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FLUCTUATIONS IN $P(\phi)_1$ PROCESSES

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Let $H = -d^2/dx^2 + P(x)$ on $L^2(\mathbb{R}, dx)$ and let $E = \inf \operatorname{spec}(H)$. Let Ω be a normalized vector with $H\Omega = E\Omega$. Let $q(t)$ be the Markov process with generator $G = \Omega^{-1}(H - E)\Omega$, which is a Brownian motion with drift. We investigate behavior of $q(t)$ as $t \to \infty$ and in particular prove that if $P(x) = a_{2m}x^{2m} + \cdots + a_0$; $a_{2m} > 0$, then

$$\limsup_{t\to\infty} \textstyle\int_t^{t+1} q(s)\, ds/(\ln t)^{1/2m} = (a_{2m})^{-1/2m}$$

with probability one. These represent fluctuations in the sense that the lim inf is $-(a_{2m})^{-1/2m}$. We obtain some weaker results for the $P(\phi)_2$ Euclidean field theory.

1. Introduction. The theory of Markov processes [7, 8, 27, 28] has generally been developed on a fairly abstract level; the prime concrete examples which have been studied are Gaussian Markov processes [5, 25, 32]. It is our goal to study a particular class of non-Gaussian processes in which interest has been aroused by developments in constructive quantum field theory [37, 45, 49]. In addition to the intrinsic interest in these processes, it is our hope that the study of these specific but non-Gaussian processes will be of use in the general theory.

We are interested in certain properties of the $P(\phi)_1$ Markov process which can be described as follows. Let

$$P(X) = \textstyle\sum_{n=0}^{2m} a_n X^n\,; \quad a_{2m} > 0 \tag{1}$$

be an arbitrary polynomial which is bounded from below. Le H be the differential operator

$$H = -\frac{d^2}{dx^2} + P(x) - E_0 \tag{2}$$

where E_0 is chosen so that H has zero as its smallest eigenvalue. General arguments [2, 33, 39] assure one that H is essentially self-adjoint on $C_0^\infty(R)$. Moreover [15, 47] there is a unique positive function $\Omega(x)$ with $H\Omega = 0$ and $\int \Omega^2\, dx = 1$. Let $d\nu = \Omega^2(x)\, dx$ and let G be the operator $\Omega^{-1}H\Omega$ on $L^2(R, d\nu)$. Then since $H\Omega = 0$, we have $G1 = 0$. Moreover, since e^{-tH} is positivity preserving, so is e^{-tG}. The $P(\phi)_1$ Markov process is the process $q(t)$ with initial distribution $d\nu(x)$ and transition function

$$E(f(q(t)) \mid q(s)) = (e^{-|t-s|G}f)(q(s))\,.$$

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This is the Markov process with generator G and *invariant* probability measure $d\nu$.

This process can also be described as a Brownian motion with drift since it satisfies

$$dq(t) = 2^{\frac{1}{2}}\, dW(t) + r(q(t))\, dt$$

where $W(t)$ is Brownian motion and $r = 2\Omega'/\Omega$.

In case $m \geqq 2$, there is another natural description of the process. For consider first the case $P(X) = \frac{1}{4}X^2$. Then $\{q(t)\}$ is a Gaussian process with covariance

$$\int q(t)q(s)\, d\mu_0 = \exp(-\tfrac{1}{2}|t - s|)$$

and is thus the familiar Ornstein–Uhlenbeck velocity process [36]. We realize $d\mu_0$ throughout this paper as a measure on $C(\mathbb{R}^1)$, the continuous functions on $\mathbb{R}$. $F_{(a,b)} = \exp(-\int_a^b \tilde{P}(q(s))\, ds)/N_{(a,b)}$ where $\tilde{P}(X) = P(X) - \frac{1}{4}X^2$, and $N_{(a,b)} = \int d\mu_0 \exp(-\int_a^b \tilde{P}(q(s))\, ds)$ is a multiplicative functional [8] over $d\mu_0$ [37], and so by the standard procedure defines a new Markov process. That this is just what we have called the $P(\phi)_1$ process above is part of the content of the Feynman–Kac formula [30, 31, 35]; see Section 4. We note that because of the above construction, what we have called the $P(\phi)_1$ process is sometimes called the $\tilde{P}(\phi)_1$ process.

A final way of describing the process is to define a measure $d\mu$ on $C(\mathbb{R}^1)$ as follows. Let $\Sigma_{(a,b)}$ denote the σ-algebra generated by $q(s)$, $a \leqq s \leqq b$. Then $d\mu \mid \Sigma_{(a,b)}$ is absolutely continuous with respect to $d\mu_0 \mid \Sigma_{(a,b)}$ with Radon–Nikodym derivative

$$\exp(-\textstyle\int_a^b \tilde{P}(q(s))\, ds)/N \cdot \Omega(q(a))\Omega(q(b))$$

where N is a normalizing constant. Then [22], the $P(\phi)_1$ process is $q(s)$ on $(C(\mathbb{R}^1), d\mu)$.

These processes can therefore be described in terms of Ω, r, P or $\tilde{P}$. We choose the description in terms of P partly because we wish to make contact with constructive field theory, and partly because our results seem to be described most naturally in terms of P (see the remarks at the end of the introduction). We note that $P - E_0$ is determined by r through the equation

$$\frac{1}{2}\frac{dr}{dx} + \frac{r^2}{4} = P - E_0\,.$$

In particular, the simple example of $r(x) = -2x^{2m+1}$ leads to $P = x^{4m+2} - (2m + 1)x^{2m}$.

Our purpose in this note is to distinguish the support of the measures $d\mu$ on $C(R^1)$ which result for different P's. These measures are disjoint since one has:

THEOREM 1. *Let (M, Σ) be a measure space with a measurable action $t \to T_t$ indexed by R. If $d\mu_1$ and $d\mu_2$ are two distinct probability measures, both ergodic for the action T_t, then they are mutually singular.*

REMARKS 1. This result is a standard one in ergodic theory. It has recently been emphasized by constructive quantum field theorists in their study of the $P(\phi)_2$ process; see Fröhlich [13], Lenard–Newman [34] and Schrader [42].

2. One proof of Theorem 1 suggested to us by H. McKean employs the individual ergodic theorem. For let $A \in \Sigma$ be a set with $\mu_1(A) \neq \mu_2(A)$. Let $S_i = \{m \in M \mid \lim_{T\to\infty} 1/T \int_0^T \chi_A(T_t m)\, dt = \mu_i(A)\}$. Then $S_1 \cap S_2 = \emptyset$ but $\mu_1(S_1) = \mu_2(S_2) = 1$.

3. A second proof depends on the fact that the finite invariant measures form a convex simplicial cone whose extreme rays are the multiples of ergodic probability measures. Moreover, two invariant measures are disjoint if and only if their greatest lower bound is 0. This is automatically true of distinct extreme rays.

Now let P_1 and P_2 be distinct polynomials with $P_1 - P_2$ nonconstant and let μ_1 and μ_2 be the corresponding $P(\phi)_1$ measures. If μ_1 were equal to μ_2, then, in particular, we would have that $(\Omega^{(1)})^2\, dx = (\Omega^{(2)})^2\, dx$ so $\Omega^{(1)}$ would equal $\Omega^{(2)}$. But since $\Omega^{-1}(\Omega'') = P - E_0$, we would have $P_1 - P_2 = \text{const}$. Thus μ_1 and μ_2 are distinct and so mutually singular.

Our direct goal is to study the fluctuations of the paths $q(t)$ on which the measures $d\mu$ are concentrated, and exhibit their dependence on the polynomial P. The ergodic theorem assures us that with probability one $1/2T \int_{-T}^{T} |q(t)|\, dt$ approaches a constant, so "on the average," the paths are bounded. On the other hand, one expects, and we will show, that $\limsup_{n\to\infty} \int_n^{n+1} q(t)\, dt = \infty$ with probability one. Our major results will involve suitable P-dependent functions $f(n)$ and $g(n)$ with $\limsup_{n\to\infty} f(n)^{-1}[\int_n^{n+1} q(t)\, dt]$ a finite nonzero constant, c, with probability one and sharper results involving further P-dependent constants, d, with $\limsup g(n)^{-1}[[\int_n^{n+1} q(t)\, dt] - cf(n)] = d$ with probability one. We will also give some similar but weaker results for certain $P(\phi)_2$ theories. While one can obtain some information about P given the process, from our results, more efficient ways of finding P are available; for example, one can use the DLR equations [22].

Results on fluctuations of Gaussian processes go back to the law of the iterated logarithm [28], and there are results on the fine structure of the fluctuations due to Erdös [9] and Watanabe [50]. Asymptotic behavior of certain non-Gaussian processes have been obtained by Gihman and Skorohod [14] for processes quite different from the ones we consider. Our work is partially motivated by, and our results generalize, the work of Collela and Lanford [3] who studied the Gaussian case and proved among other things that for the case $P(X) = \frac{1}{4}X^2$, with μ_0-probability one,

$$\limsup_{t\to\infty} q(t)/(\ln t)^{\frac{1}{2}} = 2^{\frac{1}{2}}\,. \tag{3}$$

Our results typically will involve growths slower than $(\ln t)^{\frac{1}{2}}$ and this is easy to understand. If $\deg P > 2$, the extra $\exp(-P(q(s)))$ gives smaller weighting to q's which have very large values and so the growth of the fluctuations is smaller.

There is another important difference in the $P(\phi)_1$ for $\deg P > 2$ as compared with the Gaussian case. If $\deg P = 2m$, we will find that

(4) $$\limsup_{n\to\infty} q(n)/(\ln n)^{1/(m+1)} = c > 0\,; \quad c < \infty\,,$$

(5) $$\limsup_{n\to\infty} \int_n^{n+1} q(s)\, ds/(\ln n)^{1/2m} = d > 0\,; \quad d < \infty\,.$$

Thus the growth of the fluctuations of $q(n)$ is much more rapid than the growth of the fluctuations of $\int_n^{n+1} q(s)\, ds$. As a result, in the non-Gaussian case, the fluctuations of $q(t)$ are very sharp. The critical difference is that in the Gaussian case $q(s) - e^{-\frac{1}{2}|t-s|}q(t)$ is independent of $q(t)$ so that if $q(t)$ is large there is only a mild restoring effect tending to make $q(s)$ smaller for s near t. In the non-Gaussian case we are learning that the restoring effect is much greater; that is, the distribution is strongly dependent on $q(t)$.

Let us next say a word about our proofs of (3)—(5). If $X_n(\omega)$ is a family of independent identically distributed random variables, then it is easy to find $\limsup X_n(\omega)/f(n)$ in terms of the distribution function for X; in fact, there is a simple intuition. In $f(n)$ is defined so that $\Pr(X(\omega) > f(n)) = n^{-1}$, then we expect that about n trials will yield a value at least as large as $f(n)$ and not one very much larger. Thus, in the independent case, one expects that $\limsup X_n(\omega)/f(n) = 1$ with probability one. Our first step (Section 2) will be to show that on account of an "exponential decoupling" built into the measures $d\mu$, the $q(n)$'s and $\int_n^{n+1} q(s)\, ds \equiv \tilde{q}(n)$ are "almost independent" as $n \to \infty$ to the extent that obtaining lim sup results is restricted to controlling the distribution functions for $q(0)$ and $\tilde{q}(0)$. Since the distribution functions are explicitly known in the Gaussian ($P(X) = aX^2$) case, we will be able to easily recover the Colella–Lanford result (3) (see Section 3). The behavior of the distribution function for $q(0)$ at infinity is known by ordinary differential equation methods [6, 26] but we will have to work to control the distribution function for $\tilde{q}(0)$ (Sections 4, 5) and, in particular, we will require recent estimates proven by J. Rosen [41]. We are then able to discuss fluctuations in the $P(\phi)_1$ (Sections 6, 7, 8) and to a lesser extent the $P(\phi)_2$ Markov processes (Section 9).

The question of whether to emphasize the labeling of the process by P, Ω or r is to some extent dependent on whether one wishes to emphasize q or $\tilde{q}$. The probability density of q is $\Omega^2\, dx$, or in terms of the drift

$$\Omega^2 = \exp\left(\int_0^x r(y)\, dy\right) \Big/ \int_{-\infty}^{\infty} \exp\left(\int_0^x r(y)\, dy\right) dx\,.$$

As we shall see, the probability distribution for $\tilde{q}$ is asymptotically expressed most naturally in terms of P.

2. The Borel–Cantelli lemma for h-mixing algebras. Let (X, Σ, μ) be a probability measure space and let h be a given function from Z^+ to $(0, \infty)$. A sequence Σ_n of σ-subalgebras of Σ is called h-mixing if and only if for any Σ_m-measurable f and any Σ_n-measurable g

(6a) $$\left|\int fg\, d\mu - \left(\int f\, d\mu\right)\left(\int g\, d\mu\right)\right| \leq h(|n - m|)\|f\|_2\|g\|_2\,;$$

see e.g. [1] for many properties of such sequences. (6a) is easily seen to be equivalent to various other statements. For example, if $P(S) = \int \chi_S \, d\mu$ for $S \subset \Sigma$, then (6a) implies

$$|P(S_n \cap S_m) - P(S_n)P(S_m)| \leqq h(|n - m|)(P(S_n)P(S_m))^{\frac{1}{2}} \tag{6b}$$

for all $S_n \in \Sigma_n$; $S_m \in \Sigma_m$. Alternatively, if Q_n is the L^2-projection onto all Σ_n-measurable functions with $\int f \, d\mu = 0$, then

$$\|Q_n Q_m\| \leqq h(|n - m|) \, . \tag{6c}$$

We will see below that the $P(\phi)_1$ process is h-mixing for $h(n) = \exp[-\alpha(n - 1)]$ for suitable α and suitable σ-algebras, Σ_n.

The following proposition generalizes the classical Borel–Cantelli lemmas:

THEOREM 2. *Let $\Sigma_n \subset \Sigma$ be h-mixing for an h with*

$$\textstyle\sum_{n=1}^{\infty} h(n) = \beta < \infty \, .$$

If $S_n \in \Sigma_n$ and we let

$$S_\infty = \textstyle\bigcap_{n=1}^{\infty} (\bigcup_{m=n}^{\infty} S_m) \tag{7}$$

be the set of points in infinitely many S_n's, then

$$P(S_\infty) = 0 \quad \textit{if} \quad \textstyle\sum_{n=1}^{\infty} P(S_n) < \infty \, , \tag{8a}$$

$$P(S_\infty) = 1 \quad \textit{if} \quad \textstyle\sum_{n=1}^{\infty} P(S_n) = \infty \, . \tag{8b}$$

PROOF. Since $x \in S_\infty$ if and only if $\sum_{n=1}^{\infty} \chi_{S_n}(x) = \infty$ and under the hypothesis (8a), $E(\sum_{n=1}^{\infty} \chi_{S_n}) < \infty$, (8a) is immediate.

To prove (8b) we appeal to a theorem of Erdös and Rényi [10] which states that $P(S_\infty) = 1$ if $\sum_{n=1}^{\infty} P(S_n) = \infty$ and

$$\liminf_{n\to\infty} \textstyle\sum_{i,j=1}^{n} P(S_i \cap S_j) / (\sum_{i=1}^{n} P(S_i))^2 = 1 \, . \tag{9}$$

We will show that h-mixing and (8b) imply (9). By the definition of h-mixing, (9) certainly follows if we can prove

$$\lim_{n\to\infty} [\textstyle\sum_{i,j=1}^{n} h(|i - j|) P(S_i)^{\frac{1}{2}} P(S_j)^{\frac{1}{2}}] / (\sum_{i=1}^{n} P(S_i))^2 = 0 \, . \tag{10}$$

But since $h(|i - j|)$ defines a bounded (convolution) operator on l^2:

$$(\textstyle\sum_{i,j=1}^{n} h(|i - j|) P(S_i)^{\frac{1}{2}} P(S_j)^{\frac{1}{2}}) \leqq \beta \sum_{i=1}^{n} P(S_i)$$

so the expression on the left of (10) is bounded by $\beta / \sum_{i=1}^{n} P(S_i)$ which goes to zero by hypothesis; (8b) is thus proven. □

REMARK. See the appendix for a proof of (8b) independent of Erdös–Rényi.

The relevance of Theorem 2 to the problems we are considering is:

THEOREM 3. *Let (M, Σ, μ) be the $P(\phi)_1$ process for some fixed polynomial P. Let Σ_n be the σ-algebra generated by $\{q(t) \mid n \leqq t \leqq n + 1\}$. Then there is $\alpha > 0$ so that $\{\Sigma_n\}$ are h-mixing with $h(m) = \exp(-\alpha(m - 1))$.*

PROOF. We will prove (6a). It is well known (e.g., [29, 40, 43]) that H has purely discrete spectrum and that ([2, 15, 47]) 0 is a simple eigenvalue of H. Thus

G has the same properties, 1 is its ground state and for any $F_1, F_2 \in L^2(R, d\nu)$

(11) $$|\langle F_1, e^{-tG}F_2\rangle - \langle F_1, 1\rangle\langle 1, F_2\rangle| \leqq e^{-t\alpha}\|F_1\|_2\|F_2\|_2$$

where $\alpha = \inf(\mathrm{spec}(H)\backslash\{0\})$. Now let $n \leqq m$ be fixed. Let $\Sigma^{\{k\}}$ be the σ-algebra generated by $q(k)$ and let E_k be the conditional expectation w.r.t. $\Sigma^{\{k\}}$. Let f_1 be Σ_{n-1}-measurable and f_2 Σ_m-measurable. Then, by the Markov property,

$$\begin{aligned} E(f_1f_2) - E(f_1)E(f_2) &= E([f_1 - E(f_1)][f_2 - E(f_2)]) \\ &= E[E_n(f_1 - E(f_1))E_m(f_2 - E(f_2))] \\ &= E(F_1(q(n))F_2(q(m))) \end{aligned}$$

where $F_1(q(n)) = E_n(f_1 - E(f_1))$ and $F_2(q(m)) = E_m(f_2 - E(f_2))$. Thus, by (11) and the definition of the Markov process:

$$\begin{aligned} |E(f_1f_2) - E(f_1)E(f_2)| &= |\langle F_1, e^{-tG}F_2\rangle - \langle F_1, 1\rangle\langle 1, F_2\rangle| \\ &\leqq e^{-|n-m|\alpha}\|f_1\|_2\|f_2\|_2 \end{aligned}$$

where we have used the fact that conditional expectation is a contraction on L^2 in the last step. □

3. The Colella–Lanford results (Gaussian process case). Let us consider the Gaussian process with

(12) $$E(q(s)q(t)) = e^{-\frac{1}{2}|t-s|}$$

which is the $P(\phi)_1$ process for $P(X) = \frac{1}{4}X^2$. We will discuss $\limsup_{n\to\infty} q(n)/(\ln n)^{\frac{1}{2}}$. A similar argument allows one to discuss $\limsup_{n\to\infty} \int_n^{n+1} q(s)\,ds/(\ln n)^{\frac{1}{2}}$ and the process when $P(X) = aX^2$ for any $a > 0$. The arguments in Section 8 would allow us to obtain the Colella–Lanford results on $\limsup_{t\to\infty} q(t)/(\ln t)^{\frac{1}{2}}$.

THEOREM 4 ([3]). *For the Gaussian process with covariance* (12), *with probability* 1,

$$\limsup_{n\to\infty} q(n)/(\ln n)^{\frac{1}{2}} = 2^{\frac{1}{2}} .$$

PROOF. Since $q(n)$ has probability distribution $(2\pi)^{-\frac{1}{2}}e^{-x^2/2}\,dx$, $P(q(n) \geqq X) = (2\pi)^{-\frac{1}{2}}\int_X^\infty e^{-y^2/2}\,dy$. It is easy to prove the estimates for $a > 0$:

(13) $$c_1(a+1)^{-1}e^{-\frac{1}{2}a^2} \leqq (2\pi)^{-\frac{1}{2}}\int_a^\infty e^{-\frac{1}{2}y^2}\,dy \leqq c_2(a+1)^{-1}e^{-\frac{1}{2}a^2}$$

for suitable c_1, c_2. Let

$$A_n = \{q(n) \geqq (2(\ln n))^{\frac{1}{2}}\} .$$

Then, by (13)

$$\mu(A_n) \geqq c_1(1 + (2(\ln n))^{\frac{1}{2}})^{-1}n^{-1}$$

so $\Sigma\mu(A_n) = \infty$ and thus by Theorems 2 and 3, $\mu(A_\infty) = 1$. Fix $\alpha < \frac{1}{2}$, $a > 0$, and let

$$B_n = \{q(n) \geqq (2(\ln n))^{\frac{1}{2}} + a(\ln n)^{-\alpha}\} .$$

Then, by (13) and $e^{-x} \leqq c_k x^{-k}$, all $k > 0$,

$$\begin{aligned} \mu(B_n) &\leqq c_2 n^{-1}\exp(-a2^{\frac{1}{2}}(\ln n)^{\frac{1}{2}-\alpha}) \\ &\leqq d_k n^{-1}(\ln n)^{-k(\frac{1}{2}-\alpha)} . \end{aligned}$$

Choosing k large we see that $\Sigma\mu(B_n) < \infty$, so by Theorems 2 and 3, $\mu(B_\infty) = 0$. The theorem is now proven. □

Actually, we have proven more, namely:

(14) $$\limsup_{n\to\infty} (\ln n)^\alpha[q(n) - (2 \ln n)^{\frac{1}{2}}] = 0$$

with probability 1 if $\alpha < \frac{1}{2}$. Similar arguments prove that with probability 1,

$$\limsup_{n\to\infty} (\ln n)^{\frac{1}{2}}[q(n) - (2 \ln n)^{\frac{1}{2}}] = \infty ,$$

or more strongly that

(15) $$\limsup_{n\to\infty} \{(\ln n)^{\frac{1}{2}}[q(n) - (2 \ln n)^{\frac{1}{2}}]\}/\ln (\ln n) = \tfrac{3}{4}2^{\frac{1}{2}} .$$

NOTE 1. (14) and (15) are statements about the fine structure of fluctuations in the path.

NOTE 2. Since $q(t)$ given by (12) is related to Brownian motion $W(t)$ by

$$q(t) = e^{-t/2}W(e^t) ,$$

Theorem 4 and (15) are related to the asymptotics of Brownian motion. For example, the continuous version of Theorem 4 is just a restatement of the law of the iterated logarithm [28], and (15) follows from results of Erdös [9] on the behavior of $W(t)$ near $t = 0$ (since $W(t) = tW(1/t)$).

4. Controlling the characteristic function at imaginary argument. Let us now fix a polynomial P and let $d\mu$ be the corresponding $P(\phi)_1$ Markov process measure. Let $q(t)$ be the corresponding process and define

(16) $$\tilde{q}(t) = \int_t^{t+1} q(s)\, ds .$$

As in Section 3, to use the Borel–Cantelli lemma we need to have information on the distribution functions

(17a) $$D(a) = \mu\{q(0) > a\} ,$$

(17b) $$\tilde{D}(a) = \mu\{\tilde{q}(0) > a\}$$

as $a \to \infty$. As a preliminary to obtaining this information for $\tilde{D}(a)$ we will study in this section the function

(17c) $$\check{C}(\alpha) = \int \exp(-i\alpha\tilde{q}(0))\, d\mu$$

when $\alpha = i\beta$ with β real as $\beta \to \infty$. Our key tools will be the Feynman–Kac formula, the variational principle and certain technical estimates of J. Rosen [41]. In our analysis, we reverse the classic approach of constructive quantum field theory [16] where probability methods are used to study operators via the Feynman–Kac formula and use operator theoretic methods (e.g., the estimates of Rosen [41] have an operator theoretic proof) to study probability measures, again via the Feynman–Kac formula.

Let H be given by (2) and let Ω be given by requiring $H\Omega = 0$, $\Omega > 0$ and

$\|\Omega\|_2 = 1$. Then the Feynman–Kac formula we need is:

$$\tilde{C}(i\beta) \equiv \int \exp(\beta\tilde{q}(0))\,d\mu = \int [e^{-(G-\beta q)}(1)]\,d\nu = (\Omega, \exp[-(H-\beta q)]\Omega) \tag{18}$$

which follows easily by Nelson's method [35] of using the Trotter product formula and the transition $\langle d\nu, G, 1\rangle$ to $\langle dx, H, \Omega\rangle$ (see e.g., [45]). Thus we define $H(\beta) = H - \beta q$, let $E(\beta)$ be its lowest eigenvalue and Ω_β the positive corresponding eigenvector for $H(\beta)$. On account of (18) and the estimates $(\psi, A\psi) \leqq \|A\|\,\|\psi\|^2$, $(A^{\frac{1}{2}}\psi, \psi)^2 \leqq \|\psi\|^2(\psi, A\psi)$, we have:

PROPOSITION 5. *For all real* β:

$$(\Omega_\beta, \Omega)^2 \exp(-E(\beta)) \leqq \tilde{C}(i\beta) \leqq \exp(-E(\beta))\,.$$

Control of $\tilde{C}(i\beta)$ is thus reduced to control of $E(\beta)$ and (Ω_β, Ω_0).

PROPOSITION 6. *For β real,* let *$P^*(\beta)$ be the Legendre transform of P* [11]:

$$P^*(\beta) = \sup_{x\in R}(\beta x - P(x))\,. \tag{19}$$

Then, letting $\deg P = 2m$:

$$P^*(\beta) - R(\beta) \leqq -E(\beta) \leqq P^*(\beta) + E_0 \tag{20}$$

where $R(\beta) = O(\beta^{(m-1)/(2m-1)})$ *as* $\beta \to \infty$.

NOTE 3. $P^*(\beta) = O(\beta^{2m/(2m-1)})$ as $\beta \to \infty$.

NOTE 4. Using methods of Combes [4], one should be able to show the quoted behavior on $R(\beta)$ is best possible.

PROOF. Since $H(\beta) = -(d^2/dx^2) + P(\beta) - \beta x - E_0$, we certainly have $E(\beta) \geqq \inf(P(x) - \beta x) - E_0 = -P^*(\beta) - E_0$, proving one half of (20).

On the other hand, for β large, let x_β be the unique point where $\beta x - P(x)$ takes its maximum so $\beta = P'(x_\beta)$ and $x_\beta = O(\beta^{1/(2m-1)})$. Letting ψ_β be the trial function

$$\psi_\beta(x) = (2\pi)^{-\frac{1}{4}}\left[\frac{P''(x_\beta)}{2}\right]^{\frac{1}{8}} \exp(-\tfrac{1}{2}(\tfrac{1}{2}P''(x_\beta))^{\frac{1}{2}}(x - x_\beta)^2)$$

(ground state for $-d^2/dx^2 + \frac{1}{2}P''(x_\beta)(x - x_\beta)^2$), it is not hard to see that

$$(\psi_\beta, H(\beta)\psi_\beta) = P(x_\beta) - \beta x_\beta + O(P''(x_\beta)^{\frac{1}{2}})$$

whence

$$E(\beta) \leqq -P^*(\beta) + O(\beta^{(m-1)/(2m-1)})\,,$$

proving the other half of (20). □

PROPOSITION 7. $\ln(\Omega, \Omega_\beta) \geqq -O(\beta^{(m+1)/(2m-1)})$ *as* $\beta \to \infty$.

PROOF. We first use an idea from [44]: Let $d\nu = \Omega^2\,dx$. Let $f_\beta = \Omega_\beta\Omega^{-1}$ so that $\int f_\beta^2\,d\nu = 1$. Then, by Jensen's inequality w.r.t. the probability measure $f_\beta^2\,d\nu$:

$$\begin{aligned}\ln(\Omega, \Omega_\beta) = \ln \int f_\beta\,d\nu &= \ln \int f_\beta^{-1}(f_\beta)^2\,d\nu \\ &\geqq -\int f_\beta^2 \ln f_\beta\,d\nu\,.\end{aligned} \tag{20a}$$

Now Rosen's supercontractive logarithmic Sobolev inequalities [41] for $G = \Omega^{-1}H\Omega$ on $L^2(\mathbb{R}, d\nu)$ assert that

(20b) $$\int f^2 \ln |f|\, d\nu \leqq c(f, (G+1)^{(m+1)/2m} f)$$

if f is normalized where the inner product is in $L^2(\mathbb{R}, d\nu)$. Moreover, if f is normalized, the spectral theorem and Hölder's inequality assert that

(20c) $$(f, (G+1)^{(m+1)/2m} f) \leqq (f, (G+1)f)^{(m+1)/2m} .$$

Thus, by (20a–c) and $(f_\beta, (G+1)f_\beta)_{L^2(d\nu)} = (\Omega_\beta, (H+1)\Omega_\beta)_{L^2(dx)}$:

(21) $$\ln(\Omega, \Omega_\beta) \geqq -c(\Omega_\beta, (H+1)\Omega_\beta)^{(m+1)/2m} .$$

Now, by Proposition 6 and Note 3,

$$\begin{aligned}(\Omega_\beta, (H+1)\Omega_\beta) &= 2(\Omega_\beta, (H(\beta) + \tfrac{1}{2})\Omega_\beta) - (\Omega_\beta, H(2\beta)\Omega_\beta) \\ &\leqq 2E(\beta) + 1 - E(2\beta) = O(\beta^{2m/(2m-1)}) .\end{aligned}$$

From this and (21), the proposition follows. □

From Propositions 5, 6 and 7, we conclude the main result of this section:

THEOREM 8. *Let P have degree $2m$ and let P^* be its Legendre transform (given by (19)). Then, for some k:*

$$\begin{aligned}\tilde{C}(\beta) &\leqq (\text{const}) \exp(P^*(\beta)) && \textit{all } \beta \\ \tilde{C}(\beta) &\geqq (\text{const}) \exp(P^*(\beta) - k\beta^{(m+1)/(2m-1)}) && \textit{all } \beta \geqq \beta_0 .\end{aligned}$$

5. The distribution function for q and $\tilde{q}$. In this section, we find the large a behavior of the functions $\tilde{D}(a)$ and $D(a)$ of (17). We first use our results from Section 4 to control $\tilde{D}$ and then the results of Hsieh–Sibuya [26] to control D. Upper bounds on $\tilde{D}$ are easy to obtain from upper bounds on $\tilde{C}$:

PROPOSITION 9. *Let f be a random variable and define $C_f(i\beta) = E(\exp(\beta f))$, $D_f(\alpha) = \Pr(f \geqq \alpha)$. Suppose that $C_f(i\beta) \leqq \exp(g(\beta))$ for some g. Then for $\alpha \geqq 0$*

$$D_f(\alpha) \leqq \exp(-g^*(\alpha))$$

where $g^*(\alpha) = \max_{\beta \geqq 0}(\alpha\beta - g(\beta))$.

PROOF. Clearly $E(e^{\beta f}) \geqq e^{\alpha\beta} D(\alpha)$ for $\alpha, \beta \geqq 0$. Thus

$$D(\alpha) \leqq \exp(g(\beta) - \alpha\beta) \leqq \exp(-\max_{\beta \geqq 0}(\alpha\beta - g(\beta)) . \qquad \square$$

The lower bound is considerably trickier and we are grateful to D. Newman for fruitful suggestions on how to go about finding lower bounds on $\tilde{D}(\alpha)$ given upper *and* lower bounds on $\tilde{C}(\beta)$. By Proposition 9 and Theorem 8, we have

$$\tilde{D}(\alpha) \leqq C \exp(-P^{**}(\alpha)) .$$

By a general result [11], P^{**} is the convex hull of P(i.e., $\{(y, x) \mid y \geqq P^{**}(x)\}$ is the convex hull of $\{(y, x) \mid y \geqq P(x)\}$); so, in particular,

(22) $$\tilde{D}(\alpha) \leqq C \exp(-P(\alpha)) \qquad \alpha > \alpha_0$$

since P is convex for large x. The basic idea of the proof below is the following. Suppose that $\tilde{D}(\alpha_1)$ is "significantly less than" $C\exp(-P(\alpha_1))$. Then since $\tilde{D}$ is monotone decreasing, the bound $\tilde{D}(\alpha) \leqq \tilde{D}(\alpha_1)$ $\alpha \geqq \alpha_1$ is better than (22) for $\alpha \in (\alpha_1, \alpha_2)$ where α_2 is determined by $\tilde{D}(\alpha_1) = C\exp(-P(\alpha_2))$. Now choose a value of β so that $\int \exp(\beta\tilde{q}(0))\,d\mu$ has its value primarily from the points where $\tilde{q}(0) \simeq \frac{1}{2}(\alpha_1 + \alpha_2)$. Then we will obtain an upper bound on $\tilde{C}(\beta)$ "considerably better" than what we would get by taking the bound (22). But the bound (22) leads essentially to an upper bound $C\exp(P^*(\beta))$ on $\tilde{C}(\beta)$ so we obtain a "considerably better" upper bound on $\tilde{C}(\beta)$ than $C\exp(P^*(\beta))$. If it is sufficiently better, we will obtain a violation of the lower bound of Theorem 8.

THEOREM 10. *If $\tilde{D}$ is the distribution function for $\int_0^1 \tilde{q}(s)\,ds$ in the $P(\phi)_1$ Markov process with* $\deg P = 2m > 2$, *then*:

$$\tilde{D}(\alpha) \leqq \text{const}\exp(-P^{**}(\alpha)) \qquad \textit{all} \quad \alpha \geqq 0$$
$$\tilde{D}(\alpha) \geqq \text{const}\exp(-P(\alpha) - Q(\alpha))$$

where $Q(\alpha) = O(\alpha^{\frac{3}{2}m+\frac{1}{2}})$ as $\alpha \to \infty$.

REMARK. For a large, $P(\alpha) = P^{**}(\alpha)$. If P is even and all a_n's are nonnegative, $P(\alpha) = P^{**}(\alpha)$ for all α.

PROOF. As we have already noted, the upper bound on D follows from Proposition 9 and Theorem 8. As a preliminary for the lower bound we note that for $\beta \geqq 0$

$$\begin{aligned} \tilde{C}(\beta) &= -\int_{-\infty}^{\infty} e^{\beta\alpha}\,d(\tilde{D}(\alpha)) \\ &= \beta\int_{-\infty}^{\infty} e^{\beta\alpha}\tilde{D}(\alpha)\,d\alpha \\ &\leqq 1 + \beta\int_0^{\infty} e^{\beta\alpha}\tilde{D}(\alpha)\,d\alpha\,. \end{aligned} \tag{23}$$

In the above, the upper bound already proven justifies the integration by parts.

To fix notation, let us rewrite (22) and the lower bound in Theorem 8 explicitly

$$\tilde{D}(\alpha) \leqq C_1\exp(-P^{**}(\alpha)) \qquad \alpha \geqq 0 \tag{22a}$$

$$\tilde{C}(\beta) \geqq C_2\exp(P^*(\beta) - k\beta^{(m+1)/(2m-1)}) \qquad \beta \geqq \beta_0\,. \tag{22b}$$

If necessary, increase β_0 so that

$$1 < \tfrac{1}{2}C_2\exp(P^*(\beta) - k\beta^{(m+1)/(2m-1)}) \qquad \beta \geqq \beta_0\,. \tag{22c}$$

Now we will pick successive constants L; A_1 such that

$$\tilde{D}(\alpha) \geqq C_1\exp(-P(\alpha) - L\alpha^{\frac{3}{2}m+\frac{1}{2}}) \qquad \alpha \geqq A_1 \tag{24}$$

concluding the proof of the theorem. Equation (22) fixes the constants C_1, C_2, k, $\beta_0 > 0$. Now choose A_0 and E so that

$$P^{**}(\alpha) = P(\alpha) \qquad \alpha \geqq A_0\,, \tag{25a}$$

$$\tfrac{1}{2}E\alpha^{2m} \leqq P(\alpha) \leqq E\alpha^{2m} \qquad \alpha \geqq A_0\,, \tag{25b}$$

$$mE\alpha^{2m-1} \leqq P'(\alpha) \leqq 2mE\alpha^{2m-1} \qquad \alpha \geqq A_0\,, \tag{25c}$$

$$m(2m-1)E\alpha^{2m-2} \leqq P''(\alpha) \leqq 2m(2m-1)E\alpha^{2m-2} \qquad \alpha \geqq A_0\,. \tag{25d}$$

Define $\bar{k}$ by:

(25e) $$\bar{k} = k(2^{2m}mE)^{(m+1)/(2m-1)} .$$

Next choose L so large that:

(26a) $$L^2(\tfrac{1}{8}(2^{2m}mE)^{-2}m(2m-1)E) \geqq \bar{k} + L ,$$

(26b) $$2^{2m}mC_1L\alpha^{\frac{3}{2}m+\frac{1}{2}}e^{-L\alpha^{m+1}} < \tfrac{1}{6}C_2 \qquad \alpha \geqq A_0 ,$$

(26c) $$2^{2m}mEC_1\alpha^{2m}e^{-L\alpha^{m+1}} < \tfrac{1}{6}C_2 \qquad \alpha \geqq A_0 ,$$

(26d) $$2^{4m+1}mEC^1\alpha^{\frac{1}{2}m-\frac{1}{2}}L^{-1}e^{-L\alpha^{m+1}} < \tfrac{1}{6}C_2 \qquad \alpha \geqq A_0 .$$

We can arrange (26b–d) by first choosing L_0 so large that for $L > L_0$ all the functions on the left of the inequalities are monotone decreasing for $\alpha \geqq A_0$ and then arranging for the inequalities to hold at $\alpha = A_0$ by increasing L further. Finally choose $A_1 \geqq A_0$ so that

(27a) $$mEA^{2m-1} \geqq \beta_0 \qquad A \geqq A_1 ,$$

(27b) $$\frac{2^{2m}}{2} EA^{2m-1} \geqq \beta_0 \qquad A \geqq A_1 ,$$

(27c) $$(2^{2m-1} - 1)EA^{2m} \geqq LA^{\frac{3}{2}m+\frac{1}{2}} \qquad A \geqq A_1 .$$

Now we claim that (24) holds. For suppose that (24) is false for some $\alpha_0 \geqq A_1$, i.e.

(28a) $$\tilde{D}(\alpha_0) \leqq C_1 \exp(-P(\alpha_0) - L\alpha_0^{\frac{3}{2}m+\frac{1}{2}}) .$$

Define α_1, $\bar{\alpha}$, $\bar{\beta}$ by:

(28b) $$P(\alpha_1) = P(\alpha_0) + L\alpha_0^{\frac{3}{2}m+\frac{1}{2}} ,$$

(28c) $$\bar{\alpha} = \tfrac{1}{2}(\alpha_0 + \alpha_1) ,$$

(28d) $$\bar{\beta} = P'(\bar{\alpha}) .$$

We will obtain an upper bound on $\tilde{C}(\bar{\beta})$ which violates (22b) and thereby establish (24) by contradiction. By (25c), $P(\alpha)$ is monotone for $\alpha > \alpha_0 \geqq A_0$, and by (25b), (27c) and the definition (28b) of α_1 $P(2\alpha_0) > P(\alpha_1)$, so $\alpha_0 < \alpha_1 < 2\alpha_0$. Since P' is monotone on $[\alpha_0, \alpha_1]$ by (25d), we have $P'(\alpha_0)(\alpha_1 - \alpha_0) \leqq P(\alpha_1) - P(\alpha_0) \leqq P'(2\alpha_0)(\alpha_1 - \alpha_0)$ from which we obtain by (25c):

(29a) $$(2^{2m}mE\alpha_0^{2m-1})^{-1}L\alpha_0^{\frac{3}{2}m+\frac{1}{2}} \leqq (\alpha_1 - \alpha_0) \leqq (mE\alpha_0^{2m-1})^{-1}L\alpha_0^{\frac{3}{2}m+\frac{1}{2}} .$$

From (25c) and $\alpha_0 \leqq \bar{\alpha} \leqq 2\alpha_0$, we also find

(29b) $$mE\alpha_0^{2m-1} \leqq \bar{\beta} \leqq 2^{2m}mE\alpha_0^{2m-1}$$

so that, in particular, $\bar{\beta} \geqq \beta_0$ by (27a) and by (25e)

(29c) $$\alpha_0^{m+1}\bar{k} \geqq k\bar{\beta}^{(m+1)/(2m-1)} .$$

Now consider the concave function

(30a) $$Q(\alpha) = \alpha\bar{\beta} - P^{**}(\alpha) .$$

By (28d) and (25a), $Q(\alpha)$ has its maximum at $\alpha = \bar{\alpha}$ so that using $P^{***} = P^*$

(30b) $$Q(\bar{\alpha}) = P^*(\bar{\beta}) .$$

Moreover, since $Q'(\bar{\alpha}) = 0$, we have for any $\alpha > \bar{\alpha}$:

$$\begin{aligned} Q(\alpha) &= Q(\bar{\alpha}) + \int_0^1 dt \int_0^t ds \frac{d^2}{ds^2} Q(s\alpha + (1-s)\bar{\alpha}) \\ &\leqq Q(\bar{\alpha}) + \tfrac{1}{2}(\alpha - \bar{\alpha})^2 \max_{\bar{\alpha}<\gamma<\alpha} Q''(\gamma) \\ &\leqq Q(\bar{\alpha}) - \tfrac{1}{2}(\alpha - \bar{\alpha})^2 m(2m-1)E\alpha_0^{2m-2} \qquad \text{(by (25d))} \end{aligned}$$

where we have used the fact that $Q'' = -P''$ when $\alpha \geqq A_0$ (by (25a)). In particular, taking $\alpha = \alpha_1$ in the above using (29a) with $\alpha_1 - \bar{\alpha} = \frac{1}{2}(\alpha_1 - \alpha_0)$ and using (29a):

$$\begin{aligned} Q(\alpha_1) &\leqq Q(\bar{\alpha}) - \frac{1}{8}\left(\frac{1}{2^{2m} mE\alpha_0^{2m-1}}\right)^2 L^2\alpha_0^{3m+1} m(2m-1)E\alpha_0^{2m-2} \\ &= Q(\bar{\alpha}) - \frac{1}{8}\left(\frac{1}{2^{2m}Em}\right)^2 L^2 m(2m-1)\alpha_0^{m+1}E\,; \end{aligned}$$

so by (26a)

$$Q(\alpha_1) \leqq Q(\bar{\alpha}) - (\bar{k} + L)\alpha_0^{m+1}\,;$$

so by (29c) and (30b)

(30c) $$Q(\alpha_1) \leqq P^*(\bar{\beta}) - k\bar{\beta}^{(m+1)/(2m-1)} - L\alpha_0^{m+1}\,.$$

Similarly

(30d) $$Q(\alpha_0) \leqq P^*(\bar{\beta}) - k\bar{\beta}^{(m+1)/(2m-1)} - L\alpha_0^{m+1}\,.$$

We have the following bounds on $\tilde{D}(\alpha)$: (22) if $0 \leqq \alpha \leqq \alpha_0$ or $\alpha \geqq \alpha_1$, and $\tilde{D}(\alpha) \leqq \tilde{D}(\alpha_0) \leqq$ (28a) if $\alpha_0 \leqq \alpha \leqq \alpha_1$. Thus by (23):

$$\tilde{C}(\beta) \leqq 1 + I_1 + I_2 + I_3\,,$$

where we have written $\int_0^\infty = \int_{\alpha_0}^{\alpha_1} + \int_0^{\alpha_0} + \int_{\alpha_1}^\infty$. Now:

$$\begin{aligned} I_1 &\leqq C_1\bar{\beta} \int_{\alpha_0}^{\alpha_1} \exp(Q(\alpha_1))\, d\alpha \\ &\leqq C_1(\alpha_1 - \alpha_0)\bar{\beta} \exp(Q(\alpha_1)) \\ &\leqq \tfrac{1}{6}C_2 \exp(P^*(\bar{\beta}) - k\bar{\beta}^{(m+1)/(2m-1)}) \end{aligned}$$

by (30c), (29a), (29b) and (26b). Moreover

$$\begin{aligned} I_2 &\leqq C_1\bar{\beta} \int_0^{\alpha_0} \exp(Q(\alpha))\, d\alpha \\ &\leqq C_1\bar{\beta}\alpha_0 \exp(Q(\alpha_0)) \\ &\leqq \tfrac{1}{6}C_2 \exp(P^*(\bar{\beta}) - k\bar{\beta}^{(m+1)/(2m-1)}) \end{aligned}$$

where the first inequality uses the fact that Q is monotone increasing on $(0, \bar{\alpha})$ and the second uses (29b), (30d), and (26c). In addition, since $Q(\alpha)$ is monotone decreasing and concave in (α_1, ∞)

$$\begin{aligned} I_3 &\leqq C_1\bar{\beta} \int_{\alpha_1}^\infty \exp(Q(\alpha))\, d\alpha \\ &\leqq C_1\bar{\beta} \int_{\alpha_1}^\infty \exp(Q'(\alpha_1)(\alpha - \alpha_1) + Q(\alpha_1))\, d\alpha \\ &\leqq C_1\bar{\beta}(P'(\alpha_1) - \bar{\beta})^{-1} \exp(Q(\alpha_1)) \\ &\leqq \tfrac{1}{6}C_2 \exp(P^*(\bar{\beta}) - k\bar{\beta}^{(m+1)/(2m-1)})\,, \end{aligned}$$

where we have used (29b), (30c), (26d) and the inequality

$$\begin{aligned} P'(\alpha_1) - \bar{\beta} &= P'(\alpha_1) - P'(\bar{\alpha}) \geqq \inf_{\alpha_0 \leqq \alpha \leqq \alpha_1} P''(\alpha)(\alpha_1 - \bar{\alpha}) \\ &= \inf_{\alpha_0 < \alpha < \alpha_1} P''(\alpha)\tfrac{1}{2}(\alpha_1 - \alpha_0) \\ &\geqq \tfrac{1}{2}m(2m - 1)E(2^{2m}mE)^{-1}L\alpha_0^{\frac{3}{2}m-\frac{1}{2}} \\ &= (2)^{-2m-1}(2m - 1)L\alpha_0^{\frac{3}{2}m-\frac{1}{2}} . \end{aligned}$$

by (25d) and (29). Finally by (22c) we have

$$1 < \tfrac{1}{2}C_2 \exp(P^*(\bar{\beta}) - k\bar{\beta}^{(m+1)/(2m-1)}) .$$

All these bounds together imply that

$$\tilde{C}(\beta) < C_2 \exp(P^*(\bar{\beta}) - k\bar{\beta}^{(m+1)/(2m-1)}) .$$

Since $\beta > \beta_0$ by (27b) and (29b), we have the required contradiction with (22b). □

Next we turn to the distribution function $D(\alpha)$. Clearly $D(\alpha) = \int_\alpha^\infty \Omega^2(x)\,dx$ by the definition of the Markov process. Fortunately the large x behavior of $\Omega(x)$ has been analyzed by Hsieh and Sibuya [26]; see also Dicke [6]. We are thus able to conclude:

THEOREM 11. *Define the polynomial* $Q(\alpha)$ *by*

$$Q(\alpha) = \int_0^\alpha (P(x))^{\frac{1}{2}}\,dx + O(\ln \alpha) \tag{31}$$

as $\alpha \to \infty$. *Then as* $\alpha \to \infty$,

$$|\ln D(\alpha) + 2Q(\alpha)| = O(\ln \alpha) .$$

PROOF. By the result of Hsieh–Sibuya [26], Ω has the property that

$$\Omega(x)/x^a \exp(-Q(x)) \to k , \quad \text{a nonzero finite constant}$$

for a certain explicit a. Let $\tilde{Q}(x) = Q(x) - a \ln x$. Then for α large:

$$\begin{aligned} (k + \varepsilon)^{-2}D(\alpha) &\leqq \int_\alpha^\infty \exp(-2\tilde{Q}(x))\,dx \\ &\leqq \int_\alpha^\infty \exp(-2\tilde{Q}(\alpha) + 2(\alpha - x)\tilde{Q}'(\alpha))\,dx \\ &= \exp(-2\tilde{Q}(\alpha))/2\tilde{Q}'(\alpha) = \exp(-2Q(\alpha) + O(\ln \alpha)) \end{aligned}$$

where we have used the fact that $\tilde{Q}$ is convex for large x. Also for α large, $\tilde{Q}$ is monotone so:

$$\begin{aligned} D(\alpha) &\geqq \delta \exp(-2\tilde{Q}(\alpha + \delta)) \\ &= \exp(-2\tilde{Q}(\alpha))\delta \exp(-2\tilde{Q}(\alpha + \delta) + 2\tilde{Q}(\alpha)) \end{aligned}$$

for any $\delta > 0$. If we choose δ so that $\tilde{Q}(\alpha + \delta) - \tilde{Q}(\alpha) = 1$, then $\delta\tilde{Q}'(\alpha) \to 1$ as $\alpha \to \infty$; so δ is certainly $\geqq [\tilde{Q}'(2\alpha)]^{-1}$ for α large, so that

$$\begin{aligned} D(\alpha) &\geqq \exp(-2\tilde{Q}(\alpha))e^{-2}\tilde{Q}'(2\alpha)^{-1} \\ &= \exp(-2Q(\alpha) + O(\ln \alpha)) . \end{aligned}$$ □

EXAMPLES. If $P(X) = x^4 + a_3x^3 + a_2x^2 + a_1x$, then $Q(x) = \frac{1}{3}x^3 + \frac{1}{4}a_3x^2 + (\frac{1}{2}a_2 - \frac{1}{8}a_3^2)x$.

REMARK. Our bounds on $\tilde{D}(\alpha)$ are clearly only sensitive to those coefficients of $P(X) = \sum_{n=0}^{2m} a_n X^n$ with $n > \frac{3}{2}m + \frac{1}{2}$ while the bounds on D are sensitive to those with $n \geqq m$; see Table 1 below:

m	$\tilde{D}$ depends on	D depends on
2	a_4	a_2, a_3, a_4
3	a_6	a_3, a_4, a_5, a_6
4	a_7, a_8	a_4, a_5, a_6, a_7, a_8 .

6. Fluctuation of the paths in $P(\phi)_1$. As in Section 3, it is easy to combine detailed bounds on D, $\tilde{D}$ and the Borel–Cantelli lemma to find the behavior of the fluctuations of the paths in the $P(\phi)_1$ process:

THEOREM 12. *With probability* 1 *in the* $P(\phi)_1$ *Markov process*:

$$\limsup \frac{q(n)}{(\ln n)^{1/(m+1)}} = \left(\frac{m+1}{2}\right)^{1/(m+1)} a_{2m}^{-1/(2m+2)}$$

$$\limsup \frac{\tilde{q}(n)}{(\ln n)^{1/2m}} = (a_{2m})^{-1/2m}$$

where $\tilde{q}(n) = \int_n^{n+1} q(s)\, ds$ *and* $P(X) = a_{2m} X^{2m} + \cdots$.

PROOF. We consider the result for $\tilde{q}$; the proof for q is similar. By Theorem 10, for any $\varepsilon > 0$ we can find α_0 so that for $\alpha > \alpha_0$:

$$\exp(-(a_{2m} + \varepsilon)\alpha^{2m}) \leqq \tilde{D}(\alpha) \leqq \exp(-(a_{2m} - \varepsilon)\alpha^{2m}) . \tag{32}$$

Let

$$A_n = \{q \mid \tilde{q}(n) > [(a_{2m} - 2\varepsilon)^{-1} \ln n]^{1/2m}\}$$
$$B_n = \{q \mid \tilde{q}(n) > ((a_{2m} + \varepsilon)^{-1} \ln n)^{1/2m}\} .$$

Thus for n large, we know by (32) that:

$$\mu(A_n) \leqq \exp(-[1 + (a_{2m} - 2\varepsilon)^{-1}\varepsilon] \ln n)$$
$$= n^{-[1+\varepsilon(a_{2m}-2\varepsilon)^{-1}]}$$
$$\mu(B_n) \geqq n^{-1} ,$$

so $\Sigma\mu(A_n) < \infty$, $\Sigma\mu(B_n) = \infty$. From Theorems 2 and 3 we conclude that with probability 1,

$$\limsup \tilde{q}(n)/(\ln n)^{1/2m} \leqq (a_{2m} - 2\varepsilon)^{-1/2m}$$
$$\limsup \tilde{q}(n)/(\ln n)^{1/2m} \geqq (a_{2m} + \varepsilon)^{-1/2m} .$$

Taking $\varepsilon = \frac{1}{2}, \frac{1}{3}, \cdots$, the result follows. □

We emphasize that since a similar proof shows that $\liminf \tilde{q}(n) = -\limsup \tilde{q}(n)$, these results are on fluctuations. We also note once more the dramatic difference between the growth of the fluctuations in q and in its average.

7. Fine structure of the fluctuations. Fix a polynomial P. Let us define numbers $c_0, \cdots, c_{[m/2-2]}$ by demanding for large y that

$$P^{-1}(y) = \left(\frac{y}{a_{2m}}\right)^{1/2m} + \sum_{i=0}^{[m/2-2]} c_i y^{-i/2m} + O(y^{-[m/2-1]/2m}),$$

and numbers $b_0, \cdots, b_{m-1}$ by:

$$h^{-1}(y) = \left(\frac{m+1}{(a_{2m})^{\frac{1}{2}}} y\right)^{1/m+1} + \sum_{i=0}^{m-1} b_i y^{-i/m+1} + O(y^{-m/m+1})$$

where $h = 2Q$ and Q is given by (31). Here $^{-1}$ denotes the inverse function. The c's and b's are computable inductively from the coefficients $\sum_{n=0}^{2m} a_n X^n$ of P by the binomial theorem and the c's only depend on a_n; $n > \frac{3}{2}m + \frac{1}{2}$ and the b's only on a_n; $n \geqq m$. Thus, e.g., if $P(x) = x^4 + a_3 x^3 + a_2 x^2 + \cdots$ then

$$h^{-1}(y) = (\tfrac{3}{2}y)^{\frac{1}{3}} - \frac{a_3}{4} + \left(\tfrac{5}{16}a_3^2 - \frac{a_2}{2}\right)(\tfrac{3}{2}y)^{-\frac{1}{3}} + \cdots.$$

THEOREM 13. *With probability* 1, *in the* $P(\phi)_1$ *Markov process*:

$$\limsup \left\{(\ln n)^{m-1/m+1}\left[q(n) - \frac{(m+1)^{1/m+1}}{(a_{2m})^{1/2m+2}}(\ln n)^{1/m+1} - \sum_{i=0}^{m-2} b_i (\ln n)^{-i/m+1}\right]\right\} = b_{m-1}$$

and

$$\limsup \left\{(\ln n)^{[m/2-2]/2m}\left[\tilde{q}(n) - \frac{(\ln n)^{1/2m}}{a_{2m}^{1/2m}} - \sum_{i=0}^{[m/2-3]} c_i (\ln n)^{-i/2m}\right]\right\} = c_{[m/2-2]}.$$

REMARKS 1. In cases where the sum has one or more terms (i.e., $m \geqq 2$ in the q case; $m \geqq 4$ in the $\tilde{q}$ case) we are actually saying one can draw two curves which approach each other asymptotically so that the maximal fluctuations asymptotically fall in between the curves.

2. The order $m - 1$ for q and $[m/2 - 2]$ for $\tilde{q}$ are determined as the best we are able to obtain given our error estimates.

PROOF. Let us prove that the second lim sup is less than $c_{[m/2-2]} + \varepsilon$ for any ε. The other parts are similar. Let

$$A_n = \left\{q \mid q(n) \geqq \left(\frac{\ln n}{a_{2m}}\right)^{1/2m} + \sum_{i=0}^{[m/2-2]} c_i (\ln n)^{-i/2m} + \varepsilon(\ln n)^{-[m/2-2]/2m}\right\}.$$

Then since $\tilde{D}(\alpha) \leqq C\exp(-P(\alpha))$ for α large (by Theorem 10):

$$\mu(A_n) \leqq \exp[-\ln n - a_{2m}^{1/2m}\varepsilon(\ln n)^{[\frac{3}{2}m+1]/2m} + O((\ln n)^{[\frac{3}{2}m]/2m})]$$
$$\leqq \frac{1}{n(\ln n)^2}$$

for n large since $\exp(-\delta x^\delta) \leqq x^{-2}$ for any γ, $\delta > 0$ and sufficiently large x. Thus $\Sigma\mu(A_n) < \infty$ so the lim sup in question is less than $c_{[m/2-2]} + \varepsilon$. □

8. Fluctuations of the continuous paths.

THEOREM 14. *Let* $\tilde{q}(t) = \int_t^{t+1} q(s)\,ds$. *Then* $\limsup_{t\to\infty} \tilde{q}(t)/(\ln t)^{1/2m} = (a_{2m})^{-1/2m}$.

REMARK. It may be possible to obtain a similar result for $q(t)$ by following some of the ideas in [3] but this would require detailed upper bounds on the probability distribution for the moduli of local Hölder continuity for the paths $q(t)$.

PROOF. Fix k. Then as above one finds

$$\limsup_{n\to\infty} \tilde{q}\left(\frac{n}{2^k}\right)\Big/[\ln(n)]^{1/2m} = (a_{2m})^{-1/2m}.$$

But since $\lim_{n\to\infty} [\ln n/\ln(n/2^k)]^{1/2m} = 1$ we find that

$$\limsup_{n\to\infty} \tilde{q}\left(\frac{n}{2^k}\right)\Big/\left[\ln\left(\frac{n}{2^k}\right)\right]^{1/2m} = (a_{2m})^{-1/2m}. \tag{33}$$

Now, let $r(n) = \int_{n/2^k}^{n+1/2^k} |q(s)|\, ds$. By following our arguments above with suitable modification (e.g., $-d^2/dx^2 + P - E - q$ is replaced by $(1/2^k)(-d^2/dx^2 + P - E - |q|)$) we find that

$$\limsup_{n\to\infty} r(n)\Big/\ln\left[\frac{n}{2^k}\right]^{1/2m} = 2^{-k(1-1/2m)}(a_{2m})^{-1/2m}$$

since the Legendre transform of $(1/2^k)P^*(\beta)$ has leading behavior $2^{k(2m-1)}a_{2m}\alpha^{2m}$. Now for any t, let t_k be the largest number of the form $n/2^k$ less than t. Then

$$|\tilde{q}(t) - \tilde{q}(t_k)| \leqq r(t_k) + r(t_k + 1)$$

so that

$$|\limsup(\ln t)^{-1/2m}\tilde{q}(t) - \limsup(\ln t_k)^{-1/2m}\tilde{q}(t_k)| \leqq 2\limsup r(n)\Big/\ln\left[\frac{n}{2^k}\right]^{1/2m}$$

i.e.,

$$|\limsup(\ln t)^{-1/2m}\tilde{q}(t) - (a_{2m})^{-1/2m}| \leqq 2\cdot 2^{-k(1-1/2m)}(a_{2m})^{-1/2m}$$

with probability 1. Since we can take $k = 1, 2, \cdots$ we obtain the result. □

9. Fluctuations in the $P(\phi)_2$ field theory. We will suppose that the reader is familiar with the $P(\phi)_2$ Euclidean field theory (see e.g., [45, 49]). In attempting to mimic the above, we must first deal with smeared fields and so try to mimic the methods of Sections 4 and 5. Since the tools of Section 4 used to obtain lower bounds, most notably the supercontractive estimates, are not available we can only obtain upper bounds on C and so only upper bounds on lim sup. In fact:

THEOREM 15. *Let μ be the measure on $C_0^\infty(R^2)'$ associated to a $:P(\phi)_2:$ theory of one of the following types:*

(a) *small coupling constant free boundary conditions* [17, 18];
(b) *$P = Q - \mu\phi^n$; Q even, n odd, μ large* [48] *with periodic boundary conditions*;
(c) *$P = a\phi^4 + b\phi^2 - \mu\phi$; $\mu \neq 0$, $a > 0$; with Dirichlet boundary conditions* [12, 13, 22, 24, 38, 45].

Let $\deg P = 2m$ *and let $f \in C_0^\infty(R^2)$. Define $\phi_f(x) = \phi(f(\cdot - x))$. Let x_n be a* *sequence of points with* $|x_n - x_m| \geqq C|n - m|$ *for some* $C > 0$. *Then for any* $\varepsilon > 0$:

$$\limsup_{n\to\infty} \phi_f(x_n)/(\ln n)^{1/2m} < \infty$$

with probability one.

REMARK. This result distinguishes μ from a free field; in fact, from any Gaussian field. The first results of this type are due to Schrader [42].

PROOF. Since the theories in question are known to possess mass gaps [17, 24, 48], this result follows by our general method given the following bound of Fröhlich [12] type:

THEOREM 16. *Under the above circumstances*:

$$\int \exp(\beta\phi(f))\, d\mu \leqq C_1 \exp(C_2 \beta^{2m/2m-1}) .$$

PROOF. We consider case (a). The others follow with suitable modifications for the different boundary conditions [22, 23]. As with Fröhlich's paper, we need only prove that

$$\pm\beta\phi(f) \leqq \hat{H}_l + O(\beta^{2m/2m-1}) \tag{34}$$

with l independent constants. Let f have support in $[-a/2, a/2] \times [-a/2, a/2]$. Then, by the general results in [44], (34) follows if we prove that

$$\hat{H}_a \pm \beta\phi(f_t) \geqq -O(\beta^{2m/2m-1}) \tag{35}$$

for each fixed t where $f_t(x) = f(x, t)$. Let $\alpha_\beta(P \pm \beta X)$ denote the energy per unit volume for the interaction $:P \pm \beta X:$ [19, 20]. By the improved linear lower bound [21]:

$$\hat{H}_a \pm \beta\phi(f_t) \geqq \text{const} - a\alpha_\infty(P \pm \beta\|f\|_\infty X) . \tag{36}$$

(35) follows from (36) and the bound [23]:

$$\alpha_\infty(P \pm \beta X) \leqq O(\beta^{2m/2m-1})$$

for suitable a. □

APPENDIX

A Borel-Cantelli lemma. In this paper we required a generalized Borel–Cantelli lemma. As we noted in the text, the case we need is covered by a general theorem of Erdös–Rényi. In this appendix we want to prove a simple result which also covers the case we need:

THEOREM. *Let* $\{A_n\}$ *be a sequence of sets in a probability measure space. Suppose that*:

$$|\mu(A_n \cap A_m) - \mu(A_n)\mu(A_m)| \leqq C_{nm}\mu(A_n)^{\frac{1}{2}}\mu(A_m)^{\frac{1}{2}} \tag{A.1}$$

with $\{C_{nm}\}$ *the matrix of a bounded operator on* l_2. *Let* $A_\infty = \bigcap_{n=1}^\infty \bigcup_{m=n}^\infty A_m$. *Then*:

$$\mu(A_\infty) = 0 \quad \text{if} \quad \sum \mu(A_n) < \infty , \tag{A.2}$$

$$\mu(A_\infty) = 1 \quad \text{if} \quad \sum \mu(A_n) = \infty . \tag{A.3}$$

PROOF. $x \in A_\infty$ if and only if x is in infinitely many A_n, if and only if $\sum \chi_n(x) = \infty$ where $\chi_n =$ characteristic function of A_n. (A.1) follows from Fubini's theorem for

$$\int \sum \chi_n \, d\mu = \sum \int \chi_n \, d\mu = \sum \mu(A_n) < \infty$$

implies $\sum \chi_n < \infty$, a.e.

Now suppose $\sum \mu(A_n) = \infty$. Let $S_N = \sum_1^N \chi_n$. Since S_N is monotone, we need only prove that for any integer k, $S_N \geqq k$ a.e.; or equivalently, that for any k, m, there is an N with $\mu\{x \mid S_N < k\} \leqq m^{-1}$. Fix k, m. Let $\mu_n = \mu(A_n)$, $f_n = \mu_n 1 - \chi_n$, $F_N = \sum_1^N f_n$; $U_N = \sum_1^N \mu_n$. Then, in $L^2(d\mu)$:

$$|(f_n, f_m)| = |\mu(A_n \cap A_m) - \mu_n \mu_m| \leqq C_{nm} \mu_n^{\frac{1}{2}} \mu_m^{\frac{1}{2}} .$$

Thus

$$\|F_N\|_{L^2}^2 = \sum_{n,m \leqq N} C_{nm} \mu_n^{\frac{1}{2}} \mu_m^{\frac{1}{2}} \leqq \|C\| U_N .$$

Thus

$$\{x \mid F_N(x) \geqq \|C\|^{\frac{2}{3}} U_N^{\frac{2}{3}}\} \leqq \|C\|^{-\frac{1}{3}} U_N^{-\frac{1}{3}} .$$

Since $U_N \to \infty$, we can find N with $\|C\|^{-\frac{1}{3}} U_N^{-\frac{1}{3}} \leqq m^{-1}$ and $U_N - \|C\|^{\frac{2}{3}} U_N^{\frac{2}{3}} \geqq k$. For this N:

$$\{x \mid S_N(x) = U_N - F_N(x) \leqq k\} \leqq m^{-1} . \qquad \Box$$

In the case that interests us, C_{nm} is a convolution operator by a function in l_1 and so is a bounded operator.

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