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Nelson's Symmetry and All That in the Yukawa$_2$ and $(\phi^4)_3$ Field Theories*

E. SEILER†

Department of Physics, Harvard University, Cambridge, Massachusetts 02138

AND

B. SIMON‡

Departments of Mathematics and Physics, Princeton University, Princeton, New Jersey 08540

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We prove Guerra's theorem, ϕ bounds and Fröhlich bounds in the Y_2 and $\phi_3{}^4$ field theories. Among our technical results of interest is a proof that $Z \neq 0$ in $\phi_3{}^4$ and that the spatially cutoff vacuum in Y_2 has a charge zero component. The two main inputs are Osterwalder–Schrader positivity in the spatial direction as well as in the time direction, and a finite renormalization of the "usual" partition function and Hamiltonian so that Euclidean and Hamiltonian counterterms match exactly.

1. INTRODUCTION

Our goal in this paper is the extension of the results proven for $P(\phi)_2$ in Guerra, Rosen, and Simon [28] (including results obtained earlier by Glimm–Jaffe [20] and Guerra [27]) to the more singular Y_2 and $\phi_3{}^4$ theories. To explain the technical difficulties we must overcome, let us briefly sketch the GRS proof of a special case of the ϕ bound:

$$-\int_{-1/2}^{1/2} \phi(x)\, dx \leqslant (H_l - E_l) + C. \tag{1}$$

By a remark of Glimm–Jaffe [20], one easily reduces the operator estimate (1) to proving

$$-E_l' \leqslant -E_l + C, \tag{2}$$

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† On leave from Max-Planck Institut für Physik und Astrophysik München.

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where E_l' is the infimum of the spectrum of $H_l + \int_{-l/2}^{l/2} \phi(x)\,dx \equiv H_l'$. Now

$$-E_l' = \lim_{t\to\infty} (1/t) \ln(\Omega_0, \exp(-tH_l')\,\Omega_0) \tag{3}$$

and by Nelson's symmetry:

$$\begin{aligned}(\Omega_0, \exp(-tH_l')\,\Omega_0) &= (\Omega_0, \exp(-(l-1)/2H_t)\exp(-\hat{H}_t)\exp(-(l-1)/2H_t)\,\Omega_0)\\ &\leqslant (\Omega_0, \exp(-(l-1)\,H_t)\,\Omega_0)\,\|\exp(-\hat{H}_t)\|\\ &= (\Omega_0, \exp(-tH_{l-1})\,\Omega_0)\,\|\exp(-\hat{H}_t)\|,\end{aligned}$$

where $\hat{H}_t = H_t + \int_{-t/2}^{t/2} \phi(x)\,dx$, and we have used $|(\psi, A\phi)| \leqslant \|\psi\|\,\|\phi\|\,\|A\|$. The linear lower bound for $\hat{H}_t$ [19] then leads to $-E_l' \leqslant -E_l + C_1$ from which one can prove (2) by showing $-E_{l-1} \leqslant -E_l + C_2$.

This argument breaks down in three places when one tries to extend it directly to Y_2:

(I) Nelson's symmetry is not true for the Hamiltonian in Y_2 as conventionally defined [15, 17, 21].

(II) It is not clear how to define an operator A so that $|(\psi, A\phi)| \leqslant \|A\|\,\|\phi\|\,\|\Psi\|$ can be used, becaues the Markov property is not known for this theory.

(III) The formula (3) cannot be proven as in $P(\phi)_2$ (or at least not with the present technology); that is the $P(\phi)_2$ proof that the vacuum for H_l' is non-orthogonal to Ω_0 does not extend.

Let us expand briefly on each of these points explaining how we will overcome them.

(I) *Failure of Nelson's Symmetry*

At first sight this seems surprising. To explain its origins, let us suppose that we choose to make second-order mass and energy renormalizations in $P(\phi)_2$ analogous to those made in Y_2 (except that they are finite). Thus we renormalize H_l by

$$H_l^{\text{ren}} = H_l - \delta E_2(l) + \delta m_2^2 \int_{-l/2}^{l/2} {:}\phi^2(x){:}\; d^2x,$$

where δm_2 is an l-independent constant and $\delta E_2(l) = -\langle V_l H_0^{-1} V_l\rangle$. Similarly, we renormalize $Z_{l,t}$ by:

$$Z_{l,t}^{\text{ren}} = \hat{Z}_{l,t}\exp(-\delta P_2(l, t)),$$

where

$$\hat{Z}_{l,t} = \int d\mu_0 \exp\left[-\int_{-l/2}^{l/2} dx \int_{-t/2}^{t/2} ds\ (:P(\phi(x, s)): + \delta m_2^2 :\phi^2(x, s):)\right]$$

$$\delta P_2(l, t) = \tfrac{1}{2} \int \left[\int_{-l/2}^{l/2} dx \int_{-t/2}^{t/2} ds :P(\phi(x, s)):\right]^2 d\mu_0 .$$

$\hat{Z}_{l,t}$ is certainly still symmetric in l and t but

$$\langle \Omega_0 , \exp(-tH_l^{\text{ren}})\, \Omega_0\rangle \neq \langle \Omega_0 , \exp(-lH_t^{\text{ren}})\, \Omega_0\rangle$$

because

$$Z_{l,t}^{\text{ren}} \neq \langle \Omega_0 , \exp(-tH_l^{\text{ren}})\, \Omega_0\rangle.$$

For, this would require that $\delta P_2(l, t) = -t\delta E_2(l)$, and one computes

$$\begin{aligned}\delta P_2(l, t) &= \tfrac{1}{2} \int_{-t/2}^{t/2} ds \int_{-t/2}^{t/2} du\, \langle \Omega_0 , V_l \exp(-\mid u - s \mid H_0)\, V_l\Omega_0\rangle \\ &= \int_0^t ds \int_0^s du\, \langle V_l\Omega_0 , \exp(-uH_0)\, V_l\Omega_0\rangle \\ &= +t\langle V_l\Omega_0 , H_0^{-1}V_l\Omega_0\rangle - \langle V_l\Omega_0 , H_0^{-1}(1 - \exp(-tH))\, H_0^{-1}V_l\Omega_0\rangle.\end{aligned}$$

Put differently, for Nelson's symmetry to hold, we need to choose the constant counterterm $\delta E_2(l)$ linear in l.

Because of the above computation, Nelson's symmetry holds for the usually defined Y_2 Hamiltonian only if an extra correction term $\langle H_0^{-1}V_l\Omega_0 , (1 - \exp(-tH_0)\, H_0^{-1}V_l\Omega_0\rangle / \langle H_0^{-1}V_t\Omega_0 , (1 - \exp(-lH_0))\, H_0^{-1}V_t\Omega_0\rangle$ is included. Our first proof of Guerra's theorem carried this correction term along in much the way the correction term in Nelson's symmetry for $P(\phi)_2$ with periodic boundary conditions is carried along [31]. McBryan [38] in proving a lower bound on the pressure in Y_2 followed a similar strategy.

Nevertheless, it is an attractive idea that if we choose a different second-order energy counterterm, we might obtain a Hamiltonian for which Nelson's symmetry is exact. Obviously, we seek a counterterm $\delta E_2(l)$ linear in l, or what is the same thing, a counterterm $\delta P_2(l, t)$ proportional to lt. For Y_2 such a choice will be possible. For $(\phi^4)_3$ such a choice is not possible, if we wish to make the theory finite, but what we will see is that we can make the choice symmetric in l_1, l_2, and t so that $\delta P_2(l_1, l_2, t) = c_1(l_1, l_2)t + c_2(l_1, l_2)$. Such a choice leads to a modified Nelson's symmetry and also allows a purely Euclidean construction of a Hamiltonian.

(II) *Absence of a Markov Property*

It is an open question whether one can develop a Euclidean fermion theory for the *interacting* Y_2 theory so that the Markov property holds although it is known that many "reasonable" approaches fail [13]. For $(\phi^4)_3$ with space–time cutoff or with only spatial cutoff, we expect that the usual Markov property will hold but its verification may be as difficult as for the infinite volume $P(\phi)_2$ theory where it is still an open question! As explained in [50], many of the arguments in [28] are essentially consequences of the Markov property in spatial directions, so we cannot directly mimic the methods of [28].

The way out of this impasse will be to systematically exploit Osterwalder-Schrader [41] positivity especially, but not exclusively, in the spatial direction. We will do this not only in proving Guerra's theorem and ϕ and Fröhlich bounds, but in giving a purely Euclidean construction of the $(\phi^4)_3$ Hamiltonian. This construction is very close to that of Guerra, Rosen, and Simon [30] for $P(\phi)_2$ but differs in that OS positivity replaces the Markov property. Just as the GRS construction is modeled on Nelson's reconstruction theorem, ours is modeled on the Osterwalder–Schrader reconstruction theorem. Interestingly enough, the effective dressing transformation in this construction will be the Hamiltonian semigroup.

(III) *Problem of Vacuum Overlap*

A key element in many proofs (where formulas like (3) are used) is that the Fock vacuum is not orthogonal to the spatially cutoff vacuum. In $P(\phi)_2$, this is proven by postivity arguments of Perron–Frobenius type [18]. It is possible to formulate such arguments for Fermion systems [26] (see also [4]), but it is not yet clear how to use this extension in Y_2 theories. What we will find, quite remarkably, is that vacuum overlap is a consequence solely of Osterwalder–Schrader positivity in the spatial direction.

For technical reasons, which we will discuss, we have been unable to extend our Y_2 proof of vacuum overlap to $(\phi^4)_3$. However, we will find an argument exploiting Nelson–Symanzik positivity that is clearly a relative of Perron–Frobenius arguments and that works to prove vacuum overlap in $(\phi^4)_3$, in the sense that the limit of the pressure as $t \to \infty$ is the negative of the relativistic energy per unit volume.

The content of this paper is as follows: In Section 2, we discuss our choice of energy counterterms which differ from the usual ones [5, 16] by finite constants and control the difference between the two families of counterterms. *Throughout this paper,* H_l and Z_l *refer to objects with our choice of energy counterterms and* $\tilde{H}_l$, $\tilde{Z}_l$ *refer to the conventional objects.* In Section 3, we discuss the Y_2 Hamiltonian working solely in a Matthews–Salam formalism and avoiding Osterwalder–

Schrader fields; and in Section 4, we construct the $(\phi^4)_3$ Hamiltonian. In Section 5, we discuss vacuum overlap. In Section 6, we discuss the $l \to \infty$ behavior of E_l; in Section 7 ϕ-bounds and in Sections 8 and 9, we discuss Fröhlich bounds.

We sent an announcement of our Y_2 results from Sections 5, 6, 7, and 8 to the Marseille conference. We learned that McBryan, using Osterwalder–Schrader positivity, announced Y_2 results for Sections 5, 6, and 7 (his argument for Section 5 is identical to ours in the basic principle used).

McBryan's results appear in [39]. By a very different method, Fröhlich [12] has proven ϕ-bounds in ϕ_3^4. We learned of Fröhlich's results verbally before we began our work.

2. Matching Euclidean and Hamiltonian Counterterms

As explained in the introduction, for the conventional objects $\tilde{Z}_{l,t} \neq \langle \Omega_0, \exp(-t\tilde{H}_l)\, \Omega_0 \rangle$. Our goal in this section is to make a further finite renormalization so that $Z_{l,t} = \langle \Omega_0, \exp(-tH_l)\, \Omega_0 \rangle$ and to discuss the situation for $(\phi^4)_3$.

We will call a sequence of ultraviolet cutoffs *acceptable* if: The free boson (and fermion propagators are finite and obey uniform bounds of the form:

$$| G_\kappa(x-y)| \leqslant C_R \exp(-\alpha \,| x - y |); \qquad | x - y | \geqslant R$$

for any $R > 0$ and constants α, C_R *independent* of κ. For example, the cutoffs obtained by replacing $(p^2 + m^2)^{-1}$ by $(p^2 + m^2)^{-1} \exp(-p^2/\kappa^2)$ or $(p^2 + m^2)^{-1} \cdot \exp(-p_1^2/\kappa^2)$ (in two dimensions) are acceptable. Convolution with an x-space C_0^∞ function is acceptable. More critically, there are acceptable cutoffs that, with conventional renormalization, lead to conventional partition functions and Hamiltonians. For example, in $(\phi^4)_3$, the cutoffs of Feldman–Osterwalder [7] are acceptable.

Theorem 2.1. *Let $\widetilde{\delta P_2}(l, t; \kappa)$ denote the conventional second-order Euclidean counterterm in Y_2 (i.e., $\int d\mu_0 \operatorname{Tr}(K_{l,t;\kappa})$ in Matthews–Salam formalism* [45]*). Then, for any acceptable cutoffs, there is a divergent constant c_κ, so that for all $l, t > 0$, $\widetilde{\delta P_2}(l, t; \kappa) - c_\kappa lt$ has a finite limit $\Delta_2(l, t)$ independent of cutoff so that*

$$| \Delta_2(l, t) - a(l + t) - b | \leqslant C(le^{-\alpha t} + te^{-\alpha l}) \tag{4}$$

for suitable constants a, b, C, and $\alpha > 0$; and all $l, t \geqslant 1$.

Remarks. 1. The form of the estimate (4) is suggested to us by results of Lenard and Newman [35] who prove similar results for the full $P(\phi)_2 - \ln Z_{l,t}$ at small coupling.

2. Our method of finding the volume expansion of δP_2 is borrowed from [28].

Proof. For each κ, $\widetilde{\delta P}_2(l, t; \kappa)$ has the form $\int F_\kappa(x-y) f(x) f(y)\,dx\,dy$, where f is the characteristic function of $(-l/2, l/2) \times (-t/2, t/2)$. Now

$$\int F_\kappa(x-y) f(x) f(y)\,dx\,dy = \int F_\kappa(x)\,g(x)\,dx,$$

where

$$g(x) = \int f(y) f(y+x)\,dy.$$

Thus:

$$\delta P_2(l, t; \kappa) = lt \int_{|x| \leqslant l/2;\,|s| \leqslant t/2} dx\,ds\,F_\kappa(x, s)(1 - (2\,|x|/l)(1 - (2\,|s|/t)).$$

Let $c_\kappa = \int F_\kappa(x, s)\,dx\,ds$; $a_\kappa = -2 \int |x|\,F_\kappa(x, s)\,dx\,ds$; $b_\kappa = 4 \int |x|\,|s|\,F_\kappa(x, s)\,dx\,ds$. Then $c_\kappa \to \infty$ as $\kappa \to \infty$, but a and b have finite limits since $F(x) \sim |x|^{-2} \ln |x|$ at zero. Moreover:

$$\widetilde{\delta P}_2(l, t; \kappa) - c_\kappa lt - a_\kappa(l + t) - b_\kappa$$
$$= -lt \int_{|x| \geqslant l/2 \text{ or } |s| \geqslant t/2} dx\,ds\,F_\kappa(x, s)(1 - (2\,|x|/l))(1 - (2\,|s|/t)).$$

As a result, the limit $\Delta_2(l, t)$ exists and

$$\Delta_2(l; t) = a(l + t) + b - lt \int_{|x| \geqslant l/2 \text{ or } |s| \geqslant l/2} dx\,ds$$
$$\times F(x, s)(1 - (2\,|x|/l))(1 - (2\,|y|/t)).$$

Since $|F(x, s)| \leqslant C_1 \exp(-\beta(|x| + |s|))$ for $|x| \geqslant 1$, one easily finds that for $l, t \geqslant 1$, $|\Delta_2(l, t) - a(l + t) - b| \leqslant C_2 lt(\exp(-l\beta) + \exp(-t\beta))$ so that (4) holds. ∎

We thus define $Z_{l,t}$, the renormalized partition function by

$$Z_{l,t} = \tilde{Z}_{l,t} \exp(+\Delta_2(l, t)),$$

or equivalently, by replacing $\widetilde{\delta P}_2$ in the usual definition by $c_\kappa lt$. We define H_l by subtracting $-c_\kappa l$ in place of $\delta E_2(l)$ in the usual renormalization. We have:

COROLLARY 2.2. $Z_{l,t} = \langle \Omega_0, \exp(-tH_l)\,\Omega_0 \rangle$ *and in particular, Nelson symmetry holds*:

$$\langle \Omega_0, \exp(-tH_l)\,\Omega_0 \rangle = \langle \Omega_0, \exp(-lH_t)\,\Omega_0 \rangle.$$

COROLLARY 2.3. *The infima of the spectra of H_l and $\tilde{H}_l$ obey:*

$$| E_l - \tilde{E}_l - a | \leqslant C \exp(-\alpha l).$$

In particular, $\lim_{l\to\infty} (-E_l/l)$ *exists if and only if* $\lim_{l\to\infty} (-\tilde{E}_l/l)$ *exists and they are equal.*

Proof. E_l and $\tilde{E}_l$ only differ by the difference of the renormalization constants, so

$$E_l - \tilde{E}_l = \lim_{t\to\infty} (1/t)\, \Delta_2(l, t) = a + O(\exp(-\alpha l)). \quad \blacksquare$$

Remark. Since only an overall constant is involved, $H_l - E_l = \tilde{H}_l - \tilde{E}_l$ so that ϕ bounds and Schwinger functions are unaffected.

Next we turn to $(\phi^4)_3$. First, we consider the second-order linear energy divergence:

THEOREM 2.4. *Let* $\widetilde{\delta P}_2(l_1, l_2, t; \kappa)$ *denote the conventional second-order Euclidean counterterm in* ϕ_3^4 *(i.e.,* $\frac{1}{2}\int d\mu_0(\int_{|x|\leqslant l_1/2; |y|\leqslant l_2/2; |s|\leqslant t/2} dx\, dy\, ds :\phi_\kappa^4(x,y,s):)^2$*) Then for any acceptable cutoffs, there are divergent constants* c_κ *and* d_κ *so that*

$$\widetilde{\delta P}_2(l_1, l_2, t; \kappa) - c_\kappa l_1 l_2 t - d_\kappa(l_1 l_2 + l_1 t + l_2 t)$$

has a finite limit $\Delta_2(l_1, l_2, t)$ *independent of cutoff so that*

$$\begin{aligned} &| \Delta_2(l_1, l_2, t) - a(l_1 + l_2 + t) - b | \\ &\quad \leqslant C(l_1 l_2 \exp(-\alpha t) + l_1 t \exp(-\alpha l_2) + l_2 t \exp(-\alpha l_1) \end{aligned} \tag{5}$$

for suitable constants a, b, C, and $\alpha > 0$, *and all* $l_1, l_2, t \geqslant 1$.

Proof. As in the case of Y_2,

$$\begin{aligned} \widetilde{\delta P}_2({}_1, l_2, t; \kappa) = l_1 l_2 t \int_{|x|\leqslant l_1/2; |y|\leqslant l_2/2; |s|\leqslant t/2} & dx\, dy\, ds\, F_\kappa(x, y, s) \\ & \times (1 - (2\,|x|/l_1))(1 - (2\,|y|/l_2))(1 - (2\,|s|/t)). \end{aligned}$$

The limiting function $F(x)$ has an $|x|^{-4}$ singularity so that both

$$c_\kappa = \int F_\kappa(x, y, s)\, dx\, dy\, ds,$$

and

$$d_\kappa = -2 \int |x|\, F_\kappa(x, y, s)\, dx\, dy\, ds$$

diverge as $\kappa \to \infty$. But

$$a = 4 \int |x| \, |y| \, F(x, y, s) \, dx \, dy \, ds,$$

and

$$b = -8 \int |x| \, |y| \, |s| \, F(x, y, s) \, dx \, dy \, ds,$$

are finite and so is

$$\Delta_2(l_1, l_2, t) = a(l_1 + l_2 + t) + b - l_1 l_2 t \int_D dx \, dy \, ds \, F(x, y, s)$$
$$\times (1 - (2 |x| / l_1))(1 - (2 |y| / l_2))(1 - (2 |s| / t)),$$

where D is the complement of $(-l_1/2, l_1/2) \times (-l_2/2, l_2/2) \times (-t/2, t/2)$. The estimate (5) follows as in the case of Y_2. ▮

THEOREM 2.5. *Let $\delta P_3(l_1, l_2, t; \kappa)$ denote the conventional third-order Euclidean counterterm in $(\phi^4)_3$ (i.e., $\frac{1}{6} \int d\mu_0 (\int_{|x| \leqslant l_1/2; |y| \leqslant l_2/2; |s| \leqslant t/2} dx \, dy \, ds \, {:}\phi_\kappa^4(x, y, s){:})^3$). Then for any acceptable cutoff, there is a divergent constant e_κ so that*

$$\delta P_3(l_1, l_2, t; \kappa) - e_\kappa l_1 l_2 t$$

has a finite limit $\Delta_3(l_1, l_2, t)$ independent of cutoff so that

$$| \Delta_3(l_1 l_2 t) - p(l_1 l_2 + l_1 t + l_2 t) - q(l_1 + l_2 + t) - r |$$
$$\leqslant D(l_1 l_2 \exp(-\gamma t) + l_1 t \exp(-\gamma l_2) + l_2 t \exp(-\gamma l_1))$$

for suitable constants p, q, r, D, and $\gamma > 0$ and all $l_1, l_2, t \geqslant 1$.

Proof. Letting a, b, c, be three vectors, we see that

$$\delta P_3(l_1, l_2, t; \kappa) = \tfrac{1}{6} \int G_\kappa^2(a - b) \, G_\kappa^2(b - c) \, G_\kappa^2(c - a) \, f(a) \, f(b) \, f(c) \, d^3a \, d^3b \, d^3c$$
$$= l_1 l_2 t \int H_\kappa(a, b) \, f_{l_1 l_2 t}(a, b) \, d^3a \, d^3b,$$

where

$$f_{l_1 l_2 t}(a, b) = (l_1 l_2 t)^{-1} \int f(c) \, f(a + c) \, f(a + b + c) \, dc,$$

with f the characteristic function of the $l_1 \times l_2 \times t$ "cube." Now, we divide the a, b parameter space into $6^3 = 216$ regions depending on whether $0 < a_1 < a_1 + b_1$, etc. In each region $f_{l_1 l_2 t}(a, b)$ has a simple form: For example, if $0 < a_1 < a_1 + b_1$, $0 < a_2 < a_2 + b_2$, $0 < a_3 < a_3 + b_3$, then

$$f_{l_1 l_2 t}(a, b) = \left(1 - \frac{2(a_1 + b_1)}{l_1}\right)\left(1 - \frac{2(a_2 + b_2)}{l_2}\right)\left(1 - \frac{2(a_3 + b_3)}{t}\right) \chi$$

with χ the characteristic function of the six conditions $0 < a_i < a_1 + b_i$ and $0 < a_i + b_i < l_i/2$ $(i = 1, 2)$, $0 < a_3 + b_3 < t/2$. Each of the 6^3 terms can be treated by the method used for Y_2, namely, by removing the latter three conditions and proving the remainder is exponentially bounded. The key fact is that the integral $\int H_\kappa(a, b)\, da\, db$ is divergent but only logarithmically so that $\int | a_i |\, H_\kappa(a, b)\, da\, db < \infty$. ∎

We now define the renormalized partition function by:

$$Z_{l_1, l_2, t} = \tilde{Z}_{l_1, l_2, t} \exp(+\Delta_2(l_1, l_2, t) + \Delta_3(l_1, l_2, t)),$$

or equivalently by the formal expression:

$$Z_{l_1, l_2, t} = \left[\int \exp\left(-\int :\phi^4: + \delta m^2 :\phi^2:\right) d\mu_0\right] \times [\exp[-(c_\kappa + e_\kappa)(l_1 l_2 t) - d_\kappa(l_1 l_2 + l_1 t + l_2 t)]].$$

We defer the definition of the Hamiltonian until Section 4.

Remarks. 1. It is clear that the methods we use above allow one to make finite *any* second-order vacuum diagram in a theory with nonvanishing bare mass by subtracting a function of the form $a_\kappa l_1 l_2 \cdots l_{n-1} t + b_\kappa(l_1 l_2 \cdots l_{n-1} + \cdots) + \cdots + q_\kappa(l_1 + \cdots + t) + r_\kappa$. Higher-order diagrams are also controlled by our methods so long as they have the following property: no subdiagram obtained by taking all lines connecting a strict subset of the vertices is divergent. If the latter property fails, the theory will also have counterterms in some m-point function that might cancel the divergence that occurs by applying our method to just the vacuum diagram.

2. While we have continually linked Z to $\tilde{Z}$, some of the previous theory is a little simpler dealing directly with Z. For example, in our proof [47] of the linear lower bounds in Y_2, we had to add a few extra remarks because the second-order energy counterterm for a union of volumes differed from that for the volume. This is not needed for Z.

3. We have done all our computations in x-space, which seems to lead to the most detailed bounds. However, some information can be obtained by looking at the p-space diagrams. For example, a second-order vacuum diagram with general cutoff g has the form $\int | \hat{g}(p)|^2\, Q_\kappa(p)\, dp$, where $Q_\kappa(p)$ is the value of the mass diagram obtained by putting momentum p in at one vertex and $-p$ at the

other. The fact that Y_2 is made finite by an infinite counterterm linear in the volume is a consequence of the fact that $Q_\kappa(p)$ is made finite by a single subtraction at $p = 0$ and the resulting renormalized Q behaves at ∞ only as $(\ln p)^2$. That $(\phi^4)_3$ also has a divergent surface term results from the fact that after renormalization Q behaves at ∞ as $|p|$ $(\ln p)$ [14]. This has a vague connection with Stückelberg [55] divergences.

4. We note for later purposes that c_κ, $e_\kappa = +\infty$, $d_\kappa = -\infty$ for $\kappa \to \infty$.

As a final aspect of our study of Δ_2 and Δ_3, we want to make a few remarks about their behavior as l and/or t go to zero. At this point they may diverge since integrals over the complement of $(-l/2, l/2) \times (-t/2, t/2)$ occur and as l or t go to zero, the singularity at $(x, s) = (0, 0)$ is felt. In fact:

THEOREM 2.6. (a) *In Y_2, as $l \to 0$, $l^{-2}\Delta_2(l, l) \to -\infty$.*

(b) *In $(\phi^4)_3$, as $t \to 0$, for any fixed l, $\Delta_2(l, t) \to +\infty$, and $\Delta_3(l, t) \to 0$.*

Proof. (a) We can write

$$l^{-2}\Delta_2(l, l) = -\int_{|x|\geqslant l/2 \text{ or } |s|\geqslant l/2} F(x, s)\, dx\, ds + \int_{|x|\leqslant l/2 \text{ and } |s|\leqslant l/2} \times F(x, s)[-(2\,|x|/l) - (2\,|s|/l) + (|x|\,|s|/l^2)] \qquad (6)$$

For an explicit constant $C > 0$:

$$|F(x, s) + C(x^2 + s^2)^{-1} \ln(x^2 + s^2)| \leqslant D(x^2 + s^2)^{-1},$$

for all $|x| \leqslant \frac{1}{2}$, $|s| \leqslant \frac{1}{2}$. Now:

$$\begin{aligned} &l^{-1} \int_0^{l/2} ds \int_0^{l/2} x \ln(x^2 + s^2)(x^2 + s^2)^{-1}\, dx \\ &= 1/2l^{-1} \int_0^{l/2} ds \int_{s^2}^{(l/2)^2+s^2} \ln(y)\, y^{-1}\, dy \\ &= l^{-1}/4 \int_0^{l/2} ds(\ln[1 + (l/2s)^2])[\ln((l/2)^2 + s^2) + \ln(s^2/2)]\, ds \\ &= 1/8 \int_0^1 dy\, [\ln(1 + y^2)][\ln(l/2) + \ln(1 + y^{-2}) + \ln(y^{-2})] \end{aligned}$$

diverges only as $\ln(l)$. Similarly $l^{-2} \int |x|\,|s| \ln(x^2 + s^2)(x^2 + s^2)^{-1}\, dx\, ds$ only diverges as $\ln(l)$. In these terms, the error from $D(x^2 + s^2)^{-1}$ is bounded. However,

$$\int_{|x|\geqslant l/2 \text{ or } |s|\geqslant l/2} (x^2 + s^2)^{-1} \ln(x^2 + s^2)\, dx\, ds$$

diverges as $(\ln l)^2$ and the error correction as $\ln l$. Thus, the leading divergence in (6) is $-(\ln l)^2$.

(b) The only terms in $\Delta_2(l, t)$ for $(\phi^4)_3$ that are nonzero as $t \to 0$ are

$$+l_1 l_2 \int_{|s| \geqslant t/2} |s| F(x, y, s)\, dx\, dy\, ds \equiv A_1(t),$$

and

$$-l_1 l_2 t \int_{|s| \geqslant t/2} F(x, y, s)\, dx\, dy\, ds \equiv A_2(t).$$

Since $F(x, y, s) \sim (x^2 + y^2 + s^2)^{-2}$ near $(x, y, s) = 0$, $A_1(t)$ diverges as $\int_{t/2}^1 s s^{-2}\, ds = O(\ln t^{-1})$ while $A_2(t) \sim t \int_{t/2}^1 s^{-2}\, ds = O(1)$ is convergent. It is easy to see that $\Delta_3(l, t)$ goes to zero. ∎

Remarks 1. We will see the significance of (b) in Section 4 below.

2. The significance of (a) is the following. We *expect* that in Y_2 as $l \to 0$, $l^{-2} \ln \tilde{Z}_{l,l} \to 0$ since this is true order by order in perturbation theory. (In fact, the nth order term goes to zero with a power going to ∞ as $n \to \infty$.) Thus, we *conjecture* that

$$\alpha_{l,l} \equiv l^{-2} \ln Z_{l,l} \to -\infty$$

as $l \to 0$. Since $\alpha_{l,t}$ is monotone increasing in l and t (see (6)) this is certainly allowed. It is rather different behavior from $P(\phi_2)$ where $\alpha_{l,l} \to 0$ as $l \to 0$ if P is normalized [28].

3. Construction of the Hamiltonian: Yukawa$_2$

Our goals in this section are twofold. First, we give purely Euclidean proofs of the convergence theorem for the Y_2 renormalized Hamiltonian (a theorem originally proven by Glimm and Jaffe [21]) and of the bounds of Glimm [15] and Schrader [44]. (We have already sketched the latter in [47] but there is a further simplification due to the vacuum overlap results of Section 5.) Second, we wish to give a "direct" proof of the Matthews–Salam formulas [36]. Previous proofs [45] have used the procedure of going through Osterwalder–Schrader fields [40], which are then integrated out. It seems to us that it is slightly unnatural to have to introduce auxiliary objects that are then eliminated as quickly as possible. Our proof will involve perturbation expansions of semigroups (Phillips ≡ iterated DuHamel expansions) and of determinants (Fredholm expansion). Boson expectations will be replaced by Euclidean fields, but fermion expectations will be kept essentially in that form. In a sense, our development is thus semi-Euclidean [1]. We remark that Osterwalder–Schrader [40] also make a perturbation expansion in their proof of their Feynman–Kac formula. Their expansion is made after using the Trotter product formula. As the reader can check, the combinatorics are simpler with a Phillips expansion than with Trotter product formula. We also note that our method can be used to give a new proof of the Feynman–Kac formula in $P(\phi)_2$ (one uses our method to prove the formula with a potential that is a bounded function of the time-zero fields and then, successively removes cutoffs on both sides).

To deal with the Phillips expansions it will be useful to use what Davies [2] calls Phillips perturbations and to very briefly review their properties.

DEFINITION. Let H_0 be a self-adjoint operator that is bounded from below. A self-adjoint operator V is called a *Phillips perturbation* of H_0 if and only if $\operatorname{Ran}(\exp(-tH_0)) \subset D(V)$ for all $t > 0$ and

$$\int_0^1 N(t)\,dt < \infty, \qquad N(t) \equiv \| V \exp(-tH_0)\|. \tag{7}$$

Remarks. 1. It is more natural if H_0 is only assumed to be the generator of an exponentially bounded strongly continuous semigroup and V a closed operator. In that case, all the results below go through with minor changes.

2. It is possible to develop a theory of quadratic forms along these lines. Condition (7) is replaced by:

$$\int_0^1 ds \int_0^1 dt\; t^{-1/2}s^{-1/2}N(t, s) < \infty; \qquad N(t, s) = \| \exp(-tH_0)\, V \exp(-sH_0)\|.$$

THEOREM 3.1. (a) *A sufficient condition for V to be a Phillips perturbation of H_0 is that $V(|H_0| + 1)^{-\alpha}$ be bounded for some $\alpha < 1$.*

(b) *A necessary condition for V to be a Phillips perturbation of H_0 is that $D(H_0) \subset D(V)$ and $V(H_0 - E)^{-1}$ is bounded for all $E \notin \sigma(H_0)$ with $\lim_{E\to-\infty} \| V(H_0 - E)^{-1}\| = 0$, i.e., that V be H_0-bounded with relative bound zero. In particular, if V is a Phillips perturbation of H_0, $H_0 + V$ is self-adjoint and semibounded on $D(H_0)$.*

(c) *If V is a Phillips perturbation of H_0, then one has the DuHamel formula:*

$$\exp(-tH) = \exp(-tH_0) - \int_0^t \exp(-(t - s)\, H)\, V \exp(-sH_0)\, ds \tag{8}$$

and the norm convergent expansion:

$$\exp(-tH) = \sum_{n=0}^{\infty} (-1)^n \int_0^t ds_1 \int_0^{t-s_1} ds_2 \cdots \int_0^{t-s_1\cdots-s_{n-1}} ds_n \\ \times \exp(-(t - s_1 - \cdots - s_n)\, H_0)\, V \exp(-s_1H_0) \cdots V \exp(-s_nH_0). \tag{9}$$

Proof. Without loss, we can suppose H_0 is positive, which we do throughout the proof.

(a) Under the hypothesis

$$N(t) \leqslant \| V(H_0 + 1)^{-\alpha} \| \|(H_0 + 1)^{\alpha} \exp(-tH_0)\| \leqslant \| V(H_0 + 1)^{-\alpha} \| \exp(t)\, t^{-\alpha}$$

so the integral in question converges.

(b) Since H_0 is positive, $N(t)$ is monotone decreasing. Thus, $\int_0^\infty \exp(tE)\, N(t)\, dt$ converges for any $E < 0$ and goes to zero as $E \to -\infty$. This essentially completes the proof.

(c) By the hypothesis and (b) the integral on the right side of (8) converges. Moreover, given $v, u \in D(H_0)$ it is not hard to see that if the right side of (8) applied to u is called $u(t)$, then

$$(d/dt)(v, u(t)) = -(Hv, u(t)).$$

This proves (8).

To prove (9), we note that after iterating the Duhamel expansion, we obtain the first n terms of the series on the right of (9) and an error that looks like the $(n+1)$ *st* term, but with $\exp(-(t - s_1 - \cdots - s_n)H)$ replacing $\exp(-(t - s_1 - \cdots - s_n)H_0)$. Let E_{n+1} denote this error and T_n denote the nth term in the series. Then

$$\| T_n \| \leqslant \int_0^t ds_1 \int_0^{t-s_1} ds_2 \cdots \int_0^{t-s_1-\cdots-s_{n-1}} ds_n\, N(s_1) \cdots N(s_n). \tag{10}$$

Clearly, we can decompose $N(s) = f(s) + g(s)$ where $\int_0^\infty |f(s)|\, ds < \frac{1}{2}$ and $0 \leqslant g(s) < C$. Thus

$$\begin{aligned}
\text{RHS of (10)} &\leqslant \sum_{m=0}^{n} \binom{n}{m} \int_{\substack{s_1+\cdots+s_n\leqslant t\\ s_i\geqslant 0}} d^n s\, g(s_1) \cdots g(s_m)\, |f(s_{m+1})| \cdots |f(s_n)| \\
&\leqslant \sum_{m=0}^{n} \binom{n}{m} \int_{\substack{s_1+\cdots+s_m\leqslant t\\ 0\leqslant s_i}} d^n s\, g(s_1) \cdots g(s_m)\, |f(s_{m+1})| \cdots |f(s_m)| \\
&\leqslant \sum_{m=0}^{n} \binom{n}{m} (1/2)^{n-m}\, C^m/m!\, t^m \\
&\leqslant D(3/4)^n,
\end{aligned}$$

where D is chosen so that $(Ct)^m/n! \leqslant D(1/4)^m$ for all m. This proves the convergence of the series in (9). Since $\| E_n \|$ is also bounded by the right side of (10), we are finished. ∎

The simplest Matthews–Salam formula is the following.

THEOREM 3.2. *Let* $\psi_\sigma(x)$, $\bar{\psi}_\sigma(x)$ *be the free time-zero relativistic Fermi field with an ultraviolet cutoff, let* $\phi_\kappa(x)$ *be the free Bose field at time zero, and* $\phi_\kappa(x, s)$ *the corresponding Euclidean field, let* $S_{F,\sigma}$ *be the cutoff* Euclidean *Fermi propagator and* $H_{0,B}$ *(resp.* $H_{0,F}$*) the free Boson (resp. Fermion) Hamiltonian of mass* m_0 *(resp.* M_0*). Define*

$$H_{\kappa,\sigma,l}(\lambda) = H_{0,B} + H_{0F} + \lambda \int_{|x|\leqslant 1/2l} dx\, \phi_\kappa(x)\, \bar{\psi}_\sigma(x)\, \Gamma\psi_\sigma(x),$$

and the integral operator:

$$K_{\kappa,\sigma,l,t} = S_{F,\sigma}(x-y)\, \Gamma\phi_\kappa(y)\, \chi_{l,t}(y).$$

Then:

$$(\Omega_0\,, \exp(-tH_{\kappa,\sigma,l}(\lambda))\, \Omega_0) = \int \det(1 - \lambda K_{\kappa,\sigma,l,t})\, d\mu_0\,. \tag{11}$$

Remarks. 1. Thus, e.g., $S_{F,\sigma}((x, s) - (y, t)) \equiv (\Omega_0\,, \bar{\psi}_\sigma(x) \exp(-|t-s|\, H_0)\, \psi_\sigma(y)\, \Omega_0)$. As in [45], Γ stands for either 1 or $i\gamma_5$.

2. On the Hilbert space $\mathscr{H}_{+1/2} \oplus \mathscr{H}_{+1/2}$ ($\mathscr{H}_{+1/2} \equiv$ Sobolev space of order 1/2), K_κ is easily seen to be trace class (see, e.g., [46]) for a.e. ϕ so that $\det(1 - \lambda K)$ is defined by the usual theory of trace class determinants [3, 25, 51].

3. Note the minus sign in $\det(1 - \lambda K)$ in (11). In our previous work on the subject [45–47] and that of McBryan [37, 38], $\det(1 + \lambda K)$ was discussed but the connection with the coupling constant in the Hamiltonian was not explicitly made. All estimates hold for *all* $\lambda \in \mathbb{R}$ (and when $\Gamma = i\gamma_5$, $\det(1 + \lambda K) = \det(1 - \lambda K)$, see [45]) so the change of sign affects no estimates.

Proof. Since $\bar{\psi}_0$ and ψ_σ are bounded operators commuting with $H_{0,B}$ we have

$$V^2 \leqslant \text{const}(H_{0,B} + 1) \leqslant \text{const}(H_{0,B} + H_{0,F} + 1),$$

where $V = \int_{|x|<1/2l} dx\, \phi_\kappa(x)\, \bar{\psi}_\sigma(x)\, \Gamma\psi_\sigma(x)$, on account of the estimate $\phi_\kappa(x)^2 \leqslant \text{const}(H_{0,l} + 1)$. Thus, by Theorem 3.1(a), V is a Phillips perturbation of H_0. Thus, by Theorem 3.1(c), the left side of (11) has a convergent power series in λ.

Similarly, the right side of (11) has a convergent power series in λ. For, it is easy to see [46] that $\int \exp(A\, \|K(\phi)\|_1)\, d\mu_0 < \infty$ where $\|\cdot\|_1$ is the $\mathscr{H}_{1/2} \oplus \mathscr{H}_{1/2}$ trace class norm (henceforth, $\|K\|_p$ will denote a $\mathscr{C}_p$ norm on this Hilbert space). Moreover, $\det(1 - \lambda K)$ is entire in λ with $|\det(1 - \lambda K)| \leqslant \exp(|\lambda|\, \|K\|_1)$, from which the analyticity in question follows.

It thus suffices to prove the coefficients in the power series expansions agree. By (9), the nth order term of the left side is:

$$(-1)^n \int_0^t ds_1 \int_0^{t-s_1} ds_2 \cdots (\Omega_0, V \exp(-s_1 H_0)\, V \exp(-s_2 H_0)\, V \cdots \exp(-s_{n-1} H_0)\, V\Omega_0)$$

$$= (-1)^n \int_{\substack{|x_i| \leqslant l/2 \\ |s_i| \leqslant t/2 \\ s_1 < \cdots < s_n}} d^n x\, d^n s\, (\Omega_{0B}, \phi_\kappa(x_1) \exp(-(s_2 - s_1) H_{0B}) \cdots \Omega_{0B})$$

$$\times (\Omega_{0F}, \bar{\psi}_\sigma(x_1)\, \Gamma \psi_\sigma(x_1) \exp(-(s_2 - s_1) H_{0F}) \cdots \Omega_{0F}).$$

Now, write the Bose expectation as a Euclidean field integral and the Fermi expectation as an explicit free Fermion–Schwinger function, i.e., a determinant of propagators, and obtain:

$$(-1)^n/n! \int_{\substack{|x_i| \leqslant l/2 \\ |s_i| \leqslant t/2}} d^n x\, d^n s \det(S_2(x_i - x_j, s_i - s_j)) \int d\mu_0 \prod_{i-1}^{n} \phi_\kappa(x_i, s_i)$$

$$= (-1)^n/n! \iint_{\substack{|x_i| \leqslant l/2 \\ |s_i| \leqslant t/2}} \det(K((x_i, s_i), (x_j, s_j)))\, d^n x\, d^n s\, d\mu_0,$$

which is the classical Fredholm expression [8] for the nth term in the expansion of $\det(1 - \lambda K)$. (The equality of this expression and the more general $\mathrm{Tr}(\Lambda^n(K))$ [51] is easy to establish given the fact that the kernel of K is continuous.) ∎

Without putting in detailed proofs, we wish to extend this theorem in three ways:

(1) We want to consider a more general Hamiltonian object:

$$(\Omega_0, \exp(-t_0 H)\, \psi^\#(x_1) \exp(-t_1 H) \cdots \psi^\#(x_n) \exp(-t_n H)\Omega_0), \tag{12}$$

where each $\psi^\#$ is a ψ_σ, $\bar{\psi}_\sigma$ or a ϕ_κ. By charge symmetry, this is zero, unless equal number of ψ_σ and $\bar{\psi}_\sigma$ appear. If equal numbers appear, say k, the perturbation expansion of the Hamiltonian object is still convergent and still given by an integral of a Bose field object times a determinant. This determinant is almost the nth term of the classical Fredholm expansion of the kth Fredholm minor $\Lambda^k(K/1 - \lambda K) \det(1 - \lambda K)$ except that for points corresponding to the Fermi fields in (12), S_F appears in place of K. Thus, one obtains formulas, of which a typical one is

$$(\Omega_0, \bar{\psi}(f) \exp(-tH)\, \psi(g)\, \Omega_0)$$

$$= \int d\mu_0 \det(1 - \lambda K)(f \otimes \delta_t, S_F(1 - \lambda K)^{-1} g \otimes \delta_0).$$

(2) We wish to allow the interaction V to include $P(\phi)_2$ terms as well as the basic Yukawa interaction. One needs to do this not only to recover Schrader's results [43] on $Y_2 + P(\phi)_2$ (this is a simple extension of our results for Y_2 and we say no more of it) but also to accomomdate the mass renormalization. The effect of adding such a term is just to add a factor $\exp(-\int :P(\phi(x, s)): dx\, ds)$ to the $d\mu_0$ integral. One way of seeing this is to include the $P(\phi)_2$ term in the H_0 factor when doing a Phillips expansion. The boson expectation is then written in terms of Euclidean fields using the $P(\phi)_2$ Feynman–Kac formula. Alternatively, one can rederive the $P(\phi)_2$ Feynman–Kac formula as in (3) below.

(3) We will not attempt a complete analysis of the generalized Y_2 theory [5] here but we note that if one does replace $\bar{\psi}_\sigma \psi_\sigma \phi_\kappa$ by $\bar{\psi}_\sigma \psi_\sigma :P(\phi_\kappa):$, then our proof breaks down since $:P(\phi_\kappa):$ will not be a Phillips perturbation if $\deg P \geqslant 2$. However, if one replaces $\bar{\psi}_\sigma \psi_\sigma :P(\phi_\kappa):$ by $\bar{\psi}_\sigma \psi_\sigma :P(\phi_\kappa): \exp(-\alpha \phi_\kappa^2)$ for any $\alpha > 0$, then we have a bounded interaction and we obtain a Matthews–Salam formula by our method. Controlling the $\alpha \to 0$ cutoff limit is a part of controlling the GY_2 theory.

We summarize remarks (1) and (2) above by:

THEOREM 3.3. *Let $H_{l,\sigma,\kappa;\mathrm{ren}}$ denote the Y_2 Hamiltonian with acceptable cutoffs and with the conventional mass renormalization, Wick ordering and the energy counterterm of Section 2. Then*:

$$(\Omega_0, \exp(-tH_{l,\sigma,\kappa;\mathrm{ren}})\, \Omega_0)$$
$$= \exp(-c_\kappa lt) \int d\mu_0 \exp\Big(-\delta m^2(\kappa) \int_{\substack{|x|\leqslant l/2 \\ 0<s<t}} :\phi^2(x, s): dx\, ds\Big) \det{}_2 (1 - K_{\sigma,\kappa,l,t}).$$

Moreover, the Schwinger functions (analytic continuation of Wightman functions to imaginary times) are given by Matthews–Salam formulas.

Remarks. 1. That Wick ordering corresponds to taking $\det_2$ can be seen either by explicitly looking at the subtraction [45] or by noticing that Wick ordering causes diagonal terms in the Fredholm expansion to be surpressed leading to Hilbert's original definition [33] of $\det_2$.

2. The above connection implies that the ultraviolet cutoff Schwinger functions obey Osterwalder–Schrader positivity in the time direction and lead to OS positivity in any direction in the limit where all cutoffs are removed.

We now want to describe how removing the Euclidean ultraviolet cutoffs allows us to remove the Hamiltonian ultraviolet cutoffs.

THEOREM 3.4. (a) *For all sufficiently large κ, σ and $l \geqslant 1$*

$$H_{l,\sigma,\kappa,\mathrm{ren}} \geqslant -c(l + 1).$$

(b) *As* $\sigma, \kappa \to \infty$, $H_{l,\sigma,\kappa;\mathrm{ren}}$ *converges in strong resolvent sense to an operator* H_l.

Remarks. 1. (a) combines results of Glimm [15] and Schrader [44]. (b) is a result of Glimm and Jaffe [21].

2. Henceforth, $E_l \equiv \inf \operatorname{spec}(H_l)$. We note that by (a):

$$-E_l \leqslant c(l+1), \qquad (l \geqslant 1). \tag{13}$$

In the proof of Theorem 3.4, we need the following:

LEMMA 3.5. *Let* $f_n \in L^p$ *for a probability measure space with* $\sup_n \|f_n\|_p < \infty$. *Suppose that* $f_n \to f$ *pointwise. Then* $f \in L^p$ *and* $f_n \to f$ *in any* L^q *norm with* $q < p$.

Proof. For any R, define $f^{(R)}$ and $f_n^{(R)}$ by

$$\begin{aligned} f_n^{(R)}(x) &= f_n(x), \qquad \text{if } |f_n(x)| < R \\ &= 0, \qquad \text{if } |f_n(x)| \geqslant R. \end{aligned}$$

Then $f_n^{(R)} \to f^{(R)}$ pointwise for any R with $\mu\{x \mid |f(x)| = R\} = 0$ and so for all but countably many R. Thus, $f_n^{(R)} \to f^{(R)}$ in any L^q; $q < \infty$ by the dominated convergence theorem. In particular,

$$\|f^{(R)}\|_p \leqslant \sup_n \|f_n^{(R)}\|_p \leqslant \sup_n \|f_n\|_p .$$

Since $|f^{(R)}| \nearrow |f|$, $f \in L^p$ by the monotone convergence theorem. The convergence in L^q follows from:

$$\|g - g^{(R)}\|_q^q \leqslant R^{(q-p)} \|g\|_p^p$$

if $q < p$ and the estimate $\|g - f\|_q \leqslant \|g^{(R)} - f^{(R)}\|_q + \|g - g^{(R)}\|_q + \|f - f^{(R)}\|_q$. ∎

Proof of Theorem 3.4. (a) will be proven in Section 5 (Theorem 5.4) using the Euclidean lower bound on vacuum amplitudes.

(b) Since the operators $\exp(-tH_{l,\sigma,\kappa;\mathrm{ren}})$ are uniformly bounded for each fixed t it suffices to prove convergence of a dense set of matrix elements and then appeal to general theory [42]. Consider matrix elements between Jost states, i.e., states of the form

$$\exp(-t_0H_0)\, \psi^{\#}(f_1) \exp(-t_1H_0)\, \psi^{\#}(f_2) \cdots \exp(-t_nH_0)\, \psi^{\#}(f_n)\, \Omega_0$$

(such states are dense; see Section 5).

Matrix elements for such states have a Matthews–Salam formula, so it suffices to prove $\Lambda^k(S_F(1-K)^{-1})\,\det_{\mathrm{ren}}(1-K)$ converges in some $L^p(Q, d\mu_0)$, $p > 1$. These converge in all $L^p, p > 1$: For these functions are bounded in all L^p ($p < \infty$) and thus, by Lemma 3.5, it is enough to prove pointwise convergence. Since K is a.e. in $\mathscr{C}_3$ (see [46]) and $\det_3(1-K)\,\Lambda^k(K(1-K)^{-1})$ is continuous (see [46, 51]), we obtain pointwise convergence by the explicit formula for $\det_{\mathrm{ren}}$. ∎

THEOREM 3.5. *In* Y_2: $(\Omega_0, \exp(-tH_l)\,\Omega_0) = (\Omega_0, \exp(-lH_t)\,\Omega_0)$.

Proof. Follows from the Euclidean invariance and the use of matched counterterms. ∎

4. CONSTRUCTION OF THE HAMILTONIAN: $\phi_3{}^4$

In this section, we will construct a Hilbert space and Hamiltonian for $\phi_3{}^4$ with a space cutoff by exploiting Feldman's result [6] on the existence of Schwinger functions. Our main idea is borrowed from the Osterwalder–Schrader reconstruction theorem. Throughout, we use the *formal symbol* $\exp(-U_{l_1 l_2 t})\,d\mu_0$ to denote the measure constructed by Feldman (who allows sharp cutoffs) with a sharp cutoff in $(-l_1/2, l_1/2) \times (-l_2/2, l_2/2) \times (-t/2, t/2)$ but renormalized only by the subtraction of our counterterm $\exp(-c_\kappa l_1 l_2 t - d_\kappa(l_1 t + l_2 t + l_1 l_2))$.

Let us begin by explaining why the matched Euclidean counterterms are so critical for this construction. One might try to construct the Hamiltonian by trying to define vectors $\exp(-tH_l)\,\Omega_0^{(l)}$ so that

$$(\exp(-tH_l)\,\Omega_0^{(l)}),\ \exp(-sH_l)\,\Omega_0^{(l)} = \int \exp(-U_{l_1 l_2, t+s})\,d\mu_0\,, \tag{14}$$

(where l will be shorthand for l_1, l_2). For the Schwarz inequality to hold, we need

$$\left(\int \exp(-U_{t+s})\,d\mu_0\right)^2 \leqslant \int \exp(-U_{2t})\,d\mu_0 \int \exp(-U_{2s})\,d\mu_0\,, \tag{15}$$

(15) holds with an ultraviolet cutoff in the spatial direction *before "energy" renormalization*. For energy renormalization not to destroy such an inequality, we need the energy counterterm to be of the form $\exp(-F(t))$ with

$$2F(t+s) \geqslant F(2t) + F(2s). \tag{16}$$

The "usual" energy counterterm violates (16) (the "transients" $\langle V\Omega_0, H_0^{-1}\exp(-tH_0)\,H_0^{-1}V\Omega_0\rangle$ obey the opposite inequality strictly by the Schwartz inequality and the nontransient terms lead to equality). But if

$$F(t) = at + b, \tag{17}$$

(16) is clearly obeyed. Notice that (17) plus symmetry forces an overall energy counterterm of the form:

$$F(l_1, l_2, t) = A(l_1 l_2 t) + B(l_1 l_2 + l_1 t + l_2 t) + C(l_1 + l_2 + t) + d.$$

We only take the first two terms because they are the only infinite terms.

We now proceed to construct the physical Hilbert space for space cutoff $l = (l_1, l_2)$ fixed. Consider formal objects:

$$\begin{aligned} &W(s_1, ..., s_{n+1}; f_1, ..., f_n) \\ &\quad = \exp(-s_1 H_l)\, \phi(f_1) \exp(-s_2 H_l)\, \phi(f_2) \cdots \exp(-s_{n+1} H_l)\, \Omega_0^{(l)}, \end{aligned} \tag{18}$$

with $s_1, ..., s_{n+1}$ *strictly* positive and $f_1, ..., f_n$, $C^\infty(\mathbb{R}^2)$ functions *with support in* $(-l_1/2, l_1/2) \times (-l_2/2, l_2/2)$. The right side of (18) is intended merely as a formal indication of what W will be. We do not claim to have objects H_l, $\Omega_0^{(l)}$ or $\phi(f_i)$! (in fact, as we will see, $\Omega_0^{(l)}$ does not exist: it will be a formal vector of infinite norm). Let $V_0(t_1, ..., t_{n+1}; f_1, ..., f_n)$ be defined formally as $W(s_0, ..., s_n; f_1, ..., f_n)$ with $t_1 = s_1$, $t_2 = s_1 + s_2, ..., t_{n+1} = s_1 + \cdots + s_{n+1}$. Finally, for C_0^∞ functions $g_1, ..., g_n$ with the property that $\sup g_i \subset [a_i, b_i]$; $0 < a_1 < b_1 < a_2 < b_2 < \cdots < b_n$ and $T = t_{n+1} > b_n$ consider the formal object:

$$\begin{aligned} &V_0(g_1, ..., g_n, T; f_1, ..., f_n) \\ &\quad = \int g_1(t_1) \cdots g_n(t_n)\, V_0(t_1, ..., t_n, T; f_1, ..., f_n)\, d^n t. \end{aligned} \tag{19}$$

Form a vector space of finite linear combinations of the formal objects $V_0(g_i, T; f_i)$. (These will be the basic objects; (18) and (19) are purely intended for heuristic purposes.) Define an inner product on this formal vector space by linearity and:

$$\begin{aligned} &(V_0(\bar{g}_n, ..., \bar{g}_1, T; \bar{f}_n, ..., \bar{f}_1), V_0(g_{n+1}, ..., g_{n+m}; S; f_{n+1}, ..., f_{n+m}) \\ &\quad = \int \prod_{i=1}^{n+m} \phi(f_i \times g_i') \exp(-U_{l_1 l_2, T+S})\, d\mu_0, \end{aligned} \tag{20}$$

where g_i' is defined by

$$\begin{aligned} g_i'(t) &= g_i(\tfrac{1}{2}(S - T) - t), \qquad i = 1, ..., n \\ &= g_i(\tfrac{1}{2}(S - T) + t), \qquad i = n + 1, ..., n + m. \end{aligned}$$

We now have the basic fact:

THEOREM 4.1. *The inner product defined by* (20) *is positive semidefinite.*

Proof. Choose acceptable cutoffs in the spatial direction. Define $H_{l,\kappa}$ as an operator on Fock space by $H_{l,\kappa} = H_0 + H_{I,\kappa}$ + full mass counterterm + $c_\kappa(l_1 l_2) + d_\kappa(l_1 + l_2)$. Define vectors in Fock space $W_\kappa(s_0, ..., s_n; f_i)$ by (18) with $\exp(-s_i H_l)$ replaced by $\exp(-s_i H_{l,\kappa})$ and $\Omega_0^{(l)}$ by $\exp(-\frac{1}{2} d_\kappa l_1 l_2)\, \Omega_0$ and $V_{0,\kappa}$ by change of variable and (19). The inner product of $V_{0,\kappa}$ is positive definite by the Fock space positive definiteness. But these inner products converge to the right side of (20) by the FKN formula and Feldman's convergence theorem. ▮

We temporily interrupt our construction of H_l to note a most important consequence of Theorem 4.1:

THEOREM 4.2. *$Z_{l,t} = \int \exp(-U_{l_1,l_2,t})\, d\mu_0$ is always nonzero and as $t \to 0$ with l fixed $Z_{l,t} \to \infty$.*

Proof. Consider first the function $\tilde{Z}_{l,t}$ defined with complete second order subtractions, defined for l fixed and $0 < t < \infty$. Extend to $t = 0$ by $\tilde{Z}_{l,0} = 1$. We claim that $\tilde{Z}$ is continuous including at $t = 0$. For $\tilde{Z}_{l,t,\kappa}$ is continuous including at $t = 0$. Moreover, we claim that $\tilde{Z}_{l,t,\kappa}$ converge uniformly for $0 \leqslant t \leqslant T$. This follows from noticing that all Feldman's estimates involve constants independent of g so long as $0 \leqslant g \leqslant 1$, $\operatorname{supp} g \subset [-l_1/2, l_1/2] \times [-l_2/2, l_2/2]) \times [0, T]$. Thus, $\tilde{Z}_{l,t}$ is nonzero for small t. Since $\Delta_2(l, t)$ is finite for all $t \neq 0$ and goes to $+\infty$ as $t \to 0$ (Theorem 2.4), and $\Delta_3(l, t)$ is finite for all t including the $t = 0$ limit, $Z_{l,t}$ is continuous for $t \in (0, \infty)$ and goes to $+\infty$ as $t \to 0$ since $Z_{l,t} = \tilde{Z}_{l,t} \exp(+\Delta_2(l, t) + \Delta_3(l, t))$.

Now by Theorem 4.1 (see the discussion of (14) and (15))

$$Z_{l,t+s}^2 \leqslant Z_{l,2t} Z_{l,2s}\,. \tag{21}$$

From this and the finiteness of Z for all t, it easily follows that Z cannot vanish for any t unless it vanishes for all t. ▮

Remark. Of course, it follows that $\tilde{Z} \neq 0$ for all l, t.

We now form a Hilbert space $\mathscr{H}_l$, by dividing out by the vectors of norm 0 and completing. $V(g_i, T, f_i)$ will denote the equivalence class of $V_0(g_i, T; f_i)$ in this space. Such vectors we will call *dressed Jost states*. By construction they are dense in $\mathscr{H}_l$.

We remark, that if one does this construction for $P(\phi)_2$, one recovers the entire Fock space, since the constructed Hilbert space will be a subset of Fock space containing Ω_0, and invariant under $\exp(-tH_l)$ and the algebra of fields $\{\phi(f) \mid \operatorname{supp} f \subset [-l/2, l/2]\}$. By standard arguments it follows that it is also invariant under $\exp(-tH_0)$ and so all of Fock space by the cyclicity of Ω_0 for $\{\phi(f), \pi(f) \mid \operatorname{supp} f \subset [-l/2, l/2]\}$. There is an open question for $(\phi^4)_3$ analogous to what we have just proven. Namely, one could define a possibly bigger Hilbert space that would include vectors with supp f_i not in $[-l/2, l/2]$ and would also include objects like $\exp(-sH_l)\exp(-tH_0)\,\phi(f)\,\Omega_0^{(l)}$. All our results carry through in this possibly bigger Hilbert space, essentially because the vacuum overlap theorem still holds (see the remarks in Section 5) but we prefer to work in the Hilbert space we have constructed. Clearly, the two spaces should be equal but we do not see a proof.

We also remark that on account of $Z_{l,t} \to \infty$ as $t \to 0$, $\lim_{t\to 0} \|\exp(-tH_l)\Omega_0^{(l)}\| = \infty$ so that there is no natural vector $\Omega_0^{(l)}$. Our Hilbert space can be interpreted in the spirit of Friedrichs [9] and Glimm [16]: $\exp(-\frac{1}{2}d_\kappa l_1 l_2)$ (remember that $d_\kappa \to -\infty$) plays the role of "wavefunction" renormalization and $\exp(-tH_l)$ of dressing transformations, $\Omega_0^{(l)}$ is similar to the formal vector used in studying Dirichlet BC in $P(\phi)_2$ [48].

Now we define $\exp(-tH_l)$ as operators on our Hilbert space in the spirit of Osterwalder–Schrader [41]:

THEOREM 4.3. *For each t, the map*

$$A_t V(g_i, T; f_i) = V(g_i^t, T + t; f_i), \qquad \textit{where} \qquad g_i^t(s) = g_i(s - t) \tag{22}$$

is well defined and extends to a bounded map on $\mathscr{H}_l$. *The family* A_t *is a one parameter, continuous, exponentially bounded semigroup, and is of the form* $\exp(-tH_l)$ *for some self-adjoint operator* H_l. H_l *and all its powers are essentially self-adjoint on the family of (linear combinations of) dressed Jost states.*

Proof. By repeated use of the positive definiteness of the inner product, one finds that

$$\| A_t V_0(g_i, T; f_i)\| \leqslant \| V_0(g_i, T, f_i)\|^{1-1/2^n} \, \| A_{2^n t} V_0(g_i T, f_i)\|^{1/2^n}. \tag{23}$$

By the exponential upper bound on (ZS) (see Feldman [6]):

$$\| A_{2^n t} V_0(g_i, T; f_i)\|^2 \leqslant \exp(c(2^n t + T))\, d, \tag{24}$$

where c is a constant independent of t, T, n, g_i and f_i and d only depends on g_i, f_i and T. Taking n to infinite in (23) and using (24) we obtain:

$$\| A_t V(g_i, T; f_i)\| \leqslant \exp(ct)\, \| V_0(g_i, T; f_i)\|.$$

Thus, A_t "lifts" to a bounded operator on $\mathscr{H}_l$ with $\| A_t \| \leqslant \exp(ct)$.

Continuity in t of $\langle V(g_i T; f_i), A_t V(h_i, T'; k_i)\rangle$ follows in the same way that continuity of $\tilde{Z}_{l,t}$ was proven. By construction, the (linear combinations of) dressed Jost states are a dense set invariant under $\exp(-tH_l)$. It follows from the spectral theorem that such a set is a core for any exponentially bounded function of H_l. ∎

In Section 7, we discuss the construction of a time zero field as a quadratic form and of $H_l + \phi(f \otimes \delta_0)$.

THEOREM 4.4 (Nelson's symmetry for $\phi_3{}^4$).

$$(\Omega_0^{(l)}, \exp(-tH_l)\, \Omega_0^{(l)}) = (\Omega_0^{(t)}, \exp(-lH_t)\, \Omega_0^{(t)}).$$

5. VACUUM OVERLAP

Let A be a semibounded self-adjoint operator on a Hilbert space $\mathscr{H}$. We say that $\phi \in A$ *overlaps the vacuum for* A if and only if (1) The support of the spectral measure for ϕ extends down to inf spec(A). (2) Moreover, if $\lambda =$ inf spec(A) is an eigenvalue of A, then $P_{\{\lambda\}}(A)\phi \neq 0$. Our goal is to prove that Ω_0 overlaps the vacuum for H_l in Y_2 and that $\exp(-tH_l)\, \Omega_0^{(l)}$ overlaps the vacuum for H_l in $(\phi^4)_3$. In Y_2, where it is known that H_l has an eigenvalue at the bottom of its spectrum [17], we can conclude that Ω_0 is not orthogonal to the corresponding eigenspace. *We emphasize that in* $(\phi^4)_3$, *it is not known that* H_l *has an eigenvalue at the bottom of its spectrum,* so our term "vacuum overlap" is somewhat artificial. We begin with a functional analytic result:

LEMMA 5.1. (a) *Suppose that* $\phi \in \mathscr{H}$ *is fixed and that there is a dense set* $D \subset \mathscr{H}$ *and* $p \geqslant 1$ *so that for any* $\psi \in D$, *there is a* α, $\eta \in \mathscr{H}$, *and* C_ψ

$$(\psi, \exp(-tA)\psi) \leqslant C_\psi(\phi, \exp(-t\alpha A)\phi)^{1/\alpha p}(\eta, \exp(-tA)\eta)^{1/q} \tag{25}$$

for all t, *where* $(1/p) + (1/q) = 1$. *Then* ϕ *couples to the vacuum for* A.

(b) *If* ϕ *couples to the vacuum for* A, *then*

$$\text{inf spec}(A) = -\lim_{t\to\infty} (1/t) \ln(\phi, \exp(-tA)\, \phi). \tag{26}$$

Proof. (26) follows from the more general result that

$$-\lim_{t\to\infty} t^{-1} \ln(\phi, \exp(-tA)\, \phi) = \text{inf supp(spectral measure for } \phi). \tag{27}$$

Given (25), we can replace A by $\hat{A} = A - \text{inf spec}(A)$ without changing the inequality. Since $\hat{A} \geqslant 0$, we then find:

$$(\psi, \exp(-t\hat{A})\psi) \leqslant C_\psi \| \eta \|^2 (\phi, \exp(-t\alpha\hat{A})\phi)^{1/\alpha p}. \tag{28}$$

Since D is dense, given ϵ we can find $\psi \in D$ so that $(\psi, \exp(-t\hat{A})\psi) \geqslant C \exp(-\epsilon t)$. Then by (28)

$$\lim_{t\to\infty} t^{-1} \ln(\phi, \exp(-t\hat{A})\, \phi) \geqslant -\epsilon p.$$

Since the limit must be negative and ϵ is arbitrary, we conclude that

$$\inf \operatorname{spec}(A) = -\lim_{t\to\infty} t^{-1} \ln(\phi, \exp(-tA)\,\phi),$$

whence, (27) yields the first half of vacuum overlap. If $\hat{A}$ has 0 as an eigenvalue, then we can find $\psi \in D$ so that $\lim_{t\to\infty}(\psi, \exp(-t\hat{A})\psi) \neq 0$, whence, by (28) $\lim_{t\to\infty}(\phi, \exp(-t\hat{A})\phi) \neq 0$ proving the second part of vacuum overlap. ∎

For our proof of vacuum overlap in Y_2, we require some elementary facts about Jost states including a Euclidean Reeh–Schlieder theorem. The vector valued distribution $\psi_1^{\#}(x_1) \cdots \psi_n^{\#}(x_n)\, \Omega_0$ for the free Relativistic Fermi–Boson field is the boundary value of a vector valued analytic function in the region $\operatorname{Im} z_1$, $\operatorname{Im}(z_2 - z_1),\ldots, \operatorname{Im}(z_n - z_{n-1}) \in V_+$, the forward light cone (see [34, 38]). By cyclicity of the vacuum, the set of linear combinations of *Jost states*, i.e., the range of those vectors is clearly dense. We call a Jost state Euclidean if and only if each z_i is Euclidean, i.e., $z_i = (y_i, it_i)$ with y_i, t_i real and moreover, the y_i's are non-coincident. We call a vector a *good Jost state* if it is an integral of Euclidean Jost states with a function in $C_0^\infty(R^{2n})$ supported in a region $\{t_1, t_2 - t_1,\ldots, t_n - t_{n-1}, x_{\pi(2)} - x_{\pi(1)},\ldots, x_{\pi(n)} - x_{\pi(n-1)}$ all positive$\}$ for some permutation Π on n letters. We say the state is supported in $(a, b) \times (c, d)$ if f is also supported in the region $a < x_i < b$, $c < t_i < d$.

LEMMA 5.2. (Euclidean Reeh–Schlieder theorem). *Fix a, b, c, d with $a < b$, $0 < c < d$. The linear combinations of good Jost states with support in $(a, b) \times (c, d)$ are dense in Fock space.*

Proof. Suppose η is orthogonal to all good Jost states with the support property. By taking the smearing functions to delta functions, η is orthogonal to all Euclidean Jost states with the support property. By analyticity, it is then orthogonal to all Jost states and so zero. ∎

Remark. This lemma is true in any Wightman theory.

THEOREM 5.3. *The Fock vacuum overlaps the vacuum for Y_2. In particular:*

$$-E_l = \lim_{t\to\infty} t^{-1} \ln Z_{l,t}$$

Proof. Let η be a good Jost state supported in $(-\frac{1}{2}l - 1, -\frac{1}{2}l) \times (0, 1)$. We will show that there is an $\tilde{\eta}$ so that:

$$\langle \eta, \exp(-tH_l)\eta\rangle \leqslant \langle \tilde{\eta}, \exp(-tH_l)\tilde{\eta}\rangle^{1/2} \langle \Omega_0, \exp(-tH_l)\Omega_0\rangle^{1/2}, \qquad (29)$$

so that the last two lemmas complete this proof. The proof of (29) is shown pictorial in Fig. 1. Since η is a good Jost state, there is an η' with

$$\langle \eta, \exp(-tH_l)\eta\rangle = \langle \eta', \exp(-lH_t)\,\Omega_0\rangle,$$

η' is supported in $[(-\tfrac{1}{2}t - 1, -\tfrac{1}{2}t) \cup (\tfrac{1}{2}t, \tfrac{1}{2}t + 1)] \times (0, 1)$. By the Schwarz inequality

$$\begin{aligned}\langle \eta', \exp(-lH_t)\,\Omega_0\rangle &\leqslant \langle \Omega_0, \exp(-lH_t)\,\Omega_0\rangle^{1/2}\langle \eta', \exp(-lH_t)\,\eta'\rangle \\ &= \langle \Omega_0, \exp(-tH_l)\,\Omega_0\rangle^{1/2}\langle \tilde{\eta}, \exp(-tH_l)\tilde{\eta}\rangle^{1/2}. \quad \blacksquare\end{aligned}$$

Notice that the above proof essentially uses OS positivity in the space direction. In Fig. 1, the empty box stands for the interaction, the shaded box for Jost states and the whole box for $\log\langle \psi, \exp(-tH_l)\psi\rangle$.

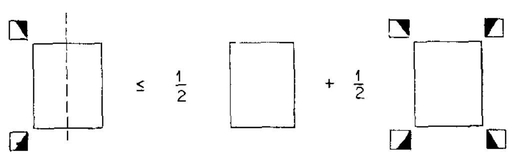

FIGURE 1.

Remark. One consequence of this theorem is that the vacuum for H_l has a charge zero component. A priori, this is far from clear.

THEOREM 5.4. *Let κ, σ be ultraviolet cutoffs obtained by convoluting with functions with support in $(-\tfrac{1}{2}, \tfrac{1}{2})$. Let $\tilde{H}_{l,\sigma,\kappa;\mathrm{ren}}$ be the Hamiltonian with cutoffs κ, σ and interaction turned on in $(-l-1, -1) \cup (0, l)$. Then for a dense set of states and all t:*

$$(\eta, \exp(-tH_{l,\sigma,\kappa;\mathrm{ren}})\eta) \leqslant C_\eta(\Omega_0, \exp(-t\tilde{H}_{l,\sigma,\kappa;\mathrm{ren}})\,\Omega_0)^{1/2}.$$

In particular, the linear lower bound for $(\Omega_0, \exp(-t\tilde{H})\,\Omega_0)$ [38, 47] *implies a uniform lower bound on $H_{l,\sigma,\kappa,\mathrm{ren}}$ for l fixed as $\sigma, \kappa \to \infty$.*

Proof. We take good Jost states, η, supported in $(-\tfrac{1}{2}l - \tfrac{3}{2}, -\tfrac{1}{2}l - \tfrac{1}{2}) \times (0, t)$. Since the interaction is a function of the fields in $(-(l/2) - (1/2), (l/2) + (1/2))$, we can use OS positivity about the line $x = -(l/2) - (1/2)$. The proof is shown pictorially in Fig. 2.

Remarks. 1. To justify using OS positivity for the interaction with an ultraviolet cutoff in the spatial direction, one uses the fact that the perturbation series in λ converges. OS positivity for the partial sums in the series is the positivity of the inner product of explicit Fock space vectors.

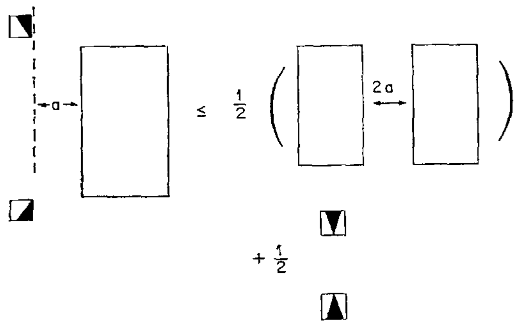

FIGURE 2.

2. Fix l, σ, κ. For any positive a, let $\tilde{H}(a)$ be the Hamiltonian with interaction in $(-l-a, -a) \cup (0, l)$. Then our method shows that:

$$-\inf \operatorname{spec}(H) \leqslant \tfrac{1}{2} \lim_{t\to\infty} t^{-1} \ln(\Omega_0, \exp(-t\tilde{H}(a))\,\Omega_0) \tag{29a}$$

for any $a >$ extent of κ, σ cutoffs. For each fixed t, $(\Omega_0, \exp(-t\tilde{H}(a))\,\Omega_0) \to (\Omega_0, \exp(-t\tilde{H})\,\Omega_0)^2$ as $a \to \infty$ by clustering for the free field, so one might try to obtain vacuum overlap for $H_{l,\sigma,\kappa;\text{ren}}$ by taking $a \to \infty$ in (29a). However, this requires an interchange of t and a limits that *cannot* be justified *purely* on the basis of clustering. This can be seen most clearly in the κ, $\sigma \to \infty$ limit where the "error term" $\sim \exp(-am_0)\exp(-tE_{2l})$ while the "direct term" $\sim \exp(-2tE_l)$. Since $-E_{2l} \geqslant 2(-E_l)$ and strict inequality is expected (and is true in $P(\phi)_2$ [28]), for any fixed a, the error dominates as $t \to \infty$.

3. By the above remark, one cannot use *just* the idea of this proof and spatial clustering to prove vacuum overlap. However, one can obtain the $a = 0$, κ, $\sigma = \infty$ version of (29a).

$$-E_l \leqslant \tfrac{1}{2} \lim_{t\to\infty} t^{-1} \ln Z_{2l,t},$$

which together with

$$\lim_{t\to\infty} t^{-1} \ln Z_{l,t} \leqslant -E_l$$

implies the major consequence of vacuum overlap:

$$\lim_{l,t\to\infty} (lt)^{-1} \ln Z_{l,t} = \lim_{l\to\infty} -E_l/l.$$

Next, we turn to $(\phi^4)_3$. In principle, our method for Y_2 should extend to $(\phi^4)_3$; after all, the method certainly works in $P(\phi)_2$. However, we have a technical problem in the extension that we do not know how to overcome. Namely, we need to prove that the dressed Jost states supported in the region $(-l_1/2, 0)(-l_2/2, l_2/2)$

are dense in $\mathscr{H}_l$. We expect that this is true but do not see the proof. Analyticity of the *dressed* Jost states in spatial variables is not even clear in $P(\phi)_2$. We will therefore develop a distinct method for $\phi_3{}^4$ which, because it relies on Nelson–Symanzik positivity, does not obviously extend to Y_2 ! As we will explain after the proof, it is a relative of Perron–Frobenius arguments. We first need an extension of the unnormalized OS positivity, Theorem 4.1:

LEMMA 5.5. *Given any function F of the positive time Euclidean fields, $\tilde{F}$ will stand for the function obtained by reflecting the fields in the $(t = 0)$-plane and taking the complex conjugate of the function. $d\nu_{l,(a,b)}$ stands for the (unnormalized) $(\phi^4)_3$ Euclidean measure with cutoff $(-l_1/2, l_1/2) \times (-l_2/2, l_2/2) \times (a, b)$ normalized only by the matched vacuum counter terms of Section 2. Then, for any polynomial bounded F, G and $a, b > 0$:*

$$\left| \int F\tilde{G}\, d\nu_{l,(-a,b)} \right|^2 \leqslant \left(\int F\tilde{F}\, d\nu_{l,(-a,a)} \right) \left(\int \tilde{G}G\, d\nu_{l,(-b,b)} \right).$$

Proof. Identical to that for Theorem 4.1 given Feldman's result [6] on the weak convergence of the $d\nu_{l,(a,b)}$, and of their moments.

THEOREM 5.6. *In $(\phi^4)_3$, for any $t > 0$, $\exp(-tH_l)\,\Omega_0^{(l)}$ couples to the vacuum for H_l.*

Proof. (We repeat our warning that we are not asserting H_l has a "vacuum," i.e., eigenvectors at the bottom of the continuum.) Let η be a dressed Jost state. η corresponds to some polynomially bounded function, F, of the positive time fields together with a time a, so that:

$$(\eta, \exp(-tH_l)\,\eta) = \int F(\tilde{F}_t)\, d\nu_{l\times(-a-t,a)},$$

where F_t is F translated t units upwards. By Nelson–Symanzik positivity, i.e., positivity of the measure $d\nu_l$:

$$\left| \int F(\tilde{F}_t)\, d\nu_{l\times(-a-t,a)} \right| \leqslant \int F^2\, d\nu_{l\times(-a-t,a)},$$

since

$$\int F^2\, d\nu = \int (\tilde{F}_t)^2\, d\nu$$

by reflection symmetry. By Lemma 5.5:

$$\left| \int F^2\, d\nu_{l\times(-a-t,a)} \right|^2 \leqslant \int F^2\tilde{F}^2\, d\nu_{l\times(-a,a)} \int d\nu_{l\times(-a-t,a+t)}.$$

Thus

$$(\eta, \exp(-tH_l)\,\eta) = C_\eta(\Omega_0^{(l)}, \exp(-(2t + 2a)\,H_l)\,\Omega_0^{(l)})^{1/2},$$

from which the result follows by Lemma 5.1. The proof is shown pictorially in Fig. 3.

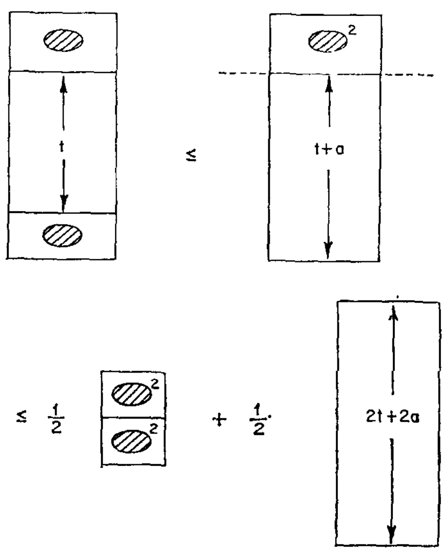

FIGURE 3.

Remark. The above argument is motivated by the result that a matrix with nonnegative elements always has as its largest eigenvector a vector that has nonnegative components and thus, a vector that is not orthogonal to any vector with strictly positive elements. One way of proving this result and its Hilbert space analog is to prove that if A is any positivity preserving operator on $L^2(M, d\mu)$ with $\mu(M) = 1$, then for any $f, g \in L^4$:

$$|(f, Ag)|^2 \leqslant (|f|^2, A1)(1, A\,|g|^2). \tag{30}$$

Equation (30) may be proved by applying the three lines lemma to the analytic function:

$$G(x) = (|f|^{2x}, A\,|g|^{2-2x}).$$

Equation (30) is clearly what we obtain from our use of NS positivity.

The three main results of the genre we are discussing that hold in $P(\phi)_2$ but that we have not been able to extend involve the vacuum overlap:

(1) uniqueness of the vacuum;

(2) prove a vacuum exists for H_l in $(\phi^4)_3$;

(3) prove that the vacuum overlap $\eta_l = -(1/l)\ln(\Omega_0, \Omega_l)$ (in $(\phi^4)_3$ $(\Omega_0^{(l)}, \Omega_l)$) is bounded from above.

In $P(\phi)_2$, (1) depends on strict positivity arguments [18, 53] (2) on either compactness arguments [18] or hypercontractivity [26] and (3) on hypercontractivity [28, 50].

6. Properties of E_l Guerra's Theorem

In Y_2, $Z_{l,t} \to 1$ as l or $t \to 0$ so one can directly mimic the $P(\phi)_2$ methods [28] in proving:

THEOREM 6.1. (a) *$E_l \to 0$ as $l \to 0$.*

(b) *$-E_l/l$ and $(lt)^{-1}\ln Z_{l,t}$ are monotone increasing in l and t and converge as $l, t \to \infty$ to the same limit α_∞.*

(c) *For any $l \geqslant 1$,*

$$-E_{l-1} \leqslant -E_l + E_1 .$$

(d) *$-E_l + \alpha_\infty l$ is monotone decreasing in l.*

Remarks. 1. As we have already discussed in Section 2, we expect $l^{-2}\ln Z_{l,l}$ and also $-l^{-1}E_l$ to diverge to $-\infty$ as $l \to 0$ as $-(\ln l)^2$.

2. We do not know how to prove that β_l is bounded below.

Proof. (a) Let $l < 1$. Then by monotonicity of $-E_l/l$ in l (see (b)):

$$-E_l \leqslant -lE_1 .$$

On the other hand, by monotonicity of $(lt)^{-1}\ln Z_{lt}$ in t:

$$\begin{aligned} -E_l &= \lim t^{-1}\ln Z_{lt} \\ &\geqslant \ln Z_{l1} = \ln\langle\Omega_0, \exp(-lH_1)\,\Omega_0\rangle, \end{aligned}$$

and $\langle\Omega_0, \exp(-lH_1)\,\Omega_0\rangle \to \langle\Omega_0, \Omega_0\rangle = 1$, as $l \to 0$.

(b) The proof is identical to $P(\phi)_2$ [28] where one uses Hölder's inequality and the spectral theorem to prove $l^{-1}\ln\langle\Omega_0, \exp(-tH_l)\,\Omega_0\rangle$ is monotone in t and then Nelson's symmetry to get monotonicity in l. The results for $-E_l$ follow from $-E_l/l = \lim_{t\to\infty}(lt)^{-1}\ln Z_{l,t}$, which comes from our overlap result.

(c) By monotonicity of $-E_l/l$ in l:

$$\begin{aligned} -E_{l-1} &= (l-1)(-E_{l-1}/l-1) \\ &\leqslant (1-(1/l))(-E_l) = -E_l + l^{-1}E_l \\ &\leqslant -E_l + E_1 . \end{aligned}$$

(d) As in $P(\phi)_2$:

$$\begin{aligned} -E_{l+a} + E_l &= \lim_{t\to\infty} t^{-1} \ln[\langle \Omega_0 , \exp(-tH_{l+a})\, \Omega_0\rangle / \langle \Omega_0 , \exp(-tH_l)\, \Omega_0\rangle] \\ &= \lim_{t\to\infty} t^{-1} \ln[\langle \Omega_0 , \exp(-(l+a)\, H_t)\, \Omega_0\rangle / \langle \Omega_0 , \exp(-lH_t)\, \Omega_0\rangle] \\ &\leqslant \lim_{t\to\infty} t^{-1} \ln \| \exp(-aH_t)\| = a \lim(-E_t/t) = a\alpha_\infty , \end{aligned}$$

so that the result holds. ∎

Because $\| \Omega_0^{(l)} \| = \infty$, these proofs do not extend to $(\phi^4)_3$; in fact, as we shall see $-E_l \to +\infty$ as $l \to 0$. However (b) and (d) (and (c) suitably modified) are all true in $(\phi^4)_3$. The key to their proof is the realization that they are really consequences of convexity of $-E_l$ as a function of l. For example, if f is convex in l and $|f(l)| \leqslant Cl$, then $\lim_{l\to\infty} f(l)/l$ exists because $(f(l) - f(l_0))/(l - l_0)$ is monotone increasing. Moreover, for any a, $f(l + a) - f(a)$ is monotone increasing in l, which leads to (c). Finally, as we shall see, $f(l) - l \lim_{x\to\infty}(f(x)/x)$ is monotone decreasing in l, which is behind (d).

THEOREM 6.2. *In $(\phi^4)_3$, $\ln Z_{l_1,l_2,t}$ and $-E_{l_1,l_2}$ are convex in each variable with the others held fixed. Moreover*:

(a) *$-E_{l_1,l_2} \to \infty$ as l_1 or l_2 goes to zero, the others being held fixed.*

(b) *$\lim_{l_1,l_2\to\infty} -E_{l_1l_2}/l_1l_2$ and $\lim_{l_1l_2t\to\infty}(l_1l_2t)^{-1} \ln Z_{l_1l_2t}$ both exist and are equal. Call the limit α_∞.*

(c) *For l_2 fixed $A(l_2) = \lim_{l_1\to\infty} -E_{l_1,l_2}/l_1$ exists and obeys*

$$A(l_2 - 1) \leqslant A(l_2) - A(2) + A(1), \qquad \text{if} \quad l_2 \geqslant 2.$$

(d) *$A(l_2) = \lim_{l_1\to\infty}(-E_{l_1,l_2} + E_{(l_1-1),l_2})$.*

(e) *For any fixed l_2, $-E_{l_1l_2} - A(l_2)\, l_1$ is monotone decreasing in l_1 and $A(l_2) - \alpha_\infty l_2$ decreasing in l_2.*

Proof. By an argument we have already given $\ln Z_{l_1,l_2,t}$ is jointly continuous in its arguments. Alternatively an upper bound and mid-convexity implies continuity. Moreover, for each fixed l_1, l_2, it is $\frac{1}{2}$-convex in t and so by continuity, convex in t. It is then convex in l_1, l_2 by symmetry. Convexity of $-E_{l_1l_2}$ and $A(l_2)$

(once we know the limit exists) then follows. Now fix l_2 and suppose that $l_1 \to 0$. Then

$$-E_{l_1 l_2} \geqslant \ln \left[\frac{\langle \Omega_0^{(l_1 l_2)}, \exp(-2H_{l_1 l_2}) \, \Omega_0^{(l_1 l_2)} \rangle}{\langle \Omega_0^{(l_1 l_2)}, \exp(-H_{l_1 l_2}) \, \Omega_0^{(l_1 l_2)} \rangle} \right]$$
$$= \ln[Z_{l_1 l_2 2}/Z_{l_1 l_2 1}]$$
$$= \ln[\tilde{Z}_{l_1 l_2 2}/\tilde{Z}_{l_1 l_2 1}] + \Delta_2(l_1, l_2, 2)$$
$$+ \Delta_3(l_1, l_2, 2) - \Delta_2(l_1, l_2, 1) - \Delta_3(l_1, l_2, 1).$$

As $l_1 \to 0$, all the terms but the Δ_2 terms go to zero, and by the proof of Theorem 2.4(b)

$$\Delta_2(l_1, l_2, t) \sim C l_2 t \ln(l_1^{-1}),$$

where C a positive constant, so that

$$-E_{l_1 l_2} \geqslant O(\ln l_1^{-1}).$$

This proves (a), (b) and the first part of (c) follow from convexity and Lemma 6.3. The second half of (c) is then a simple consequence of the convexity of $A(l_2)$ in l_2. Parts (d) and (e) then follow from Lemma 6.4. ∎

LEMMA 6.3. *Let $f(x_1, \ldots, x_n)$ be defined in the region where all $x_i \geqslant 1$ and obey*

$$f(x_1, \ldots, x_n) \leqslant C x_1 \cdots x_n$$

in that region. Suppose f is convex in each variable when the others are held fixed. Then:

$$\lim_{x_1, \ldots, x_n \to \infty} f(x_1, \ldots, x_n)/x_1 \cdots x_n$$

exists independently of how the x_i go to infinity.

Proof. Let $\tilde{f}(x_1, \ldots, x_n) = f(x_1, \ldots, x_n) - C x_1 \cdots x_n$. Then $\tilde{f}$ is negative and convex in each variable the others held fixed. By convexity,

$$[\tilde{f}(x_1, \ldots, x_n) - \tilde{f}(1, x_2, \ldots, x_n)]/(x_1 - 1)$$

is monotone increasing in x_1 if $x_2, \ldots, x_n$ are held fixed. But then

$$(x_1 - 1)^{-1} \tilde{f}(x_1, \ldots, x_n) = (x_1 - 1)^{-1}(\tilde{f}(x_1, \ldots) - \tilde{f}(1, \ldots)) + (x_1 - 1)^{-1} \tilde{f}(1, \ldots)$$

is monotone in x_1 if $x_2, \ldots, x_n$ are held fixed since $\tilde{f}$ is negative. Thus, by symmetry, $\prod_{i=1}^n (x_1 - 1)^{-1} \tilde{f}(x_1, \ldots, x_n)$ is monotone increasing in the x_i. Since it is negative, it has a limit independent of the order in which the x_i go to infinity. But then $\prod_{i=1}^{n} x_i^{-1}\tilde{f}(x_i)$ has a limit and therefore, so does $f(x_1, ..., x_n)/x_1 \cdots x_n$. ∎

LEMMA 6.4. *Let $f(x)$ be convex in the region $1 \leqslant x < \infty$ and linearly bounded from above. Let $a = \lim_{x\to\infty} f(x)/x$. Then*

(a) *$f(x) - f(x-1) \to a$ as $x \to \infty$,*

(b) *$f(x) - ax$ is monotone decreasing.*

Proof. Since f is convex, it has a right-hand derivative $(D^+f)(x)$ at every point and

$$f(x) - f(y) = \int_y^x (D^+f)(t)\, dt.$$

Moreover, $(D^+f)(x)$ is monotone increasing. From this monotonicity and

$$x^{-1}(f(x) - f(1)) = x^{-1} \int_1^x (D^+f)(t)\, dt,$$

it follows that $(D^+f)(t) \to a$ monotonically as $t \to \infty$. Thus

$$f(x) - f(x-1) = \int_{x-1}^{x} (D^+f)(t)\, dt \to a,$$

and for $x > y$

$$f(x) - f(y) = \int_y^x (D^+f)(t)\, dt \leqslant a(x-y),$$

so that

$$f(x) - ax \leqslant f(y) - ay. \quad ∎$$

We note that since we have precise control on the difference between the objects with conventional renormalization and those with matched counterterms (Theorems 2.1, 2.4, and 2.5), we have:

THEOREM 6.5. *In Y_2 (resp. ϕ_3^4) the limits $\lim_{l\to\infty} -\tilde{E}_l/l$ (resp. $\lim_{l_1 l_2\to\infty} -\tilde{E}_{l_1 l_2}/l_1 l_2$) and $\lim_{l,t\to\infty}(lt)^{-1} \ln \tilde{Z}_{l,t}$ (resp. $\lim_{l_1 l_2 t\to\infty}(l_1 l_2 t)^{-1} \ln \tilde{Z}_{l_1 l_2 t}$) exist and are equal to each other and also to α_∞.*

Remark. In addition, $\lim(-E_l - \alpha_\infty l)$ exists if and only if $\lim(-\tilde{E}_l - \alpha_\infty l)$ exists and the limits differ by an explicit constant.

7. ϕ-BOUNDS

In this section and the next we prove ϕ bounds and Fröhlich bounds in Y_2 and $\phi_3{}^4$ (these bounds were first proved in $P(\phi)_2$ by Glimm–Jaffe [20] and Fröhlich [10], respectively). It is known that modulo technical difficulties and hypotheses these bounds are equivalent (see Fröhlich, [10] for $\phi \to F$ and [11] for $F \to \phi$). We give distinct (but related) proofs for two reasons: In Y_2, we wish to prove Fermion Fröhlich bounds; it is not quite clear how to obtain these from ϕ-bounds alone even using the fact that smeared relativistic Fermi fields are bounded. On the other hand, the passage $F \to \phi$ requires additional hypotheses that certainly hold, but that we wish to avoid.

By the methods of this section and the next, one can obtain bounds for $:\phi^2:$ and various nonlocal functions of the field in Y_2 and $\phi_3{}^4$ theories.

As previously in this paper, our replacement for the Markov property used at this point in $P(\phi)_2$ proofs is OS positivity: for ϕ-bounds in spatial directions and for Fröhlich bounds in space and time directions. We note that the joint use of OS positivity in space and time directions to obtain operator bounds in $P(\phi)_2$ has been exploited recently by Glimm and Jaffe [23] and by Glimm, Jaffe, and Spencer [24].

As in the $P(\phi)_2$ case, we prove operator bounds by proving bounds on difference of vacuum energies [20]. However, the passage from the energy bounds to operator bounds is more subtle due to operator theoretic questions. We begin by describing this passage in $(\phi^4)_3$. Similar considerations work also in Y_2:

1. Fix $f \in C_0{}^\infty(R^2)$. One defines an operator "$H_l + \phi(f)$" on $\mathscr{H}_l$ by giving the semigroup between dressed Jost states. For example:

$$(\exp(-aH_l)\,\Omega_0^{(l)}, \exp(-t(H_l + \phi(f)))\exp(-bH_l)\,\Omega_0^{(l)})$$
$$= \int \exp(\phi(f \otimes \chi_{(0,t)}))\, d\nu_{l \times (-a, t+b)}\,.$$

The boundedness of the operator so defined follows as in Section 4 using a slight improvement of the $\phi_3{}^4$ lower bound of Glimm–Jaffe [22]. (Such a bound is implicit in some of the proofs in Feldman's paper [6]; see especially [6, Theorem 22].)

$$\left(\int \exp(\phi(f))\, d\nu_g\right) \leqslant \exp(C[A(g) + A(f)])$$

for $0 \leqslant g \leqslant 1$, $|f| \leqslant a$ where C is an a-dependent constant and $A(f) \equiv$ volume $\{x \mid \operatorname{dist}(x, \operatorname{supp} f) \leqslant 1\}$.

2. For $f \in C_0{}^\infty(R^2)$, one defines $\phi(f)$ as a quadratic form with form domain equal to the dressed Jost states. This is done by using analyticity arguments very

similar to those in [41]. A priori $\exp(+sH_l)\,\phi(f)\exp(-sH_l)$ is defined between fixed Jost states if we smear in s with a function having support in a small interval whose size depends on the Jost states. But one can show analyticity in s and so define $\phi(f)$ between the Jost states.

3. Between Jost states, one proves that

$$\exp(-t(H_l + \phi(f)))$$
$$= \exp(-tH_l) + \int_0^t \exp(-s(H_l + \phi(f)))\,\phi(f)\exp(-(t-s)\,H_l)\,ds,$$

and then that $H_l + \phi(f)$ is an extension of the form sum $H_l + \phi(f)$ on Jost states.

4. A bound $-E(H_l + \phi(f)) \leqslant -E(H_l) + a$ implies that between Jost states

$$\begin{aligned}\phi(f) &\leqslant H_l + \phi(f) - E(H_l + \phi(f)) - \phi(f)\\ &\leqslant H_l - E(H_l) + a.\end{aligned}$$

Thus, on a form core for H_l we obtain

$$\pm\phi(f) \leqslant H_l - E_l + a.$$

5. The above considerations lead to ϕ-bounds. They prove that the form sum $H_l + \phi(f)$ is a closed quadratic form on $Q(H_l)$ but they do not prove that this form sum equals "$H_l + \phi(f)$" as constructed in step 1. We suspect this could be proven but do not pause to do so.

THEOREM 7.1. *In Y_2, one has*

$$\pm\phi(f) \leqslant \hat{H}_l + c_1 \|f\|_{-1}^2 + c_2$$

for all $f \in C_0^\infty(R)$ with supp $f \subset [-\frac{1}{2}, \frac{1}{2}]$ *all $l \geqslant 1$ and suitable constants c_1, c_2. Here, $\hat{H}_l = H_l - E_l$ and $\|f\|_{-1}^2 = \int |\hat{f}(k)|^2(k^2+1)^{-1}\,dk$.*

Proof. Let $E_l(f)$ be the ground state energy for $H_l + \phi(f)$. As in the above remarks, we need only prove

$$-E_l(f) \leqslant -E_l + c_1\|f\|_{-1}^2 + c_2\,. \tag{31}$$

Let F be the function obtained by translating f by $\frac{1}{2}$ unit and taking the sum of the translation and its reflection about $l = 0$. We first claim that

$$-E_l(f) \leqslant -\tfrac{1}{2}E_{l-1} - \tfrac{1}{2}E_{l+1}(F). \tag{32}$$

This follows as in the proof of coupling to the vacuum and is shown pictorially in Fig. 4:

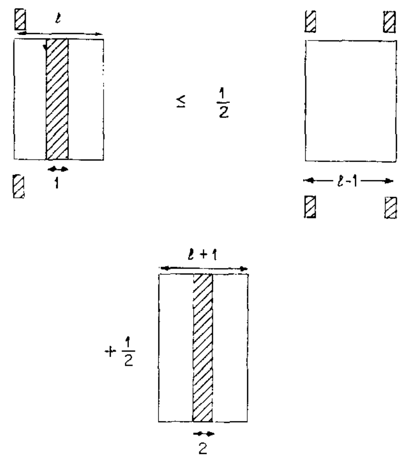

FIGURE 4.

In Fig. 4, the effect of the Jost states drop out in the $t \to \infty$ limit. By iterating (32) we obtain

$$-E_l(f) \leqslant (1 - (1/2^n)(-E_{l-1}) + (1/2^n)(-E_{l+2^n-1}(F_n)), \tag{33}$$

where F_n is obtained by iterating the passage from $f \to F$ n times. Now by coupling to the vacuum (which holds for $H_l + \phi(F_n)$ if $n \geqslant 1$ since F_n is symmetric)

$$\begin{aligned}
-E_{l+2^n-1}(F_n) &= \lim_{t\to\infty} t^{-1} \ln \int \exp(\phi(F_n \otimes \chi_{(0,t)})) \operatorname{det}_{\text{ren}}(1 + K_{L,t})\, d\mu_0 \\
&\leqslant \lim_{t\to\infty} t^{-1} \ln \left[\int \det(1 + K_{L,t})^2\, d\mu_0\right]^{1/2} \left[\int \exp(\phi(F_n \otimes \chi_{(0,t)})^2\, d\mu_0\right]^{1/2} \\
&\leqslant C_3 L + 2^n C_1 \|f\|_{-1}^2,
\end{aligned}$$

where $L = l + 2^n - 1$. Taking $n \to \infty$ in (33), we obtain

$$-E_l(f) \leqslant -E_{l-1} + C_3 + C_1 \|f\|_{-1}^2 .$$

Equation (31) results by using $-E_{l-1} \leqslant -E_l + E_1$ and taking $C_2 = C_3 + E_1$. ∎

If one can pass to the infinite volume limit, then Theorem 7.1 extends by the method of Glimm and Jaffe [20]. By using translation invariance in that limit, one easily obtains

$$\pm\phi(f) \leqslant \|f\|_S(\hat{H}_l + 1)$$

for a suitable Schwartz space norm $\|\cdot\|_S$. However, one can do much better than the usual $\|\cdot\|_S$ at least, as involves behavior at infinity. One first proves a suitable "improved" linear lower bound as in [29]:

THEOREM 7.2. *Fix a semibounded polynomial P, a Y_2 coupling constant, λ, and Γ ($= 1$ or $i\gamma_5$). Let H_l, $d\mu_{l,t}$ be the Hamiltonian and cutoff (unnormalized) Euclidean measure for $Y_2 + P(\phi)_2$. Let $\alpha_\infty(\mu)$ denote the pressure for this theory with a $-\mu\phi$ term added, i.e.,*

$$\alpha_\infty(\mu) = \lim_{l\to\infty} -E\left(H_l - \mu\int_{-l/2}^{l/2}\phi(x)\,dx\right)\Big/ l. \tag{34}$$

Then for any $f\in C_0^\infty(R)$ with $\operatorname{supp} f \subset [-l/2, l/2]$:

$$H_l - \phi(f) \geqslant -\int_{-l/2}^{l/2}\alpha_\infty(f(x))\,dx \tag{35}$$

and for any $f\in C_0^\infty(R^2)$ with $\operatorname{supp} f \subset [-l/2, l/2]\times[-t/2, t/2]$:

$$\ln\left[\left|\int \exp(\phi(f))\,d\mu_{l,t}\right|\right] \leqslant \int_{|x|\leqslant l/2,\,|s|\leqslant t/2}\alpha_\infty(f(x,s))\,dx\,ds. \tag{36}$$

Remarks. 1. In this theorem and the one immediately following we intend to allow both the pure $P(\phi)_2$ case ($\lambda = 0$) and the pure Y_2 case ($P = 0$). Theorem 7.2 is, of course, not new in the $P(\phi)_2$ case; (35) is from [29] and (36) from [30], in fact, we just follow the proofs from there. Theorem 7.3 below is a slight improvement of existing $P(\phi)_2$ results.

2. Since Theorem 7.2 depends on monotonicity of $-E_l/l$, it does not extend to ϕ_3^4, but using the monotonicity implicit in the proof of Lemma 6.3, we expect it should be possible to prove a substitute suitable for extending Theorem 7.3 to ϕ_3^4.

Proof (following [29]). For (35), it suffices to consider the case where f is piecewise constant, i.e., $-(l/2) = a_0 < a_1 < \cdots < a_n = l/2$, and $f(x) = \alpha_i$ on (a_{i-1}, a_i). Then, by Nelson's symmetry:

$$\begin{aligned}(\Omega_0, \exp(-t(H_l - \phi(f)))\,\Omega_0) &= \left(\Omega_0, \prod_{i=1}^n \exp(-(a_i \cdots a_{i-1})(H_t - \alpha_i\phi(\chi_t))\,\Omega_0\right)\\ &\leqslant \prod_{i=1}^n \exp(-[E(H_t - \alpha_i\phi(\chi_t))(a_i \cdots a_{i-1})]\\ &\leqslant \prod_{i=1}^n \exp[+t\alpha_\infty(\alpha_i)(a_i - a_{i-1})]\\ &= \exp\left(t\int_{-l/2}^{l/2}\alpha_\infty(f(x))\,dx\right).\end{aligned}$$

This proves (35). In (36), we consider f's of the form $f(x, s) = g_i(x)$ for $a_{i-1} \leqslant s < a_i$ and repeat the argument of (35). ∎

We now have:

THEOREM 7.3. *Fix* $f \in C_0^\infty(R)$ *and fix* P, λ, Γ *as in Theorem* 7.2. *Then for all* l *with* $\operatorname{supp} f \subset [-l/2, l/2]$:

$$\pm\phi(f) \leqslant \hat{H}_l + c_l(f),$$

where $c_l(f)$ *is a constant which obeys*

$$\lim_{l\to\infty} c_l(f) = \int_{-\infty}^{\infty} [\alpha_\infty(f(x)) - \alpha_\infty(0)]\, dx. \tag{37}$$

In particular, if the infinite volume limit of the Wightman functions exists, then for any $f \in \mathscr{S}$:

$$\pm\phi(f) \leqslant \hat{H}_\infty + \int_{-\infty}^{\infty} [\alpha_\infty(f(x)) - \alpha_\infty(0)]\, dx. \tag{38}$$

Remark. It is not hard to see that for large μ, $\alpha_\infty(\mu) \leqslant O(\mu^2)$ for pure Y_2 and $\alpha_\infty(\mu) \leqslant O(\mu^{1+m(d)})$ with $m(d) = (\deg P - 1)^{-1}$ if $\deg P \geqslant 2$, and for small μ, $\alpha_\infty(\mu) \leqslant O(\mu)$ in general and $\alpha_\infty(\mu) \leqslant O(\mu^2)$ in case P is even and $\Gamma = i\gamma_5$ *and* we are below the critical point as defined by continuity of $\alpha_\infty'(\mu)$ at $\mu = 0$. These bounds allow the passage from (37) to (38) (for $f \in \mathscr{S}$) and also lead to bounds of the form

$$\pm\phi(f) \leqslant \|f\|_L(\hat{H}_l + 1),$$

where $\|f\|_L$ is a sum of L^p norms. For example, in pseudoscalar Y_2 below the (putative) critical point

$$\pm\phi(f) \leqslant (\hat{H}_l + c_1 \|f\|_2^2),$$
$$\pm\phi(f) \leqslant c_2 \|f\|_2 (\hat{H}_l + 1).$$

Proof. By following the proof Theorem 7.1 but using Theorem 7.2 to bound the factor $-E_{l+2^n-1}(F_n)$ in (33), we obtain, if $\operatorname{supp} f \subset [-a/2, a/2]$:

$$-E_l(t) \leqslant -E_{l-a} + 2^{-n} \int_{l-a+2^n a}^{l-a+2^n a} \alpha_\infty(F_n(x))\, dx$$

$$\xrightarrow[(n\to\infty)]{} -E_{l-a} + \int_{-a}^{a} \alpha_\infty(f(x))\, dx.$$

This implies $\pm\phi(f) \leqslant \hat{H}_l + c_l(f)$ with

$$c_l(f) = -E_{l-a} + E_l + \int_{-a}^{a} \alpha_\infty(f(x))\, dx.$$

Equation (37) follows by using Lemma 6.4.

$$E_{l-a} - E_l \to a\alpha_\infty(0), \qquad \text{as} \quad l \to \infty.$$

As explained in the remark, (38) follows from (37). ∎

For $(\phi^4)_3$, the ϕ-bounds that arise naturally from the methods we have been using do not quite have the form one might guess. We have (supp $f \subset [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$)

$$\pm\phi(f) \leqslant \hat{H}_{l_1 l_2} + c_{l_1 l_2}(f), \tag{39}$$

but we do not prove that $c_{l_1 l_2}(f)$ is bounded for all $l_1 l_2 \geqslant 2$; rather, we only prove that $\limsup_{l_1\to\infty} c_{l_1 l_2}(f)$ is bounded for all $l_2 \geqslant 2$. That this is the form of the ϕ-bounds should not be too surprising. For Fröhlich's bounds in $P(\phi)_2$ only hold if one direction is taken to infinity. The key moral is that we get l-independent bounds in one direction, if all but one direction are taken to infinity. We note that bounds on lim sup $c_{l_1 l_2}(f)$ for free boundary conditions lead to bounds on $c_{l_1 l_2}(f)$ for Dirichlet BC: this is just a transfer to $\phi_3{}^4$ of a remark of Fröhlich for $P(\phi)_2$ [10, 48]. (See the next section for a brief discussion of $\phi_3{}^4$ with Dirichlet BC).

THEOREM 7.4. *Consider $\phi_3{}^4$ with fixed coupling constant. Let $f \in C_0{}^\infty(R^2)$ have support in $[-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$. Then for $l_1, l_2 \geqslant 2$:*

$$\pm\phi(f) \leqslant \hat{H}_{l_1, l_2} + c_{l_1, l_2}(f), \tag{39}$$

where $c_{l_1 l_2}(f)$ is only dependent on $\|f\|_\infty$ and

$$\lim_{l_1\to\infty} c_{l_1 l_2}(f) \leqslant c(f) \tag{40}$$

for all $l_2 \geqslant \frac{1}{2}$, where $c(f)$ is only dependent on $\|f\|_\infty$.

Proof. By the method of Section 5, $\exp(-t(H_l - \phi(f)))\, \Omega_0^{(l)}$ overlaps the vacuum for $H_l - \phi(f)$ so $-E_l(f)$ is given as the limit as $t \to \infty$ of an object whose cross section at fixed t is given in the first part of in Fig. 5. We reflect n times in the l_1 direction and then m times in the l_2 direction to obtain:

$$\begin{aligned} -E_{l_1 l_2}(f) &\leqslant (1 - (1/2^n))(-E_{(l_1-1), l_2}) + (1/2^n)\, E_{l_1+2^n-1, l_2}(F_n) \\ &\leqslant (1 - (1/2^n))(-E_{(l_1-1, l_2)}) + (1 - (1/2^m))(1/2^n)\, E_{l_1+2^n-1, l_2-1} \\ &\quad + 2^{-m} 2^{-n} E_{l_1+2^n-1, l_2+2^m-1}(F_{n,m}) \end{aligned}$$

(shown pictorially in Fig. 5).

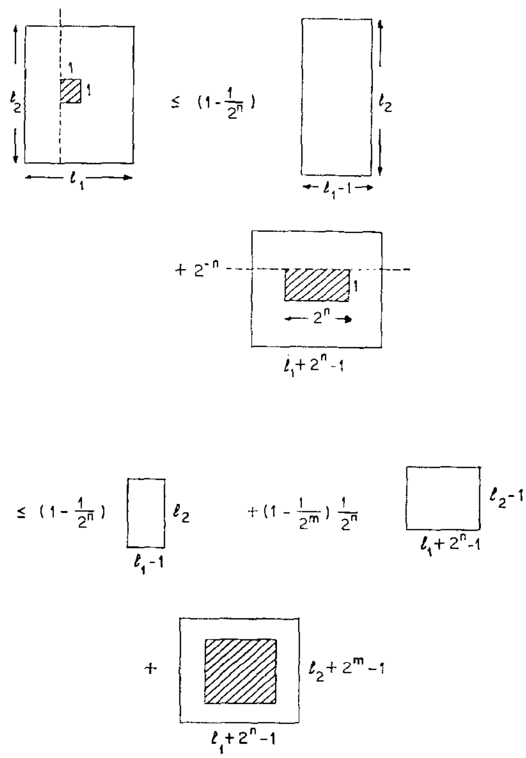

FIGURE 5.

Taking first $m \to \infty$ and then $n \to \infty$ and using the general $\phi_3{}^4$ linear lower bound, we find

$$-E_{l_1l_2}(f) \leqslant -E_{l_1-1,l_2} + A(l_2-1) + c_0(f);$$

where $A(l_2)$ is given by Theorem 6.2(c). Equation (39) now holds with

$$c_{l_1l_2}(f) = -E_{l_1-1,l_2} + E_{l_1,l_2} + A(l_2-1) + c_0(f).$$

Using Theorem 6.2(d),

$$\lim_{l_1\to\infty} c_{l_1l_2}(f) = -A(l_2) + A(l_2-1) + c_0(f), \tag{41}$$

so that (40) follows from Theorem 6.2(c).

Remark. Using Lemma 6.4 and (41) one easily sees that

$$\lim_{l_2\to\infty} (\lim_{l_1\to\infty} c_{l_1 l_2}(f)) = c_0(f) - \alpha_\infty(0).$$

Thus, bounds of the type occuring in Theorem 7.3 will follow from a bound $c_0(f) \leqslant \int_{|x_1|\leqslant 1, |x_2|\leqslant 1} \alpha_\infty(f(x))\, d^2x$, which we certainly expect to hold.

8. Fröhlich Bounds

In this section, we want to very briefly describe the method for proving Fröhlich bounds in Y_2 and ϕ_3^4.

THEOREM 8.1. *Fix coupling constants in Y_2. Let*

$$S_{l,t}(f_1, f_2, ..., f_k; g_1, ..., g_m; h_1, ..., h_m)$$

denote the Schwinger function with space-time cutoffs in $(-l/2, l/2) \times (-t/2, t/2)$ and $f_1, ..., f_k$ smearing functions for ϕ, $g_1, ..., g_m$ for $\bar{\psi}$ and $h_1, ..., h_m$ for ψ. Then, for a constant C independent of l, f, g, h, and all $l \geqslant 1, f, g, h$, with supp $f_i(g_i, h_i) \subset [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$:

$$\limsup_{t\to\infty} | S_{l,t}(f_i, g_j, h_j)| \leqslant C(k!)^{1/2} \prod \|f_i\|_{-1} \|g_i\|_{-1/2} \|h_i\|_{-1/2}, \tag{42}$$

where

$$\|f\|_{-\alpha} = \int |\hat{f}(k)|^2 (k^2 + m^2)^{-\alpha}\, d^2k.$$

Remarks. 1. Bounds of this form for fixed l, t were first proven in [45]. Bounds on $Z_{l,t}S_{l,t}$ with a volume dependent constant occur in [38, 47] and are used below.

2. $[-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$ can be replaced by $[-a/2, a/2] \times [-a/2, a/2]$ if C is taken suitably a-dependent.

3. The bounds in (42) are not in the form of temperedness bounds but in the next section we prove temperedness bounds for $\limsup_{l\to\infty} \limsup_{t\to\infty} S_{l,t}$ in a form suitable for the Osterwalder–Schrader reconstruction theorem (see Theorem 9.3).

4. Equation (42) holds also for $Y_2 + P(\phi)_2$ theories. For pure $P(\phi)_2$ theories the $\|f_i\|_{-1}$ norms seems to be an improvement on the best local bounds [48] which have the form $\|f\| = \int |\hat{f}(k)|^2(k_1^2 + m^2)^{-1}\, d^2k$.

5. Similar results hold for generating functions for S.

Proof. Pictorially the proof is already shown in Fig. 5 suitably reinterpreted (change l_1 to t and l_2 to l): Namely, we start with $Z_{l,t}S_{l,t}$ and reflect n times in the t direction and then m-times in the l direction and obtain:

$$| Z_{l,t}S_{l,t} | \leqslant (Z_{l,t-1})^{1-(1/2^n)} (Z_{l-1,t+2^n-1})^{(1/2^n)(1-(1/2^m))}$$
$$\times [(ZS)_{l+2^m-1,t+2^n-1} (f_i^{(n,m)} \cdots)]^{(1/2^n)(1/2^m)}.$$

By the volume divergent bounds on (ZS) [38, 47], the last factor is bounded by $\exp(c[1 + (l-1) 2^{-m}][1 + (t-1) 2^{-n}])(k!)^{1/2} \prod \| f_i \|_{-1} \| g_j \|_{-1/2} \| h_j \|_{-1/2}$ and so taking $n \to \infty$ and $m \to \infty$:

$$| Z_{l,t}S_{l,t} | \leqslant Z_{l,t-1} \exp(-E_{l-1})\, c(k!)^{1/2} \left(\prod \| f_i \|_{-1} \| g_j \|_{-1/2} \| h_j \|_{-1/2}\right).$$

Dividing by $Z_{l,t}$ and using Lemma 6.4 to see that $Z_{l,t-1}/Z_{l,t} \to \exp(E_l)$ and Theorem 6.1 to see that $-E_{l-1} + E_l \leqslant E_1$:

$$\limsup_{t\to\infty} | S_{l,t} | \leqslant C'(k!)^{1/2} \prod \| f_i \|_{-1} \| g_j \|_{-1/2} \| h_j \|_{-1/2}$$

with $C' = c \exp(E_1)$. ∎

Similarly, we have:

THEOREM 8.2. *Fix coupling constants in* $\phi_3{}^4$. *Let* $S_{l_1 l_2 t}(f_1, ..., f_k)$ *denote the Schwinger function with space–time cutoffs in* $(-l_1/2, l_1/2) \times (-l_2/2, l_2/2) \times (-t/2, t/2)$ *and smearing functions* $f_1, ..., f_k$. *Then for a constant* C *independent of* l_1 *and* f *and all* $l_1 \geqslant 2$, f_i *with* $\operatorname{supp} f_i \subset [-\frac{1}{2}, \frac{1}{2}] \times [-\frac{1}{2}, \frac{1}{2}]$:

$$\limsup_{l_2;\, t\to\infty} | S_{l_1 l_2 t}(f_i)| \leqslant Ck! \prod_{i=1}^{k} \| f_i \|_\infty . \tag{43}$$

Proof. We first note that by applying Cauchy estimates to the bound $\int \exp(\phi(f))\, d\mu_{l_1 l_2 t} \leqslant \exp(C(l_1 l_2 t))$ for all $f \in C_0{}^\infty$ with $\| f \|_\infty \leqslant 1$ and $\operatorname{supp} f \subset [-l_1/2, l_1/2] \times \cdots$ (see point 2 in Section 7) we find that:

$$|(ZS)_{l_1 l_2 t} (f_1, ..., f_k)| \leqslant Ck_1! \cdots k_m! \prod_{i=1}^{k} \| f_i \|_\infty , \tag{44}$$

where each f_i is supported in a unit cube with integral center, k_1 in one cube, ..., k_m

in another cube, $k_1 + \cdots + k_m = k$. Reflecting m times in the t direction, n in the l_2 direction and j in the l_1 direction and then using (44) we find:

$$
\begin{aligned}
&|(ZS)_{l_1 l_2 t}(f_1, ..., f_k)| \\
&\quad \leqslant (Z_{l_1 l_2 t-1})^{(1-(1/2^n))} (Z_{l_1 (l_2-1), t+2^n-1})^{(1/2^n)(1-(1/2^m))} \\
&\qquad \times (Z_{l_1-1, l_2+2^m-1, t+2^n-1})^{(1/2^n)(1/2^m)(1-(1/2^j))} [C^{(1/2^n)+(1/2^m)+(1/2^j)}]\, k! \prod_{i=1}^{k} \|f_i\|_\infty .
\end{aligned}
$$

Taking $n \to \infty$, then $m \to \infty$, and then $j \to \infty$:

$$|(ZS)_{l_1 l_2 t}(f_1, ..., f_k)| \leqslant Z_{l_1 l_2 (t-1)} \exp(-E_{l_1, l_2-1}) \exp(A(l_1 - 1))\, k! \prod_{i=1}^{k} \|f_i\|_\infty .$$

Dividing by $Z_{l_1 l_2 t}$ and taking $t \to \infty$, we get

$$\limsup_{t\to\infty} | S_{l_1 l_2 t}(f_i)| \leqslant \exp(E_{l_1, l_2} - E_{l_1, l_2-1} + A(l_1 - 1)\, k! \prod_{i=1}^{k} \|f_i\|_\infty . \quad (45)$$

Now taking $l_2 \to \infty$ and using $E_{l_1 l_2} - E_{l_1, l_2-1} \to -A(l_1)$ and $A(l_1 - 1) - A(l_1) \leqslant A(1) - A(2)$, we obtain (43) with $C = \exp(A(1) - A(2))$. ▮

Remarks. 1. In (43), we can replace $Ck!$ by $CD^k k!$ for a $D > 0$ and C dependent on D.

2. Equation (45) follows using Fröhlich's method [10] from the ϕ-bounds of Section 7. We note that it is not necessary to have a vacuum for Fröhlich's method to work. All one needs is that $(\Omega_0^{(L)}, \exp(-tH_l)\Omega_0^{(L)})/(\Omega_0^{(L)}, \exp(-(t-1)H_l)\Omega_0^{(L)}) \to \exp(-E_l)$, which follows from the spectral theorem and coupling to the vacuum (or by Lemma 6.4).

As it stands, (43) does not involve Schwartz space norms. In the next section we will remedy this by using improved volume divergent bounds. Alternatively one has:

THEOREM 8.3. *If $S_{l_1 l_2 t}(f_i)$ has a translation invariant limit as $l_1, l_2, t \to \infty$, then for all $f_i \in C_0^\infty$*

$$| S_\infty(f_1, ..., f_k)| \leqslant Ck! \prod_{i=1}^{k} \|f_i\|_S ,$$

for a suitable Schwartz space norm $\|f_i\|_S$.

Proof. The key fact is to note that by *NS* positivity, if each f_i is supported in some unit cube, we have:

$$| S_\infty(f_1, ..., f_k)| \leqslant \prod_{i=1}^{k} [S_\infty(f_i, ..., f_i)]^{1/k}$$

$$\leqslant Ck! \prod_{i=1}^{k} \| f_i \|_\infty .$$

Thus, writing $f_i = \sum_{j \in Z^3} f_i^{(j)}$, where $f^{(j)}$ has support in the unit cube about j:

$$| S_\infty(f_1, ..., f_k)| \leqslant Ck! \prod_{i=1}^{k} \left(\sum_j \| f_j^{(j)} \|_\infty\right)$$

$$\leqslant Ck! \prod_{i=1}^{k} \| f_i \|_S ,$$

since

$$\sum_j \| f^j \|_\infty \leqslant [\sup \|(2 + x^2)^2 f\|_\infty] \sum_j (1 + | j |^2)^{-2}$$

is a Schwartz space norm.

The point of Theorem 8.3, of course, is that it gives one bounds in a form suitable for the Osterwalder–Schrader reconstruction theorem [41]. We remark that modulo technical details involving convergence of the lattice approximation in $(\phi^4)_3$ with Dirichlet BC, one can construct $(\phi^4)_3$ infinite volume field theories as follows: The key observation, which we learned from Herbst in the fall of 1974, is that the mass counter term in $(\phi^4)_3$ can be taken to be the same with free or Dirichlet BC (This is not true for the second-order energy counter term. Thus "conditioning" estimates [30, 31] break down). As a result, the Dirichlet Schwinger functions should be monotone in region and bounded by the corresponding free Schwinger functions. Our bounds then allow the passage to an infinite volume theory obeying all the OS axioms (except perhaps clustering) and thus a Wightman theory obeying all the axioms (except perhaps uniqueness of vacuum). If a cluster expansion can be developed at large external field (in the spirit of Spencer [54]), then one should be able to use Lee–Yang arguments [53] to obtain uniqueness of the vacuum and nonzero mass gap for any $(\phi^4)_3$–$\mu\phi$ theory as in two dimensions [32, 49].

9. Improved Volume Divergent Bounds on Schwinger Functions

We, in a previous paper [47], and independently McBryan [38], have proven volume divergent bounds on unnormalized Y_2 Schwinger functions, in the form

$$(ZS)_{l,t}(f_1,\dots,f_k;g_1,\dots,g_m;h_1,\dots,h_m) \leqslant C_1^{k+2m}\prod_{\alpha}(k_\alpha!)^{1/2}\exp(C_2lt)\prod\|f_j\|_{-1}\|g_i\|_{-1/2}\|h_i\|_{-1/2}, \tag{46}$$

where l and t are integers and each test function has support in some unit square obtained by breaking $[-l/2, l/2]\times[-t/2, t/2]$ into unit squares k_α functions in square α. One expects (46) to hold with $C_2 = \alpha_\infty$, the Y_2 pressure and this is our goal in this section. We prove this not so much for its own sake but because when put into our machine for proving Fröhlich bounds, it yields bounds on $\limsup_{l\to\infty}\limsup_{t\to\infty} S_{l,t}$ suitable for the Osterwalder–Schrader reconstruction theorem. The knowledgable reader will notice that our philosophy in proving the new bound is closely connected to that in [29] and that related ideas appear in the recent paper of Glimm, Jaffe, and Spencer [24].

Theorem 9.1. *In Y_2, (46) holds with $C_2 = \alpha_\infty$.*

Remark. The new C_1 is related to the old C_1, C_2 (call them $\tilde{C}_i$) by $C_1 = \tilde{C}_1\exp(\tilde{C}_2 - \alpha_\infty)$.

Proof. We suppose first that l is an even integer and that the functions are symmetric about $x = 0$ (i.e., for every function supported in $x < 0$, its reflection occurs as another function). The improved bound in this situation easily leads to the improved bound in the general case by applying the Schwartz inequality (OS positivity) in the spatial direction.

Let $B_1,\dots,B_t$ denote the interaction and any trial functions in each strip $-(t/2)\leqslant s\leqslant -(t/2)+1,\dots,(t/2)-1\leqslant s\leqslant (t/2)$. Let $\tilde{B}_i^{(2m)}$ denote the object with interaction region $(-l/2, l/2)\times(-m, m)$, with copies of B_i and its reflection alternated. Then, we claim that:

$$\langle B_1\cdots B_t\rangle \leqslant \langle B_1^{(2)}\rangle^{1/2}\langle B_t^{(2)}\rangle^{1/2}\prod_{i=2}^{t-1}\lim_{n\to\infty}\langle B_i^{(2^n)}\rangle^{1/2^n}. \tag{47}$$

To prove (47), we define an operator O_i between Jost states by $(\psi, O_i\eta) = \langle\psi_i^- B_i\eta_i^+\rangle$ where ψ_i^- means putting the Jost state in the time interval $(-\infty, -(t/2)+i-1)$ and η_i^+ at time interval $((t/2)+i, \infty)$ and $\langle\cdots\rangle$ denotes a suitable Schwinger function. Exploiting the usual (OS) iteration and (46), O_i is a bounded operator, and since B_i is symmetric in the spatial direction, Ω_0 couples to the vacuum for $-\ln | O_i |$ by the method of Section 5. Thus, for any Fock space vectors:

$$(\psi, O_i\eta) \leqslant \| \psi \| \| \eta \| \limsup_{n\to\infty} \langle \tilde{B}_i^{(2^n)} \rangle^{1/2^n}.$$

In particular:

$$\langle B_1 \cdots B_t \rangle \leqslant \langle \widetilde{B_1 \cdots B_{t/2}^{(2)}} \rangle^{1/2} \langle \widetilde{B_{(t/2)+2} \cdots B_t^{(2)}} \rangle^{1/2} \limsup_{n\to\infty} \langle \tilde{B}_{(t/2)+1}^{(2^n)} \rangle^{1/2^n}$$

$$\leqslant \langle \widetilde{B_1 \cdots B_{(t/2)-1}^{(2)}} \rangle^{1/2} \langle \widetilde{B_{(t/2)+2} \cdots B_t^{(2)}} \rangle^{1/2}$$

$$\times \lim_{n\to\infty} \langle \tilde{B}_{t/2}^{(2^n)} \rangle^{1/2^n} \lim_{n\to\infty} \langle \tilde{B}_{(t/2)+1}^{(2^n)} \rangle^{1/2^n},$$

where in the second step we deal with the operator defined by $B_{t/2}^{(2)}$. Iterating this argument, we obtain (47). Since

$$\langle \tilde{B}_i^{(2)} \rangle^{1/2} = \langle \Omega_0, O_i \Omega_0 \rangle \leqslant \lim_{n\to\infty} \langle \tilde{B}_i^{(2^n)} \rangle^{1/2} \tag{48}$$

for a suitable operator O_i, we obtain

$$\langle B_1 \cdots B_t \rangle \leqslant \prod_{i=1}^{t} \lim_{n\to\infty} \langle \tilde{B}_i^{(2^n)} \rangle^{1/2^n}. \tag{49}$$

Repeating this argument in the space direction, we find

$$(ZS)_{l,t} \leqslant \prod_{i=1}^{k} \prod_{j=1}^{l} A_{ij},$$

where A_{ij} is a lim sup as $n, m \to \infty$ of a square, which is $2^n \times 2^m$ copies of the stuff in box (i, j) and its reflections raised to the $2^{-n}2^{-m}$ power. If there is no test function in box i, j, then $A_{ij} = \exp(\alpha_\infty)$. If there is stuff in box i, j, then by (46) $A_{ij} \leqslant (k_{ij}!)^{1/2}$ ($\prod$ test function norms) $C_1^{k_{ij}+m_{ij}+n_{ij}} \exp(C_2)$, so that the improved (46) follows. ∎

THEOREM 9.2. *Let $f_1, \ldots, f_k$, $g_1, \ldots, g_m$, $h_1, \ldots, h_m$ be functions supported in any unit squares with centers at integral points. Then*

$$\limsup_{l\to\infty} \limsup_{t\to\infty} | S_{l,t}(f_i, g_j, h_j) |)$$

$$\leqslant C_1^{k+2m} \left(\prod_\alpha k_\alpha!\right)^{1/2} \prod \| f_i \|_{-1} \| g_j \|_{-1/2} \| h_j \|_{-1/2}. \tag{50}$$

Proof. Let N denote the right side of (50). Suppose that the f's, g's, and h's are inside a $(-l_0/2, l_0/2) \times (-t_0/2, t_0/2)$. By following the proof of Theorem 8.1, but using the improved bounds of Theorem 9.2, we find:

$$(Z_{l,t} S_{l,t}) \leqslant Z_{l,t-t_0} \exp(-t_0 E_{l-l_0}) \exp(\alpha_\infty l_0 t_0)\, N,$$

so that

$$\limsup_{t\to\infty} |\ _{l,t}| \leqslant N \exp(+t_0[-E_{l-l_0} + E_l + \alpha_\infty l_0]),$$

which yields (50) since

$$-E_l + E_{l-l_0} \to l_0\alpha_\infty . \quad \blacksquare$$

THEOREM 9.3. *Fix a $Y_2 + P(\phi)_2$ theory. There exists a Schwartz space norm, $\|\cdot\|$, so that for $f_i, h_j, g_j \in C_0^\infty(R^2)$:*

$$\limsup_{l\to\infty} \limsup_{t\to\infty} | S_{l,t}(f_i, g_j, h_j)| \leqslant (k!)^{1/2} \prod_{i,j} \|f_i\| \, \|g_j\| \, \|h_j\|. \tag{51}$$

Remark. The point of (51) is that it is a suitable input for the Osterwalder–Schrader reconstruction theorem [41].

Proof. Since $\prod_\alpha (k_\alpha)! \leqslant k!$, (51) follows from (50) if $\|f\| = \sum_\alpha \|f_\alpha\|_{-1/2}$ with f_α the restriction of f to square α. $\blacksquare$

We make a few remarks about extending these ideas to ϕ_3^4. The volume dependent bounds as essentially proven by Feldman [6] take the form

$$| Z_{l_1 l_2 t} S_{l_1 l_2 t}(f_1, ..., f_n)| \leqslant C_1^n \left(\prod_\alpha n_\alpha!\right)\left(\prod_{i=1}^n \|f_i\|_\infty\right) F(l_1, l_2, t), \tag{52}$$

where $F(l_1 l_2 t) = \exp(Cl_1 l_2 t)$. One might hope that this extends to the case $F(l_1, l_2, t) = \exp \alpha_\infty(l_1 l_2 t)$ but this is not true. Equation (47) still holds but (48) is no longer true. Thus, boundary squares require different treatment. One finds thus that:

THEOREM 9.4. *In ϕ_3^4, (52) holds where*

$$\begin{aligned} F(l_1 l_2 t) = \exp(&\alpha_\infty(l_1 - 2)(l_2 - 2)(t - 2) + 2A(1)[(l_1 - 2)(l_2 - 2) \\ &+ (l_1 - 2)(t - 2) + (l_2 - 2)(t - 2)] - 4E(1, 1)[l_1 + l_2 + t - 6] \\ &+ 8 \ln Z(1, 1, 1)). \end{aligned}$$

However, in applying the improved bound to get a Fröhlich bound, we deal with a big $(2^n l_1^{(0)}) \times (2^m l_2^{(0)}) \times (2^k t_{(0)})$ square so boundary terms do not matter, and we find:

THEOREM 9.5. *In ϕ_3^4 for a suitable Schwartz space norm:*

$$\limsup_{l_1\to\infty} \limsup_{l_2\to\infty} \limsup_{t\to\infty} | S_{l_1 l_2 t}(f_1, ..., f_n)| \leqslant (n!) \prod_{i=1}^n \|f_i\|.$$

10. A Remark on Half-Dirichlet Energies in $P(\phi)_2$

In proving $:\phi^j:$ bounds [23] and related bounds [24] in the infinite volume limit, an estimate of the form $-E_{l-1} \leqslant -E_l + C$ would be useful, where $-E_l$ is the half-Dirichlet energy as defined in [30, 31, 48]. By a clever argument [23, 24], one can avoid this estimate, but it is natural to ask if it is true. We will prove such an estimate here. Our proof exploits the fact we have emphasized already in this paper that estimates $-E_{l-1} \leqslant -E_l + C$ follow from convexity of $-E_l$ in l. Actually, $-E_l$ will not be convex in l, but rather $-E_l + f(l)$ will be convex for an explicit function $f(l)$. The term $f(l)$ arises from corrections to Nelson's symmetry, so that the situation for half-Dirichlet energies is somewhat akin to the situation for half-periodic energies in $P(\phi)_2$ [31].

THEOREM 10.1. *Let E_l be the half-Dirichlet energy in $P(\phi)_2$. Then $-E_l = G(l) - f(l)$, where $G(l)$ is convex in l and*

$$f(l) = -(4\pi)^{-1} \int_0^\infty \ln(1 - \exp(-2l\mu(k)))\, dk. \tag{53}$$

Proof. In terms of the machinery of [48, Sects. VII.1 and VII.3], there is an "idealized" vector $\eta_0^{(l)}$ like $\Omega_0^{(l)}$ in $\phi_3{}^4$ so that:

$$Z_{l,t}^{HD} = (\eta_0^{(l)}, \exp(-tH_l)\, \eta_0^{(l)})/(\eta_0^{(l)}, \exp(-tH_{0,l})\, \eta_0^{(l)}),$$

where H_l (resp. $H_{0,l}$) is the interacting (resp. free) HD Hamiltonian. An explicit computation shows that

$$(\eta_0^{(l)}, \exp(-tH_{0,l})\, \eta_0^{(l)}) = \prod_{n=1}^{\infty} (1 - \exp[-2t\mu(k_n^{(l)})])^{-1/2},$$

where $k_n^{(l)} = 2\pi n/l$. Thus, using Nelson's symmetry, we see that

$$-E_l = \lim_{t\to\infty} t^{-1} \ln Z_{t,l}^{HD} = G(l) - f(l),$$

where $G(l) = \lim_{t\to\infty} t^{-1} \ln(\eta_0^{(t)}, \exp(-lH_t)\, \eta_0^{(t)})$ is convex and

$$f(l) = -\tfrac{1}{2} \lim_{t\to\infty} t^{-1} \sum_{n=1}^{\infty} \ln[1 - \exp(-2l\mu(k_n^{(t)}))]$$

is given by (53). ∎

Notice that $f(l)$ has to following properties:

(a) $f(l) > 0$ all l,
(b) $f(l) = O(\exp(-\alpha l))$ at infinity,
(c) $f(l)$ is monotone decreasing,
(d) $f(l)$ is convex.

COROLLARY 10.2. *Let E_l be the half-Dirichlet energy in $P(\phi)_2$. Then for all $l > 1$*

$$-E_{l-1} \leqslant -E_l + E_2 - E_1 + a,$$

where a is a constant independent of P and l (but dependent on the bare mass).

Proof. On account of the convexity of $G(l)$,

$$-E_{l-1} \leqslant -E_l + E_2 - E_1 + a_l,$$
$$a_l = f(l) - f(l-1) + f(1) - f(2).$$

Since f is decreasing $f(l) - f(l-1) < 0$, so $a_l < f(1) - f(2) \equiv a$. ∎

COROLLARY 10.3. *If $-E_l$ is the half-Dirichlet energy in $P(\phi)_2$, then $\lim_{l\to\infty} -E_l - \alpha_\infty l$ exists (but it may be $-\infty$) and the limit is nonpositive.*

Proof. $\lim_{l\to\infty} f(l) = 0$ and by convexity $\lim_{l\to\infty} G(l) - \alpha_\infty l$ exists and is nonpositive. ∎

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Note added in proof.

1. The Y_2 "matched" counterterms are used already by O. McBryan and Y. Park in [56].

2. Y. Park, (Bielefeld preprint) has independently established the bound $\tilde{Z} \neq 0$ in $(\varphi^4)_3$ by a method similar to ours.

3. F. Guerra in [57] has proven the $P(\varphi)_2$ case of Theorems 7.3 and 9.2 essentially by the method we use. His work clearly precedes ours.

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