GLOBAL SUPPORT PROPERTIES OF STATIONARY ERGODIC PROCESSES

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Nelson [7] has made the deep observation that a variety of quantum fields analytically continued to imaginary time are represented by stationary, ergodic, generalized Markov processes (see also [12, 14]). Recently, there has been some interest in determining the support properties of the measure in the canonical model of the process on $C_0^\infty(\mathbf{R}^n)'$ given by Minlos' theorem [4]. The analysis has begun with the free Euclidean field, i.e., the Gaussian process over $C_0^\infty(\mathbf{R}^n)$ with covariance

$$\int q(f)q(g)\, d\mu_0 = \langle f, (-\Delta + m_0^2)^{-1} g\rangle.$$

The main results found for μ_0 are:

(0) ([2]) Those distributions equal to a signed measure on some open set have measure 0 if $n \geq 2$.

(1) ([1, 2]) If $\tilde{\Delta}$ is the Laplacian in $n - 1$ dimensions, then μ_0 is supported on $(-\tilde{\Delta} + 1)^\alpha H$ if $\alpha > n/4 - 1/2$ and H is the set of locally Hölder continuous functions.

(2) ([2]) If $f \in C_0^\infty(\mathbf{R}^n)$ and $q_f(x) = q(f(\cdot - x))$, then with μ_0-probability one:

$$\overline{\lim_{x\to\infty}}\, q_f(x)/\sqrt{\ln |x|} = C(f, m_0, n)$$

where C is an explicit constant only depending on f, m_0, n.

(3) ([9]) μ_0 is supported by $(-\tilde{\Delta} + 1)^\alpha (1 + x^2)^{n/4} [\log (2 + x^2)]^{\beta/2} L^2(\mathbf{R}^n)$ if $\alpha > \frac{1}{4}n - \frac{1}{2}$ and $\beta > 1$ and by its complement if $\alpha > \frac{1}{2}n - 1$ and $\beta < 1$.

We have studied the extension of these results to the $P(\phi)_1$ and $P(\phi)_2$ Markov fields [7, 6, 14]. (0), (1) extend (or should extend) to these theories since they are known or believed to be locally absolutely continuous to the free theory. (This local absolute continuity is known for all $P(\phi)_1$ theories [6] and for "small coupling" $P(\phi)_2$ [8]). We will examine the analog of (2) in detail elsewhere [11] using partly methods from [10]. This note had its genesis in an attempt to extend the result (3) of M. Reed and L. Rosen to these interacting fields. We have found that their result only depends on the ergodicity of the process associated with μ_0.

Our results all follow from Theorem 1 which is a simple consequence of the ergodic theorem:

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THEOREM 1. *Let* $(\Omega, \mathcal{A}, \mu)$ *be a probability space on which* $\mathbf{R}^\nu$ *acts ergodically as a continuous group of measure-preserving transformations,* $T^x (x \in \mathbf{R}^\nu)$. *If* $0 \leq h$ *with* $0 < E(h) < \infty$, *then:*

$$\int_{\mathbf{R}^\nu} \frac{T^x(h)\, d^\nu x}{(1+x^2)^{\nu/2}[\log(2+x^2)]^\beta} \begin{array}{ll} < \infty \text{ a.e.} & \text{for } \beta > 1 \\ = \infty \text{ a.e.} & \text{for } \beta \leq 1. \end{array}$$

Proof. Let us denote the integral in question by $F(\omega)$. Then for $\beta > 1$, we have the stronger result that $F(\omega) \in L^1(d\mu)$ since, by Fubini's theorem:

$$E(F) = E(h) \int_{\mathbf{R}^\nu} \frac{d^\nu x}{(1+x^2)^{\nu/2}[\log(2+x^2)]^\beta} < \infty$$

If $\beta \leq 1$, the multi-parameter individual ergodic theorem [15] tells us that

$$\lim_{R \to \infty} (1+R^2)^{-\nu/2} \int_{|x_i| < R} T^x(h)\, d^\nu x = 2^\nu E(h), \text{ a.e.}$$

Hence for almost every $\omega \in \Omega$, there is some $R(\omega)$ with

$$(1+y^2)^{-\nu/2} \int_{|x_i| < y} T^x(h)(\omega)\, d^\nu x \geq E(h) \quad \text{if} \quad y \geq R(\omega).$$

But for some C and all x:

$$(1+x^2)^{-\nu/2}[\log(2+x^2)]^{-\beta} \geq C \int_{y \geq (\max|x_i|)} (1+y^2)^{-\nu/2-\frac{1}{2}}[\log(2+y^2)]^{-\beta}\, d^1 y.$$

Thus:

$$\begin{aligned} F(\omega) &\geq C \int_{\mathbf{R}^\nu} [T^x(h)](\omega) \left[\int_{y \geq \max|x_i|} (1+y^2)^{-\nu/2-\frac{1}{2}} lg^{-\beta}\, dy \right] d^\nu x \\ &= C \int_0^\infty ((1+y^2)^{-\frac{1}{2}} lg^{-\beta})[1+y^2]^{-\nu/2} \int_{|x_i| \leq y} T^x(h)(\omega)\, d^\nu x \\ &\geq C \left[\int_{R(\omega)}^\infty (1+y^2)^{-\frac{1}{2}}[\log(2+y^2)]^{-\beta}\, dy \right] E(h) = \infty. \end{aligned}$$

The following corollaries apply Theorem 1 to the $P(\phi)_1$ and $P(\phi)_2$ Markov processes. The reader should consult [11, 14] for the definition of these processes.

COROLLARY 1. *For the* $P(\phi)_1$ *Markoff process* $X(t)$

$$\int_{\mathbf{R}^1} \frac{|X(t)|^\nu\, d^1 t}{(1+t^2)^{\frac{1}{2}}(lg(2+t^2))^\beta} \begin{array}{ll} < \infty \text{ a.e.} & \text{if } \beta > 1 \\ = \infty \text{ a.e.} & \text{if } \beta \leq 1 \end{array}.$$

Proof. Since the generator of the process is a second order differential operator with a one-dimensional eigenspace for its lowest eigenvalue, translations act continuously and ergodically.

Corollary 1 now follows from Theorem 1 if we set $h = |X(0)|^p$ and note that by standard estimates, $0 < E(h) < \infty$.

COROLLARY 2. *For the $P(\phi)_1$ Markoff process $X(t)$ the measure μ (on $C(\mathbf{R})$) is supported on $(1 + t^2)^{1/2p}(lg(2 + t^2))^{\beta/p}L^p(R^1)$ if $\beta > 1$ and on its complement if $\beta \leq 1$.*

Proof. This is merely a restatement of Corollary 1.

COROLLARY 3. (generalizing [9]) *Let μ_0 be the free Euclidean field of mass m on $\mathbf{R}^\nu$ (with $m > 0$ if $\nu \leq 2$ and $m \geq 0$ if $\nu \geq 3$). Let $\tilde{\Delta}$ be the Laplacian with respect to some $\mathbf{R}^{\nu-1} \subset \mathbf{R}^\nu$. Let $\alpha > \frac{1}{4}\nu - \frac{1}{2}$. Then μ_0 (on $C_0^\infty(\mathbf{R}^\nu)'$) is supported by*

$$(1 - \tilde{\Delta})^\alpha(1 + x^2)^{\nu/2p}[\log (2 + x^2)]^{\beta/p}L^p(\mathbf{R}^\nu)$$

if $\beta > 1$ and by its complement if $\beta \leq 1$.

Proof. μ_0 is the measure corresponding to the Gaussian process, ϕ, indexed by $H_{-1,m}(\mathbf{R}^\nu)$, the set of distributions $f \in \mathcal{S}'(\mathbf{R}^\nu)$ with $\int |\hat{f}|^2 (p^2 + m^2)^{-1} d^\nu p < \infty$. In particular, $(1 - \tilde{\Delta})^{-\alpha} \delta_0 \in H_{-1,m}(\mathbf{R}^\nu)$ so that $\phi((1 - \tilde{\Delta})^{-\alpha} \delta_0)$ is a Gaussian random variable. We can thus take $h = |\phi((1 - \tilde{\Delta})^{-\alpha} \delta_0)|^p$ and use Theorem 1.

COROLLARY 4. *Let μ be the measure for a $P(\phi)_2$ field theory with unique vacuum (in particular, a theory with small coupling constant* [5] *or one with $P(X) = X^4 - \lambda X$, $\lambda \neq 0$* [13]). Then for any $\gamma > \frac{1}{4}$ and $\alpha > \frac{1}{2}$, *μ is supported by:*

$$(1 - \Delta)^\gamma(1 + x^2)^{1/p}[\log (2 + x^2)]^{\beta/p}L^p(\mathbf{R}^2)$$

or by

$$(1 - \tilde{\Delta})^\alpha(1 + x^2)^{1/p}[\log (2 + x^2)]^{\beta/p}L^p(\mathbf{R}^2)$$

if $\beta > 1$ and their complements if $\beta \leq 1$.

Proof. As in Corollary 3, we need only prove that $E(|\phi(h_\gamma)|^p) < \infty$ and $E(|\phi(f_\alpha)|^p) < \infty$ where h_γ is the function whose Fourier transform is $(1 + p^2)^{-\gamma}$ and f_α the function whose Fourier transform is $(1 + p_1^2)^{-\alpha}$. Write $h_\gamma = \sum_{n \in \mathbf{Z}^2} h_\gamma^{(n)}$ where $h_\gamma^{(n)}$ has support in the unit square about the point $n \in \mathbf{Z}^2 \subset \mathbf{R}^2$. Since $(1 + p^2)^{-\gamma}$ is analytic in a tube, h_γ falls off exponentially and by integration by parts, h_γ is C^∞ away from $x = 0$. Thus by Fröhlich's bounds [3] in the simplest form $\sum_{n \neq 0} \phi(h_\gamma^{(n)}) \in \bigcap_{p<\infty} L^p(\mathcal{S}'(\mathbf{R}^2), d\mu)$ for any γ. To control the central square, we need the estimate proven as in [3] or [14]:

$$E(\cosh (\phi(f))) \leq C_1 \exp \left(C_2 \int_{-\infty}^{\infty} ||f_t||_{-1}^2 \, dt\right)$$

for f with support in $\{\langle x, t\rangle \mid |x| \leq \frac{1}{2}\}$ where $f_t(x) = f(x, t)$ and $||g||_{-1}^2 = \int_{-\infty}^\infty |\hat{g}(k)|^2 (k^2 + 1)^{-1} dk$. It is not hard to see that to prove $\int_{-\infty}^\infty dt \, ||h_{\gamma,t}^{(0)}||_{-1}^2 < \infty$ it is sufficient to prove that $\int_{-\infty}^\infty dt \, ||h_{\gamma,t}||_{-1}^2 < \infty$. But by the Plancherel theorem,

$$\int_{-\infty}^{\infty} ||h_{\gamma,t}||_{-1}^{2}\, dt = (\text{const}) \int (k_1^2 + 1)^{-1}(k^2 + 1)^{-2\gamma}\, d^2k$$

which is finite if $\gamma > \frac{1}{4}$. Thus $\phi(h_\gamma) \in \bigcap_{p<\infty} L^p$ if $\gamma > \frac{1}{4}$. A similar argument based on Fröhlich's time zero bounds proved the $\alpha > \frac{1}{2}$ result.

For the case $p = 2$, we can extend Corollary 3 to the $P(\phi)_2$ theory:

COROLLARY 5. *Let μ be a measure for a $P(\phi)_2$ theory with mass gap (in particular, a theory with small coupling constant). Let $\tilde{\Delta} = -(d^2/dx_1^2)$. Then μ is supported by*

$$(1 - \tilde{\Delta})^{\alpha}(1 + x^2)^{1/2}(\log (2 + x^2))^{\beta/2}L^2(\mathbf{R}^2)$$

if $\beta > 1$, $\alpha > 0$ and its complement if $\beta \leq 1$, $\alpha > 0$.

Proof. As in Corollary 3, we need only prove that $\phi((1 - \tilde{\Delta})^{-\alpha}\, \delta_0) \in L^2(\mathbf{R}^2)$. But in a theory with mass gap and canonical commutation relations

$$E(\phi(f)^2) \leq C \int |\hat{f}(k)|^2\, (k^2 + 1)^{-1}\, d^2k$$

by arguments in [12, 14].

Finally, by the method used in Corollaries 3, 4:

COROLLARY 6. *Let μ be a Borel probability measure on $C_0^{\infty}(\mathbf{R}^{\nu})'$ for which the translations act invariantly and ergodicially. Suppose that μ is supported by the locally L^p functions and that $\int (\int_{|x|\leq 1} dx\, |T(x)|^p)\, d\mu(T) < \infty$. Then μ is supported by $(1 + x^2)^{\nu/2p} \log (2 + x^2)^{\beta/p}L^p(\mathbf{R}^{\nu})$ if $\beta > 1$ and its complement if $\beta \leq 1$.*

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