Lemma A.2. $\ln|z|$ *is subharmonic on* $\mathbb{C}$.

Proof. Since $\ln|z|$ is rotation invariant and the inequality is obvious at $a = 0$ we need only prove that for any $a > 0$:

$$\ln a \leqq \frac{1}{2\pi} \int_0^{2\pi} \ln|a + r e^{i\theta}|\, d\theta .$$

Now $\ln(a+z)$ is analytic if $|z| < a$ so $\ln|a+z| = \mathrm{Re}(\ln(a+z))$ is harmonic in $|z| < a$ by Lemma A.1. Thus for $r < a$

$$\ln a = \frac{1}{2\pi} \int_0^{2\pi} \ln|a + r e^{i\theta}|\, d\theta .$$

Since $\ln|a+re^{i\theta}| = \ln|ae^{-i\theta}+r|$ we see that for $a<r$

$$\ln r = \frac{1}{2\pi}\int_0^{2\pi} \ln|a+re^{i\theta}|\,d\theta\,.$$

Letting $r\downarrow a$ and appealing to the monotone convergence theorem, this last equality holds when $r=a$. Thus $\frac{1}{2\pi}\int_0^{2\pi} \ln|a+re^{i\theta}|\,d\theta = \ln(\max(r,a)) \geqq \ln a$. □

Theorem A.3. *If f is analytic on $\Omega\subset\mathbb{C}$, then $\ln|f|$ is subharmonic.*

Proof. We need only prove for any a and r with $D=\{z|\,|z-a|\leqq r\}\subset\Omega$, $\ln|f|$ is subharmonic on D^{int}. If f is identically 0, then $\ln|f|\equiv-\infty$ is clearly subharmonic. Otherwise, we can find $z_1,\ldots,z_k$ so that $g(z)=f(z)/\prod_{i=1}^{k}(z-z_i)$ is analytic and non-vanishing in D^{int}. Thus $\ln g$ is analytic in D^{int} so that, by Lemma A.1, $\ln|g|=\mathrm{Re}(\ln g)$ is harmonic and so subharmonic in D^{int}. Since $\ln|f|=\ln|g| + \sum_{i=1}^{k}\ln|z-z_i|$, $\ln|f|$ is a sum of subharmonic functions in D^{int} and so subharmonic. □

We will also require two results on families of subharmonic functions.

Theorem A.4. *If $\{u_\alpha\}_{\alpha\in I}$ is a family of subharmonic functions on some fixed $\Omega\subset\mathbb{C}$ and $u\equiv\sup u_\alpha$ is upper semicontinuous on Ω, then u is subharmonic on Ω.*

Proof. The mean value inequality follows by taking the sup over α of

$$\frac{1}{2\pi}\int u(a+re^{i\theta}) \geqq \frac{1}{2\pi}\int u_\alpha(a+re^{i\theta}) \geqq u_\alpha(a)\,. \quad \square$$

Theorem A.5. *If u_n is a sequence of functions subharmonic in Ω and pointwise monotone nonincreasing, then $u=\lim_n u_n$ is subharmonic in Ω.*

Proof. $u(x)<a$ if and only if $u_k(x)<a$ for some n so $\{x|u(x)<a\}=\bigcup_n\{x|u_n(x)<a\}$ is open so u is upper semicontinuous. The mean value property follows by appealing to the monotone convergence theorem. □

As a final result concerning subharmonic functions we will prove the following:

Theorem A.6. *If $G(z)$ is superharmonic and non-negative in the region $\{z|\mathrm{Re}\,z>0\}$ and if $G(x)\geqq b$ for $a\leqq x\leqq a+2$, then*

$$G(x)\geqq \frac{b}{\ln(2(a+1))}\,\frac{x}{a+1}$$

for $0<x<a$.

Remark. The constant in the above bound is not optimal but one cannot do better that linearly in x as $x\to0$ as the example $G(z)=b\,\mathrm{Re}(z/a)$ shows.

To prove Theorem A.6, we first note a general minimum principle for superharmonic functions.

Theorem A.7. *If f is a function superharmonic in a bounded open region, Ω, and lower semicontinuous in $\bar{\Omega}$, then*

$$\inf_{x\in\bar{\Omega}} f(x) = \inf_{x\in\partial\Omega} f(x)\,.$$

Proof. Since $\bar{\Omega}$ is compact and f is lower semicontinuous, there exists x_0 with $f(x_0)=a\equiv\inf_{x\in\bar{\Omega}} f(x)$ and $\{x|f(x)=a\}$ is closed. Suppose $x_0\in\Omega$. Then since $f(x_0)\geqq\frac{1}{2\pi}\int f(x_0+re^{i\theta})\,d\theta, f(x_0+re^{i\theta})=a$, a.e. in θ and so by lower semicontinuity for all θ. Thus $\{x\in\Omega|f(x)=a\}$ is open so if $x_0\in\Omega$, $f(x)=a$ on a component of Ω and so by lower semicontinuity at points of $\partial\Omega$. □

Corollary A.8. *If Ω is a bounded open region so that*
(1) *g is subharmonic in Ω, upper semicontinuous in $\bar{\Omega}$,*
(2) *f is superharmonic in Ω, lower semicontinuous in $\bar{\Omega}$,*
(3) *$f\geqq g$ on $\partial\Omega$, then $f\geqq g$ in all of Ω.*

Proof. Apply Theorem A.7 to $f-g$. □

Proof of Theorem A.6. Let Ω_1 be the open ellipse with center $a+1$, foci at a and $a+2$ and semi major axis $a+1$. Let Ω be Ω_1 with $\{x|a\leqq x\leqq a+2\}$ removed. Let f be the function on $\bar{\Omega}$ which is equal to $G(z)/b$ on $\bar{\Omega}\setminus\{0\}$ and 0 at 0. The theorem now immediately follows from Corollary A.8 and the lemma below:

Lemma A.9. *Let Ω be the ellipse with center at* 0, *semi major axis $\alpha>1$ and foci at ± 1 with $[-1,1]$ removed. Then there exists a function g on $\bar{\Omega}$ with the following properties:*
(1) *$g=0$ on $\partial\Omega\setminus[-1,1]$,*
(2) *$g=1$ on $[-1,1]$,*
(3) *g is continuous on $\bar{\Omega}$, harmonic on Ω,*
(4) $g(x)\geqq(\ln 2\alpha)^{-1}\left(1-\frac{x}{\alpha}\right)$ *for* $1\leqq x\leqq\alpha$.

Proof. Consider first the function $h(z)=z+\sqrt{z^2-1}$ on $\mathbb{C}\setminus[-1,1]$ where $\sqrt{\ }$ is the branch positive for $z>1$. Then $h(z)$ is non-vanishing, analytic in $\mathbb{C}\setminus[-1,1]$ and $|1/h(z)|\to 0$ as $z\to\infty$, $\to 1$ as $z\to[-1,1]$. By the maximum modulus principle, $|h(z)|>1$ on all of $\mathbb{C}\setminus[-1,1]$.

Now since $h(z)^{-1}=z-\sqrt{z^2-1}$, if $h(z)=e^{iw}$ then $z=\cos w$. It follows that if $w=u+iv$, $z=x+iy$ then $x=\cos y\cosh v$, $y=\sin u\sinh v$ so that $\frac{x^2}{(\cosh v)^2}+\frac{y^2}{(\sinh v)^2}=1$, i.e. $v=$ const., equivalently $|h(z)|=$ const, on ellipses with center 0 and foci ± 1.

Let $F(z)=\ln|h(z)|$, for $z\in\mathbb{C}\setminus[-1,1]$, $F(z)=0$ if $z\in[-1,1]$. Then F is continuous and by Lemma A.1 it is harmonic on $\mathbb{C}\setminus[-1,1]$ since h is a non-vanishing analytic function inside any closed disc in $\mathbb{C}\setminus[-1,1]$ and $F=\mathrm{Re}(\ln h)$. Moreover, by an elementary computation $F'(x)>\frac{1}{x}$ for $x\notin(1,\infty)$ so that for $\alpha>1$

$$F(\alpha)-F(x)\geqq\frac{1}{\alpha}(\alpha-x)=1-\frac{x}{\alpha}$$

for $x\in(1,\infty)$. In addition $F(\alpha)\leqq\ln(2\alpha)$ for $\alpha>1$. Taking $g(z)=F(\alpha)^{-1}[F(\alpha)-F(z)]$, the lemma is proven.

Remark. The function $\ln|z+\sqrt{z^2-1}| = \ln(\text{Arc}\cos z)$ and the related ellipses enter naturally in the theory of Legendre series, see e.g. [29], and this theory was our motivation for the choice above.

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