On finite mass renormalizations in the two-dimensional Yukawa model*

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(Received 16 May 1975)

In the Mathews–Salam formulas for the (space–time cutoff) Schwinger functions of Y_2, no restriction on finite mass renormalizations for the boson is necessary.

1. INTRODUCTION

In an earlier paper[1] it was shown that the Matthews–Salam formulas[2] can be used to construct the Schwinger functions for Y_2, at least in the presence of a space–time cutoff. Unfortunately, this could be shown only under a certain restriction on the finite renormalization of the boson mass: It had to be essentially nonnegative. But in Glimm[3,4] and Glimm and Jaffe[5] it is shown that for the semiboundedness of the Y_2 Hamiltonian such a restriction is unnecessary. This is achieved by separating out contributions coming from low fermion momenta and estimating them in a way different from the estimate for the high momentum contribution. Since only the latter need renormalization the counterterms can be made smaller by choosing the "lower momentum cutoff" high enough. In this paper we carry through the same idea in the Euclidean framework invented by Matthews and Salam.[2]

2. IMPROVED INTEGRABILITY ESTIMATES

We use the notation of Ref. 1. In particular, we write for $A \in \mathcal{C}_{n+1}$ [that is, $\mathrm{Tr}(A^*A)^{(n+1)/2} < \infty$]

$$\det{}_{(n)}(1+A) = \det\left[(1+A)\exp\left(\sum_{k=1}^{n}\frac{(-1)^k}{k}A^k\right)\right]. \qquad (2.1)$$

If A is a (bounded) linear operator on a Hilbert space $\mathcal{H}$, we denote by $\Lambda^m(A)$ the operator induced by A on $\Lambda^m(\mathcal{H})$, the m-fold antisymmetric tensor product of $\mathcal{H}$.

Our main result is

Theorem 2.1:

$$u = \left\|\Lambda^m\left(\frac{1}{1+\lambda K}\right)\det{}^{(0)}_{\mathrm{ren}}(1+\lambda K)\right\|_{\mathcal{H}},$$

$$\times \exp\left((M^2\int :\phi^2(x): g^2(x)d^2x\right) \in \bigcap_{1\le p<\infty} L^p(d\mu_0)$$

for all $M^2 \in \mathbb{R}$, $\lambda \in \mathbb{R}$ ($\det^{(0)}_{\mathrm{ren}}$ denotes the renormalized determinant defined in Ref. 1 with the finite mass renormalization parameter M^2 appearing there put equal to 0).

From Theorem 2.1 we get immediately

Corollary 2.2: The finite volume Schwinger functions exist and fulfill

$$|S_g^{(M)}(h_1,\ldots,h_n;f_1,\ldots,f_m;g_1,\ldots,g_m)|$$

$$\le c_1\exp(nc_1+mc_2)\,\Gamma(n+\tfrac{1}{2})^{1/2}\prod_{r=1}^{n}\|h_r\|_{-1}\prod_{i=1}^{m}\|f_i\|_{\mathcal{H}^*}\|g_i\|_{\mathcal{H}^*} \qquad (2.2)$$

where the constants c_1, c_2, c_3 may depend on g and M.

Proof of the Corollary: By the Matthews–Salam formula [cf. (4.15) of Ref. 1] the left-hand side of (2.1) is

$$\left|\int\left(\frac{1}{\sqrt{p^2+m^2}}f_1\wedge\cdots\wedge\frac{1}{\sqrt{p^2+m^2}}f_m,\ \Lambda^m\left(\frac{1}{1+\lambda K}\right)\right.\right.$$

$$S_Fg_1\wedge\cdots\wedge S_Fg_m)_{\Lambda^m(\mathcal{H})}\prod_{r=1}^{n}\phi(h_r)$$

$$\left.\det{}^{(0)}_{\mathrm{ren}}(1+\lambda K)\exp[M^2\int :\phi^2:(x)g^2(x)\,d^2x]\,d\mu_0(\phi)\right|$$

$$\le \int u\left|\prod_{r=1}^{n}\phi(h_r)\right|d\mu_0\prod_{i=1}^{m}\|f_i\|_{\mathcal{H}^*}\|g_i\|_{\mathcal{H}^*}. \qquad (2.3)$$

To prove the theorem we have to split the operator K into two parts:

$$K = S_F\Gamma\phi g = L_\xi + H_\xi \qquad (2.4)$$

where

$$L_\xi = S_{F,\xi}\Gamma\phi g, \qquad (2.5)$$

$$S_{F,\xi} = \frac{1}{(2\pi)^2}\int_{|p|\le\xi}\frac{\not p+m}{p^2+m^2}\exp(ipx)\,d^2p. \qquad (2.6)$$

The crucial estimate to separate the contributions from L_ξ and H_ξ is contained in the following:

Lemma 2.3: Let $L \in \mathcal{C}_1$, $H \in \mathcal{C}_3$. Then

$$\left\|\Lambda^m\ \frac{1}{1+L+H}\ \det\{(1+L+H)\exp[-(L+H)+H^2/2]\}\right\|^2$$

$$\le \det{}_{(2)}(1+O_{H+})\exp[8\|L\|_1 - \tfrac{1}{2}\mathrm{Tr}(H^*H)^2 - 2\mathrm{Re}\,\mathrm{Tr}H^2H^*]$$

$$\times\exp(3m/2), \qquad (2.7)$$

where $O_H = H + H^* + H^*H$ and O_{H+} its nonnegative part. The proof is given in Sec. 3.

Corollary 2.4:

$$u \le \det{}_{(2)}(1+O_{H_\xi+})^{1/2}$$

$$\times\exp\left[-\frac{\lambda^4}{4}\mathrm{Tr}(H_\xi^*H_\xi)^2 - \lambda^3\,\mathrm{Re}\,\mathrm{Tr}H_\xi^*\exp\left(-\frac{\lambda^2}{2}\mathrm{Tr}_{\mathrm{reg}}:H_\xi^2:\right)\right]$$

$$\times \exp\left(4\lambda\|L_\zeta\|_1 + \frac{\lambda^2}{2}\langle \mathrm{Tr}\, L_\zeta^2\rangle \exp[M^2 \int :\phi^2(x): g^2(x)\, d^2x]\right). \tag{2.8}$$

where $\mathrm{Tr}_{\mathrm{reg}} :H_\zeta^2:$ is defined as in Ref. 1, that is, with subtraction of the full formal counterterm $\int d^2p/(p^2+m^2)$; hence

$$\mathrm{Tr}_{\mathrm{reg}} :K^2: = \mathrm{Tr}_{\mathrm{reg}} :H_\zeta^2: + 2\mathrm{Tr} :H_\zeta L_\zeta: + \mathrm{Tr} :L_\zeta^2:. \tag{2.9}$$

Next we deal with the low momentum part:

Lemma 2.5:

$$\exp(4\lambda\|L_\zeta\|_1) \in \bigcap_{1\le p<\infty} L^p(d\mu_0).$$

Proof: (a) $L_\zeta \in \mathcal{C}_1$ because it can be factored into two Hilbert–Schmidt operator. Let θ_ζ be the projection onto momenta $|p| \le \zeta$; χ a function which is 1 on supp g and fulfills

$$\int |p|^3 |\tilde{\chi}(p)|^2 d^2p < \infty, \tag{2.10}$$

$$A = \frac{1}{p+m}\theta_\zeta \chi(p^2+m^2), \tag{2.11}$$

$$B = \frac{1}{p^2+m^2}\phi_g. \tag{2.12}$$

Then $L_\zeta = A \cdot B$,

$$A^*A = \sqrt{p^2+m^2}\,\chi \frac{\theta_\zeta}{(p^2+m^2)^{1/2}}\chi(p^2+m^2), \tag{2.13}$$

$$B^*B = \frac{1}{(p^2+m^2)^{1/2}}\phi_g \frac{1}{(p^2+m^2)^{3/2}}\phi_g \tag{2.14}$$

(recall that we are working in $\mathcal{H} = \mathcal{H}_{1/2} \oplus \mathcal{H}_{1/2}$).

Then it is easy to see that

$$\mathrm{Tr} A^*A < K < \infty \quad (K \text{ independent of } \phi), \tag{2.15}$$

$$\mathrm{Tr} B^*B = (\phi, C\phi)_1, \tag{2.16}$$

where C is trace class in $\mathcal{H}_1$ (actually

$$C = \frac{1}{k^2+\mu^2} gEg\Big), \tag{2.17}$$

where E is a multiplication operator in momentum space:

$$E(k) = \int d^2p \frac{1}{[(p+k/2)^2+m^2]^{1/2}[(p-k/2)^2+m^2]^{3/2}}. \tag{2.18}$$

By using the numerical inequality

$$x \le \tfrac{1}{2}(1/\delta + \delta x^2) \quad (x \in \mathbf{R},\ \delta > 0), \tag{2.19}$$

we obtain

$$\exp(4\lambda\|L_\zeta\|_1) \le \exp(4\lambda\|A\|_2\|B\|_2) \le \exp[2\lambda\|A\|_2(1/\delta + \delta(\phi, C\phi)_1)]. \tag{2.20}$$

The right-hand side is in L^p for small enough δ.

The high momentum part is estimated essentially as in Ref. 1. But there is a modification because our estimate (2.7) involves $\det_{(2)}(1+O_{H+})$ instead of $\det_{(3)}(1+O_{H+})$. If we estimate

$$w_\zeta(\phi) = \det_{(2)}(1+O_{H+})^{1/2}\exp\left(-\frac{\lambda^4}{4}\mathrm{Tr}(H_\zeta H_\zeta)^2\right) \times \exp(\lambda^3\, \mathrm{Re}\,\mathrm{Tr} H_\zeta^2 H_\zeta^*) \tag{2.21}$$

by introducing a cutoff κ in ϕ and consider

$$\ln w_\zeta(\phi) - \ln w_\zeta(\phi_\kappa), \tag{2.22}$$

we get, using Lemmas 3.1, 3.2 of Ref. 1,

$$\begin{aligned}\ln w_\zeta(\phi) &- \ln w_\zeta(\phi_\kappa)\\ &\le \frac{\lambda^2}{4}[\mathrm{Tr}(H_\zeta^*(\phi)H_\zeta(\phi))^2 - \mathrm{Tr}(H_\zeta^*(\phi_\kappa)H_\zeta(\phi_\kappa))^2]\\ &+ \lambda^3[\mathrm{Tr} H_\zeta(\phi)^2 H_\zeta^*(\phi) - \mathrm{Tr} H_\zeta(\phi_\kappa)^2 H_\zeta^*(\phi_\kappa)]\\ &+ \|O_{H_\zeta(\phi)} - O_{H_\zeta(\phi_\kappa)}\|_4 \sum_{k=0}^{3} c_k \|H_\zeta(\phi)\|_4^k \|H_\zeta(\phi_\kappa)\|_4^{3-k}\\ &+ \tfrac{1}{3}\mathrm{Tr}(O^3_{H_\zeta(\phi)-} - O^3_{H_\zeta(\phi_\kappa)-});\end{aligned} \tag{2.23}$$

The only new term is the last one. We have to show that

$$\int |\mathrm{Tr}(O^3_{H_\zeta(\phi)-} - O^3_{H_\zeta(\phi_\kappa)-})|^2 d\mu_0 = O(\kappa^{-\epsilon}). \tag{2.24}$$

This follows from the following two lemmas.

Lemma 2.6: Let $A, B \in \mathcal{C}_3$ be self-adjoint. Then

$$|\mathrm{Tr}(A_+^3 - B_+^3)| \le \|A-B\|_4(\|A\|_{8/3}^2 + \|A\|_{8/3}\|B\|_{8/3} + \|B\|_{8/3}^2).$$

Proof: Denote by λ_i, μ_i the eigenvalues of A_+, B_+, respectively (ordered decreasingly):

$$\begin{aligned}|\mathrm{Tr}(A_+^3 - B_+^3)| &= |\textstyle\sum(\lambda_i - \mu_i)(\lambda_i^2 + \lambda_i\mu_i + \mu_i^2)| \le (\sum|\lambda_i - \mu_i|^4)^{1/4}\\ &\times (\textstyle\sum(\lambda_k^2 + \lambda_k\mu_k + \mu_k^2)^{4/3})^{3/4}\\ &\le \|A-B\|_4(\|A\|_{8/3}^2 + \|A\|_{8/3}\|B\|_{8/3} + \|B\|_{8/3}^2)\end{aligned}$$

by Lemma 3.2 of Ref. 1.

We next require a general interpolation theorem for the spaces $\mathcal{C}_p$; explicitly, the following three-lines theorem:

Proposition: Let K_z be an analytic operator-valued function in the strip $S = \{z \in \mathbb{C} \mid a < \mathrm{Re} z < b\}$, weakly continuous on the closure of S with $(\varphi, K_z\psi)$ bounded for a dense set of φ and ψ. Suppose that $K_{a+iy} \in \mathcal{C}_{p_0}$ for all real y, $K_{b+iy} \in \mathcal{C}_{p_1}$ for all real y with $\alpha = \sup_y \|K_{a+iy}\|_{p_0} < \infty$ and $\beta = \sup_y \|K_{b+iy}\|_{p_1} < \infty$. Then for any $z \in \bar{S}$, $K_z \in \mathcal{C}_{p_t}$ with $\ln\|K_z\|_{p_t} \le t\ln\beta + (1-t)\ln\alpha$ where $t = (\mathrm{Re} z - a)b - a$ and $p_t^{-1} = tp_1^{-1} + (1-t)p_0^{-1}$.

Interpolation theorems fall into three closely related types: three line lemmas, Riesz–Thorin theorems, and Stein theorems. Kunze[6] proved a general Riesz–Thorin theorem for $\mathcal{C}_p$ spaces and Calderon[7] made a general analysis of interpolation spaces. By combining these works, one gets the Proposition above (see, e.g., Reed and Simon[8], Appendix to Sec. IX.4). The proposition has been independently discovered by Gohberg, Krein, and Krein and the reader can find a self-contained proof on pp. 137-139 of Ref. 9.

Lemma 2.7: For every $\epsilon > 0$, K, H_ζ, O_k, O_{H_ζ} are in $\mathcal{C}_{2+\epsilon}$ a.e.

Proof: We give the proof for K; for the other cases, the proof is analogous. We apply the proposition with

$$z \mapsto K_z = \frac{p+m}{(p^2+m^2)^z}\phi_g \tag{2.25}$$

on the strip

$$S = \{z \in \mathbb{C} : \tfrac{3}{4}+\delta \le \mathrm{Re} z \le 1+\delta\} \quad (0<\delta<\tfrac{1}{4}). \tag{2.26}$$

Since it is easy to see that $K_z \in \mathcal{C}_4$ a.e. for $\mathrm{Re} z = \frac{3}{4}+\delta$ and $K_z \in \mathcal{C}_2$ a.e. for $\mathrm{Re} z = 1+\delta$ it follows that $K = K_1 \in \mathcal{C}_{2/(1-2\delta)}$ a.e. and

$$\ln\|K\|_{2/(1-2\delta)} \le 4\delta \ln\|K_{3/4+\delta}\|_4 + (1-4\delta)\ln\|K_{1+\delta}\|_2. \tag{2.27}$$

Lemma 2.6 and 2.7 allow to estimate the non-Gaussian part of $w_\zeta(\phi)$ in the same way as this is done in Ref. 1 (essentially Nelson's argument). The Gaussian parts coming from H_ζ are as in Ref. 1

$$u_\zeta = \exp\left(\frac{\lambda^2}{2}\mathrm{Tr}_{\mathrm{reg}} : H_\zeta^* H_\zeta :\right).$$

Lemma 2.8:

$$u_\zeta = \exp\left(\frac{\lambda^2}{2}\mathrm{Tr}_{\mathrm{reg}} : H_\zeta^* H_\zeta :\right) = \exp[-(\lambda^2/2)(\phi, B_\zeta\phi)_1] \tag{2.28}$$

where B_ζ is a positive Hilbert–Schmidt operator on $\mathcal{H}_{+1}$ and

$$B_\zeta \ge \frac{1}{k^2+\mu^2}\pi\ln(1+\zeta^2/m^2)g^2 \tag{2.29}$$

Proof:

$$H_\zeta = (S_F - S_{F,\zeta})\phi_g, \tag{2.30}$$

$$B_\zeta = \frac{1}{k^2+\mu^2} g G_\zeta g, \tag{2.31}$$

$$G_\zeta = -\int d^2p\left(\frac{\theta(p_+^2-\zeta^2)}{(p_+^2+m^2)^{1/2}}\frac{1}{(p_-^2+m^2)^{1/2}} - \frac{1}{p^2+m^2}\right) \tag{2.32}$$

($p_\pm = p \pm k/2$). Using $ab \le \frac{1}{2}(a^2+b^2)$ we get

$$G_\zeta \ge \tfrac{1}{2}\int d^2p\left(\frac{1}{p^2+m^2} - \frac{\theta(p_+^2-\zeta^2)}{p_+^2+m^2}\right) + \tfrac{1}{2}\int d^2p\left(\frac{1}{p^2+m^2} - \frac{1}{p_-^2+m^2}\right) = \tfrac{1}{2}\int_{|p|\le\zeta} d^2p\,\frac{1}{p^2+m^2} = \pi\ln(1+\zeta^2/m^2). \tag{2.33}$$

(The last equality follows because

$$\int d^2p\left(\frac{1}{p^2+m^2} - \frac{1}{p_-^2+m^2}\right) = \int d^2p\left(\frac{1}{(p+k/4)^2+m^2} - \frac{1}{(p-k/4)^2+m^2}\right)$$

$= 0$ by symmetry.)

Now it is clear that we only have to choose ζ such that

$$\pi\lambda^2\ln(1+\zeta^2/m^2) \ge M^2 \tag{2.34}$$

to make $u_\zeta \cdot \exp(M^2\int :\phi^2: g^2\,dx) \in \bigcap_{1\le p<\infty} L^p$. This completes the proof of Theorem 2.1.

3. DETERMINANT INEQUALITIES

In this section we make use of ideas of Lieb[10] and Kato (private communication) in order to prove the crucial lemma 2.3.

Lemma 3.1: Let A, B be linear operators from a M-dimensional (real or complex) Hilbert space $\mathcal{H}_M$ to N-dimensional Hilbert space $\mathcal{H}_N$. Then

$$\|\Lambda^m(1/A^*B)\det(A^*B)\|^2 \le \|\Lambda^m(1/A^*A)\det(A^*A)\|\,\|\Lambda^m(1/B^*B)\det(B^*B)\|. \tag{3.1}$$

Remark: Note that $\Lambda^m(A^{-1})\det A$ is a polynomial in the matrix elements of A, so (3.1) makes sense (and is true) also for singular A or B.

Proof: We use the polar decompositions

$A = U|A|$, $B = V|B|$ where $|A| = (A^*A)^{1/2}$, $B = (B^*B)^{1/2}$;

U, V are partial isometries. Then $A^*B = |A|C|B|$ where $C = U^*V$ is a contraction in $\mathcal{H}_M$. The left-hand side of (3.1) is then bounded by

$$\|\Lambda^m(1/|A|)\det|A|\|^2\|\Lambda^m(1/|B|)\det|B|\|^2 \times \|\Lambda^m(1/C)\det C\|^2 \tag{3.2}$$

and the last factor is ≤ 1 as can be seen by replacing C by $|C|$ (unitaries do not matter); then $\|\Lambda^m(1/|C|)\times\det|C|\|$ is the product of the m largest eigenvalues of $|C|$. The first two factors in (3.2) give the right-hand side of (3.1).

Lemma 3.2: Let A_i, B_i ($i=1,\ldots,n$) be linear operators from $\mathcal{H}_M$ to $\mathcal{H}_M$. Then

$$\|\Lambda^m(1/\textstyle\sum_i A_i^*B_i)\det\sum_k A_k^*B_k\|^2 \le \|\Lambda^m(1/\sum_i A_i^*A_i)\det\sum_k A_k^*A_k\|\,\|\Lambda^m(1/\sum_i B_i^*B_i) \times\det\sum_k B_k^*B_k\|. \tag{3.3}$$

Proof: This is a special case of Lemma 3.1 ($\mathcal{H}_N = \oplus_{k=1}^n \mathcal{H}_M$).

Remark: Lemma 3.2 has been brought to our attention by Lieb who first proved it for the special case $m=0$ and then proved a general result[10] from which (3.3) follows easily. The idea of the proof given here (the reduction to Lemma 3.1) is due to Kato (private communication).

Lemma 3.3: Let A, B be trace class operators on a Hilbert space $\mathcal{H}$. Then

$$\left\|\Lambda^m\left(\frac{1}{1+A+B}\right)\det(1+A+B)\right\|^2 \le \left\|\Lambda^m\left(\frac{1}{|1+A|+|B|}\right)\det(|1+A|+|B|)\right\| \times\left\|\Lambda^m\left(\frac{1}{|1+A|+W|B|W^{-1}}\right)\det(|1+A|+W|B|W^{-1})\right\| \tag{3.4}$$

with a unitary W. (For $m=1$ this result also can be found in Lieb.[10])

Proof: If $\mathcal{H}$ has finite dimension, this is a special case of Lemma 3.2 with $A_1 = B_1 = 1$, $A_2 = U|A|^{1/2}$, $B_2 = |A|^{1/2}$, $A_3 = V|B|^{1/2}$, $B_3 = |B|^{1/2}$ $(A = U|A|,\ B = V|B|,\ W = U^*V)$. The infinite-dimensional case follows from an easy limiting argument (see Appendix, Proposition 2).

Lemma 3.4: If $0 \le A \le 1$ is a $\mathcal{C}_{n+1}$ operator on $\mathcal{H}$ (i.e., $\mathrm{Tr}|A|^{n+1} < \infty$), then $\|\Lambda^m[1/(1-A)]\det_{(n)}(1-A)\| \le \exp[m(1+\frac{1}{2}+\cdots+1/n)]$. (3.5)

Proof: If $\alpha_1 \ge \alpha_2 \ge \cdots \ge 0$ are the eigenvalues of A, the left-hand side is

$$\prod_{i\ge1}(1-\alpha_i)\exp(\alpha_i+\alpha_i/2+\cdots+\alpha_i^n/n)\prod_{i\le m}\frac{1}{1-\alpha_i}$$

$$\le \prod_{i\le m}\exp(\alpha_i+\alpha_i^2/2+\cdots+\alpha_i^n/n)$$

$$\le \exp[m(1+1/2+\cdots+1/n)].$$

Lemma 3.5 (= Lemma 2.3): For $A \in \mathcal{C}_3$, $B \in \mathcal{C}_1$

$$\left\|\Lambda^m\left(\frac{1}{1+A+B}\right)\det((1+A+B)\exp(-A+A^2/2))\right\|^2$$

$$\le \det_{(2)}(1+O_{A+})\exp(6\|B\|_1+\tfrac{3}{2}m-\tfrac{1}{2}\|A\|_4^4-2\,\mathrm{Re}\,\mathrm{Tr}A^2A^* \tag{3.6}$$

where $O_{A+} = (A + A^* + A^*A)_+$.

Proof: It is sufficient to give the proof for $A, B \in \mathcal{C}_1$ (see Appendix, Proposition 4). By Lemma 3.3 it suffices to estimate

$$X = \left\|\Lambda^m\left(\frac{1}{1+C+|B|}\right)\det(1+C+|B|)\right\| \tag{3.7}$$

where $C = |1+A| - 1 = C_+ - C_-$ $(\Rightarrow 0 \le C_- \le 1)$. With

$$D = \frac{1}{\sqrt{1+C_+}}|B|\frac{1}{\sqrt{1+C_+}},\quad E = \frac{1}{\sqrt{1+D}}C_-\frac{1}{\sqrt{1+D}}$$

we have (since $C_+C_- = C_-C_+ = 0$)

$$X \le \det(1+C_+)\left\|\Lambda^m\left(\frac{1}{1+D-C_-}\right)\det(1+D-C_-)\right\|$$

$$\le \det(1+C_+)\det(1+D),$$

$$\left\|\Lambda^m\left(\frac{1}{1-E}\right)\det(1-E)\right\| \le \det(1+C_+)\det(1+D)$$

$$\times\exp(\tfrac{3}{2}m)\exp(-\mathrm{Tr}E-\mathrm{Tr}E^2/2) \tag{3.8}$$

by Lemma 3.4.

Now

$$\mathrm{Tr}E = \mathrm{Tr}(1+D)^{-1}C_- \ge \mathrm{Tr}(1-D)C_- \ge \mathrm{Tr}C_- - \mathrm{Tr}D\|C_-\|$$

$$\ge \mathrm{Tr}C_- - \mathrm{Tr}|B|, \tag{3.9}$$

$$\mathrm{Tr}E^2 = \mathrm{Tr}\left[\left(1-\frac{D}{1+D}\right)C_-\right]^2$$

$$= \mathrm{Tr}C_-^2 - 2\,\mathrm{Tr}\frac{D}{1+D}C_-^2 + \mathrm{Tr}\left(\frac{D}{1+D}C_-\right)^2$$

$$\ge \mathrm{Tr}C_-^2 - 2\,\mathrm{Tr}D \ge \mathrm{Tr}C_-^2 - 2\,\mathrm{Tr}|B|. \tag{3.10}$$

Because of $1 + O_{A+} = |1+A|^2P_+ + (1-P_+) = (1+C)^2P_+ + (1-P_+) = (1+C_+)^2$ (where P_+ is the projection associated with C_+) we get

$$X \le \det(1+O_{A+})^{1/2}\det(1+|B|)\exp(2\,\mathrm{Tr}|B|+3m/2)$$

$$\times\exp(-\mathrm{Tr}C_- - \mathrm{Tr}C_-^2/2) = \det_{(2)}(1+O_{A+})^{1/2}\det(1+|B|)$$

$$\times\exp(2\,\mathrm{Tr}|B|+3m/2)$$

$$\times\exp(-\mathrm{Tr}C_- - \tfrac{1}{2}\mathrm{Tr}C_-^2 + \tfrac{1}{2}\mathrm{Tr}O_{A+} - \tfrac{1}{4}\mathrm{Tr}O_{A+}^2) \tag{3.11}$$

We claim

$$\mathrm{Tr}C_- + \tfrac{1}{2}\mathrm{Tr}C_-^2 \ge \tfrac{1}{2}\mathrm{Tr}O_{A-} + \tfrac{1}{4}\mathrm{Tr}O_{A-}^2. \tag{3.12}$$

Proof: Since $(1-C_-)^2 = 1 - O_{A-}$,

$$\mathrm{Tr}(C_- + \tfrac{1}{2}C_-^2) = \mathrm{Tr}(1-\sqrt{1-O_{A-}}) + \tfrac{1}{2}\mathrm{Tr}(1-\sqrt{1-O_{A-}})^2$$

$$= 2\mathrm{Tr}(1-\sqrt{1-O_{A-}}) - \tfrac{1}{2}\mathrm{Tr}O_{A-} \ge \tfrac{1}{2}\mathrm{Tr}O_{A-} + \tfrac{1}{4}\mathrm{Tr}O_{A-}^2$$

$(\sqrt{1-x} \le 1 - x/2 - x^2/8$ for $0 \le x \le 1)$.

Therefore,

$$X \le \det_{(2)}(1+O_{A+})^{1/2}\exp(3\|B\|_1 + \tfrac{3}{2}m)$$

$$\times(\tfrac{1}{2}\mathrm{Tr}O_A - \tfrac{1}{4}\mathrm{Tr}O_A^2). \tag{3.13}$$

The lemma now follows from

$$\tfrac{1}{2}\mathrm{Tr}O_A - \tfrac{1}{4}\mathrm{Tr}O_A^2 - \mathrm{Re}\,\mathrm{Tr}A + \tfrac{1}{2}\mathrm{Re}\,\mathrm{Tr}A^2$$

$$= -\tfrac{1}{4}\|A\|_4^4 - \mathrm{Re}\,\mathrm{Tr}A^2A^*. \tag{3.14}$$

ACKNOWLEDGMENT

We are grateful to E. Lieb for making the results of Ref. 10 available to us before publication and to T. Kato for communicating to us the idea for the simple proof of Lemma 3.2. One of us (E.S.) wishes to thank the Institute for Advanced Study for its hospitality.

Remark: When this work was essentially completed we received a paper by O.A. McBryan[11] on the same subject which also proves Theorem 2.1; the method used there, an expansion of the low momentum interaction, is quite different.

APPENDIX: CONTINUITY OF CERTAIN OPERATOR FUNCTIONS

Proposition 1: For $A \in \mathcal{C}_1$

$$\left\|\Lambda^m\left(\frac{1}{1+A}\right)\det(1+A)\right\| \le \exp(\|A\|_1 + m). \tag{A1}$$

(This estimate is not best possible: The factor e^m can be eliminated.)

Proof:

$$\left\|\Lambda^m\left(\frac{1}{1+A}\right)\det(1+A)\right\|$$

$$= \sup_{\{e_i, f_k\}}\left|(e_1\wedge\cdots\wedge e_m, \Lambda^m\frac{1}{1+A}f_1\wedge\cdots\wedge f_m)\right.$$

$$\left.\times\det(1+A)\right| \tag{A2}$$

where the sup is over orthonormal systems of vectors e_i, f_i $(i = 1, \ldots, m)$. If we denote by C_k the operator which maps $u \in \mathcal{H}$ into $e_k(f_k, u)$ $(k = 1, \ldots, m)$, then

$$
\begin{aligned}
&\langle e_1 \wedge \cdots \wedge e_m, \Lambda^m\left(\frac{1}{1+A}\right) f_1 \wedge \cdots \wedge f_m\rangle \det(1+A) \\
&= \frac{\partial^m}{\partial\lambda_1 \cdots \partial\lambda_m} \det\left(1 + A + \sum_{k=1}^{m} \lambda_k C_k\right) \\
&= \left(\frac{1}{2\pi i}\right)^m \oint_{|\lambda_1|=1} \frac{d\lambda_1}{\lambda_1^2} \cdots \oint_{|\lambda_m|=1} \frac{d\lambda_m}{\lambda_m^2} \det\left(1 + A + \sum_{k=1}^{m} \lambda_k C_k\right). \qquad \text{(A3)}
\end{aligned}
$$

Using the well-known inequality

$$|\det(1+B)| \le \exp(\|B\|_1) \qquad \text{(A4)}$$

we obtain (A1).

Proposition 2: For $A, B \in \mathcal{C}_1$

$$
\begin{aligned}
&\left\|\Lambda^m\left(\frac{1}{1+A}\right)\det(1+A) - \Lambda^m\left(\frac{1}{1+B}\right)\det(1+B)\right\| \\
&\le \|A-B\|_1 \exp(\|A\|_1 + \|B\|_1 + m + 1). \qquad \text{(A5)}
\end{aligned}
$$

Proof: Without loss of generality we can assume $A \ne B$. Consider then

$$F(t) = \Lambda^m\left(\frac{1}{1+A+t(B-A)}\right)\det(1+A+t(B-A)). \qquad \text{(A6)}$$

The left-hand side of (A5) is bounded by

$$\int_0^1 \|F'(t)\|\, dt \le \sup_{t\in[0,1]} \|F'(t)\|. \qquad \text{(A7)}$$

By Cauchy's formula we have

$$\|F'(t)\| = \left\|\frac{1}{2\pi i}\oint_{|\tau|=\zeta} \frac{dt}{\tau^2} F(t+\tau)\right\| \le \frac{1}{2\pi}\oint_{|\tau|=\zeta} \left|\frac{dt}{\tau^2}\right| \|F(t+\tau)\|. \qquad \text{(A8)}$$

Choosing $|\tau| = \zeta = (\|A-B\|_1)^{-1}$ and using proposition 1, we obtain

$$
\begin{aligned}
\|F(t+\tau)\| &\le \exp[\|(1-t)A + tB + \tau(B-A)\|_1 + m] \\
&\le \exp(\|A\|_1 + \|B\|_1 + m + 1) \qquad \text{(A9)}
\end{aligned}
$$

and, therefore,

$$\|F'(t)\| \le \|A-B\|_1 \exp(\|A\|_1 + \|B\|_1 + m + 1) \qquad \text{(A10)}$$

which proves the assertion by (A6).

Proposition 3: The function

$$R_n : \mathcal{C}_n \to \mathcal{C}_1$$

$$A \mapsto R_n(A) = (1+A)\exp\left(-\sum_{k=1}^{n-1}\frac{(-A)^k}{k}\right) - 1 \qquad \text{(A11)}$$

is continuous.

Proof: $R_n(A) = A^n G(A)$ where G is an entire function. Therefore

$$\|R_n(A) - R_n(B)\|_1 \le \|A^n - B^n\|_1 \|G(A)\| + \|B^n\|_1 \|G(A) - G(B)\|. \qquad \text{(A12)}$$

Repeated use of Hölder's inequality for operators gives

$$\|A^n - B^n\|_1 \le \|A-B\|_n \sum_{k=0}^{n-1} \|A\|_n^k \|B\|_n^{n-1-k}. \qquad \text{(A13)}$$

Proposition 4: The function

$L_n^m : \mathcal{C}_n \to \mathcal{L}(\Lambda^m\mathcal{H})$ (bounded operators on $\Lambda^m\mathcal{H}$),

$$A \mapsto L_n^m(A) = \Lambda^m\left(\frac{1}{1+A}\right)\det{}_{(n-1)}(1+A)$$

is continuous.

Proof: $L_n^m(A) = \Lambda^m\left(\exp\sum_{k=1}^{n}\frac{(-A)^k}{k}\right)$,

$$\Lambda^m\left(\frac{1}{1+R_n(A)}\right)\det(1+R_n(A)). \qquad \text{(A14)}$$

The first factor is obviously continuous; Proposition 5 then follows from Proposition 3 and Proposition 4.

Remark: For special cases ($m = 0$ or 1) most results of this appendix can be found in Refs. 9 and 12.

*This work was partially supported by USNSF grants GP-40768X and GP-39048.

†On leave from the Max Planck Institute für Physik und Astrophysik, München, Germany.

‡A Sloan Foundation Fellow.

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