

where $A(0) = 0$, $B(1) = 0$,

$$A(x)^{(K-1)/K} + B(x)^{(K-1)/K} = C,$$

$$A'(x)B(x) - B'(x)A(x) = 1,$$

$$\ln C = -\frac{K-1}{2K} \ln \left(\frac{K}{K-1} \frac{[\Gamma(K/(K-1))]^2}{\Gamma(2K/(K-1))} \right). \quad (14)$$

(8) Substituting (13) and (14) back into the expression for $W[F]$ and integrating by parts gives

$$W[F_0] = -1 - 2K \ln C,$$

and substituting this result into (10) recovers the formula in (4) for the behavior of A_n when n is large.

We are currently working on extending these methods to higher-dimensional field theories and hope to use them eventually to obtain the radius of convergence of $F_1(\alpha)$.

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⁷A careful treatment of propagators which are not simple exponentials is given in Ref. 6.

⁸The use of Laplace's method here emphasizes that the behavior of perturbation theory in large order reflects the semiclassical content of the theory. A brief discussion of this point is given in Ref. 1.

Phase Transitions in the Quantum Heisenberg Model

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We rigorously prove that in three or more dimensions, the nearest-neighbor, simple-cubic, ferromagnetic, quantum Heisenberg model of spin $S (= \frac{1}{2}, 1, \dots)$ has a phase transition at nonzero temperature.

The quantum Heisenberg model represents one of the simplest models in which ferromagnetic behavior occurs. From a variety of intuitions including high-temperature expansions¹ and spin-wave approximations,² it has been believed for many years that the three-dimensional model has

a first-order phase transition in the magnetic field at sufficiently low temperatures. Despite its obvious physical interest, a rigorous proof of this has not been available—the only previous rigorous results on phase transitions in quantum lattice systems are proofs of the absence of phase

transitions in suitable models in external field,³ the proof of absence of long-range order in two-dimensional isotropic models,⁴ and the proof of a phase transition for highly anisotropic models.⁵ In this Letter we wish to announce and sketch a rigorous proof of the occurrence of spontaneous magnetization at nonzero temperatures in near-neighbor, simple-cubic, ferromagnetic quantum Heisenberg models in three or more dimensions.

To describe the ν -dimensional model precisely, consider a box Λ with sides of length L_1, \dots, L_ν . At each point, α , with integral coordinates in Λ we have an independent quantum spin \vec{s}_α (with three components $s_\alpha^{(i)}$) of some fixed angular momentum S ($=\frac{1}{2}, 1, \dots$). The basic Hamiltonian is given by⁶

$$H_\Lambda = \sum_{\alpha \in \Lambda} \sum_{i=1}^{\nu} (S^2 - \vec{s}_\alpha \cdot \vec{s}_{\alpha+\delta_i}), \quad (1)$$

where δ_i is the unit vector with i th component +1 and we intend periodic boundary conditions, i.e., if α is at a face so that $\alpha+\delta_i \notin \Lambda$, $\alpha+\delta_i$ is interpreted as $\alpha - (L_i - 1)\delta_i$. Each pair of points is counted once in (1). In the box Λ , one forms the Gibbs state

$$\langle A \rangle_\Lambda = \text{Tr}[A \exp(-\beta H_\Lambda)] / \text{Tr} \exp(-\beta H_\Lambda). \quad (2)$$

We will prove directly that for $\beta > \tilde{\beta}_c(S)$ (to be defined below),

$$\lim_{\Lambda \rightarrow \infty} |\Lambda|^{-2} \langle [\sum_{\alpha \in \Lambda} \vec{s}_\alpha]^2 \rangle > 0, \quad (3)$$

which is one notion of phase transition. It is not difficult⁷ to go from (3) to other notions of phase transition such as nondifferentiability of the free energy or discontinuity of magnetization in external field. Our upper bound, $\tilde{\beta}_c(S)$, on the critical β is given by the implicit equation

$$\sqrt{\frac{2}{3}}(S+1) = (2\pi)^{-\nu} \int \coth(\sqrt{\frac{2}{3}}\tilde{\beta}_c E_p S) d^\nu p, \quad (4a)$$

where

$$E_p = \nu - \cos p_1 - \dots - \cos p_\nu, \quad (4b)$$

and $2SE_p$ is the energy of a momentum p spin wave; the integral is over the hypersphere $|p_i| \leq \pi$, $i=1, \dots, \nu$. Since $\coth x$ is monotone decreasing and goes to 1 as $x \rightarrow \infty$, it is not hard to see that (4a) has a finite solution as long as $\nu \geq 3$ (in order that the integral converge at $p=0$) and the parameter S obeys $S+1 > \sqrt{\frac{3}{2}}$ (i.e., for all physical values $S = \frac{1}{2}, 1, \dots$). In Table I, we list for $\nu=3$ the function $\mu(S) \equiv \tilde{\beta}_c(S)S(S+1)$ obtained by a numerical analysis.⁸ It is easy to deduce from

TABLE I. The second column lists our rigorous upper bounds on inverse transition temperatures in three dimensions. The third column lists $\beta_c(S)S(S+1)$ according to Ref. 1.

S	$\mu(S) \equiv \tilde{\beta}_c(S)S(S+1)$	Series expansion value (Ref. 1)
1/2	1.354	0.8880 ± 0.0009
1	0.965	0.7660 ± 0.0020
3/2	0.872	0.7308 ± 0.0013
2	0.832	
5/2	0.810	0.7084 ± 0.0006
3	0.797	
7/2	0.788	
4	0.783	
9/2	0.778	
5	0.775	
∞	0.758	0.6916 ± 0.0002

the convexity of $\coth x$ that $\nu\tilde{\beta}_c(S)$ is a decreasing function of ν , as it should be.

The variable μ is convenient since $\mu(S)$ has a finite limit $\mu(\infty) = \frac{3}{2}(2\pi)^{-\nu} \int E_p^{-1} dp$. This is reasonable on the basis of limit theorems⁹ which relate the limit $S \rightarrow \infty$ to the classical Heisenberg model, i.e., a model with Hamiltonian (1) but with \vec{s}_α a vector on a sphere of radius $S=1$ rather than an operator. It has recently been proven by Fröhlich, Simon, and Spencer¹⁰ (henceforth FSS) that this classical lattice system has a phase transition—their upper bound on the critical β is just $\mu(\infty)$ which is also the transition temperature for the spherical model. The starting point of our proof is the basic strategy of FSS; namely in the classical case, one proves the following two things.

(A) An upper bound on the Fourier transform of $\langle \vec{s}_\alpha \cdot \vec{s}_\beta \rangle$ of the following form: Let

$$\hat{s}_p^{(j)} = |\Lambda|^{-1/2} \sum_{\alpha \in \Lambda} s_\alpha^{(j)} e^{ip \cdot \alpha}$$

for p obeying $p_i L_i = 2\pi n_i$ ($n_i = -\frac{1}{2}L_i + 1, \dots, \frac{1}{2}L_i$ if L_i is even). Then, in the classical case, FSS show that

$$\sum_{j=1}^3 \langle \hat{s}_p^{(j)} \hat{s}_{-p}^{(j)} \rangle \leq \frac{3}{2\beta E_p}. \quad (5)$$

(B) A trivial equality (Parseval),

$$|\Lambda|^{-1} \sum_{p,j} \langle \hat{s}_p^{(j)} \hat{s}_{-p}^{(j)} \rangle = \langle \vec{s}_0 \cdot \vec{s}_0 \rangle = S^2. \quad (6)$$

Equations (5) and (6) together imply that for $S^2\beta > \mu(\infty)$,

$$\lim_{|\Lambda| \rightarrow \infty} |\Lambda|^{-1} \langle \hat{s}_0^2 \rangle > 0.$$

[Note that $\langle \hat{s}_0^2 \rangle = |\Lambda|^{-1} \langle [\sum_{\alpha \in \Lambda} s_\alpha]^2 \rangle$.]

In carrying this strategy to the quantum case, the first problem that arises is that (5) can no longer hold, for (5) would imply that with Λ fixed and $\beta \rightarrow \infty$, $\langle \vec{s}_\alpha \cdot \vec{s}_\gamma \rangle$ would approach a constant. But it does not; it approaches S^2 if $\alpha \neq \gamma$ and $S(S+1)$ if $\alpha = \gamma$. The key to solving this problem is to deal with a different, but natural, two-point function. Define the *Duhamel two-point function* of A and B by

$$(A, B) \equiv [\text{Tr}(e^{-\beta H})]^{-1} \left[\int_0^1 dt \text{Tr}(A e^{-t\beta H} B e^{-(1-t)\beta H}) \right] \\ = \frac{\partial^2}{\partial \mu \partial \lambda} \text{Tr}(e^{-\beta H + \mu A + \lambda B}) \Big|_{\mu=\lambda=0} [\text{Tr}(e^{-\beta H})]^{-1}.$$

This natural object written in a different form is the inner product used in Bogoliubov's inequality.¹¹ It has been discussed recently by several authors in this form.^{12,13}

The analog to step (A) in our quantum proof⁷ is to show that, if L_1, \dots, L_ν are even,

$$\sum_{j=1}^3 \langle \hat{s}_p^{(j)}, \hat{s}_{-p}^{(j)} \rangle \leq \frac{3}{2\beta E_p}. \quad (5')$$

The proof of (5') is related to the FSS proof of (5) with two important changes: (a) We must avoid use of the usual kind of transfer matrix which exists in the classical case but not the quantum case. (b) We must extend a certain integral inequality to the noncommutative case.

Having overcome the first difficulty and established (5') we are faced with another; the analog of (6) is

$$|\Lambda|^{-1} \sum_{p,j} \langle \hat{s}_p^{(j)} \hat{s}_{-p}^{(j)} \rangle = S(S+1). \quad (6')$$

(5') involves the Duhamel two-point function while (6') involves the ordinary thermal two-point function. The usual inequality¹¹

$$(A^*, A) \leq \frac{1}{2} [\langle A^* A \rangle + \langle A A^* \rangle], \quad (7)$$

goes in the wrong direction. The second step in our proof⁷ is a generalization¹⁴ of an inequality of Roepstorff.¹³ Let $f(x)$ be the convex function for $x \geq 0$ given by

$$f(x \tanh x) = x^{-1} \tanh x.$$

Then

$$(A^*, A) \geq \frac{1}{2} [\langle A^* A \rangle + \langle A A^* \rangle] f(c), \quad (8a)$$

where

$$c = \beta \langle [A^*, [H, A]] \rangle / 2 [\langle A^* A \rangle + \langle A A^* \rangle]. \quad (8b)$$

(8) should be applicable to more than just our problem. It is the necessary ingredient needed to obtain a bound on the thermal two-point function from (5'). The proof of (8) relies on the fact that f is convex and that the integrand in the definition of the Duhamel two-point function (A^*, A) is the Laplace transform of a positive measure.

Using (8) and (5'), properties of f , and the elementary bound

$$\beta \sum_{j=1}^3 \langle [\vec{s}_p^{(j)}, [H, \vec{s}_{-p}^{(j)}]] \rangle = 4\beta E_p (|\Lambda| \nu)^{-1} \sum_{\alpha \in \Lambda} \sum_{i=1}^{\nu} \langle \vec{s}_\alpha \cdot \vec{s}_{\alpha + \delta_i} \rangle \leq 4\beta E_p S^2, \quad (9)$$

one has for $p \neq 0$ that

$$\langle \hat{s}_{-p} \cdot \hat{s}_p \rangle \leq S \sqrt{\frac{3}{2}} \coth x_p, \quad (5'')$$

$$x_p = \sqrt{\frac{2}{3}} \beta S E_p. \quad (10)$$

Equation (5'') is the quantum analog of (5). (6') and (5'') imply

$$|\Lambda|^{-1} \sum_{p \neq 0} \langle \hat{s}_p \cdot \hat{s}_{-p} \rangle \leq \sqrt{\frac{3}{2}} S |\Lambda|^{-1} \sum_{p \neq 0} \coth x_p. \quad (6'')$$

As $|\Lambda| \rightarrow \infty$, (4) is obtained.

We close with two related remarks. It seems to us that in the second part, (B), of our proof we have neither given away any factors nor have we had to make any restrictions on the model. However, in the first part, (A), our method of proof has restricted us to nearest-neighbor coupling and a simple cubic lattice. The factor of $\frac{3}{2}$ can be interpreted as coming from $\frac{1}{2}kT$ per component summed over three components. Since

the components are related, there are really only two degrees of freedom, so $\frac{3}{2}$ might become $\frac{2}{2} = 1$ at low temperatures.

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¹⁴Equation (8) with $f(c)$ replaced by $c^{-1}(1 - e^{-c})$ and $A = A^*$ appears in Ref. 13. Since one can show that $f(c) \geq c^{-1}(1 - e^{-c})$, (8) improves the result of Ref. 13 in two ways. Where we use the fact that the integrand in the definition of (A^*, A) is the Laplace transform of a positive measure, Roepstorff uses the weaker fact that the integrand is log convex.

Threshold Production of Positive Pions and the Form of the πNN Vertex*

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Recent measurements of (p, π^+) reactions near threshold allow us to restrict substantially the possible forms of the empirical nonrelativistic πNN vertex.

The theoretical description of (p, π^+) reactions is presently dominated by two disparate viewpoints: The first¹ supposes that the incident proton shakes off a pion and the resulting neutron is captured into a single-particle orbital of the residual nucleus, a process which is analogous to (d, p) stripping and is therefore called pionic stripping. The average influence of the target nucleons is taken into account through distortion of the incident and outgoing waves. The second school² supposes that at the large momentum transfers of pionic stripping, single-nucleon wave functions have too little strength, so that there is a substantial gain to be made through sharing the momentum equally among several nucleons (incoherent multinucleon model). In a recent review I have discussed in detail my reasons for preferring the pionic stripping to the incoherent multinucleon model.³ Assuming the former to be correct, there nevertheless remains a nontrivial problem of distinguishing the effect of coherent rescattering (distortion) from those of the nuclear structure as represented by the single-particle wave function. In trying to disentangle these effects it is essential to know the form of the πNN vertex amplitude. Unfortunately, as several authors have shown, the nonrelativistic reduction of the πNN coupling is made

ambiguous by our ignorance of the "small" components of (four-component) nucleon wave functions in the usual nuclear-structure theories.⁴

The object of this note is to explore to what extent the *phenomenological* form of the πNN vertex is fixed by our present experimental knowledge. The measurements at 185 MeV⁵ of $^{40}\text{Ca}(p, \pi^+)^{41}\text{Ca}(\text{g.s.})$ and $^{10}\text{B}(p, \pi^+)^{11}\text{B}(\text{g.s.})$, when combined with more recent results at 154-MeV bombarding energy,⁶ tend to distinguish relatively clearly between the two most popular phenomenological nonrelativistic πNN effective Hamiltonians. To see how this comes about, we note first that the amplitude for pionic stripping, in the distorted-wave Born approximation, has the form

$$\int d^3x \varphi_{\vec{k}, (-)^*}(\vec{x}) \langle f | \bar{\psi}_n(\vec{x}) O \psi_p(\vec{x}) | \bar{\Psi}_{i, p}^{- (+)} \rangle, \quad (1)$$

where

$$|\bar{\Psi}_{i, p}^{- (+)}\rangle = [1 + (E + i\eta - H)^{-1}(H - V_{\text{coul}} - E)] \times a^\dagger(\vec{p}) |i\rangle \quad (2)$$

is the solution of the Schrödinger equation with asymptotic Coulomb-wave behavior; $a^\dagger(\vec{p})$ is a creation operator for the incident proton, with barycentric momentum \vec{p} ; $\varphi_{\vec{k}, (-)^*}(\vec{x})$ is the scattering wave function of the outgoing pion, which has