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Coupling Constant Analyticity for the Anharmonic Oscillator*

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WITH AN APPENDIX BY

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We study the analytic properties of the singular perturbation theory for $p^2 + x^2 + \beta x^4$. We prove rigorously several of the properties of the energy levels, previously found by C. Bender and T. T. Wu using methods of unknown validity. In particular: (a) $E_n(\beta)$ has a "global" third-order branch point at $\beta = 0$, i.e., any path of continuation which winds three times around $\beta = 0$ and circles clockwise about all branch points, returns E_n to where it started from and a path that makes one turn around does not. (b) On the three-sheeted surface, $\beta = 0$ is not an isolated singularity; thus, there are infinitely many singularities. (c) The singularities have $\pm 270°$ as asymptotic phase. We also show that the perturbation series is asymptotic uniformly in any sector $|\arg \beta| < \theta$ with $\theta < 3\pi/2$. We extend these results to many dimensional oscillators and x^{2m} perturbations.

Finally, we study the Padé approximants formed from the Rayleigh–Schrödinger series. Since $E_n(\beta)$ has no singularities in the cut plane $|\arg \beta| < \pi$ (using results of references [37, 38]), the Padé approximants $f^{[N,N+j]}$ converge as $N \to \infty$ for j fixed, and yield E_n. A numerical analysis of the Padé approximants, studying the convergence to $E_n(\beta)$ is presented.

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I. Introduction

It is our purpose here to study the energy levels of the Hamiltonian $p^2+x^2+\beta x^4$. For β real and positive the Hamiltonian is well-defined and the energy levels are functions analytic in β in a neighborhood of the positive real axis. We are interested in the study of the analytic continuation of these functions and in particular, in the nature of the singularity at $\beta = 0$ which the folklore (and common sense) tells us exists. This is an interesting question for at least three reasons.

The first involves the general mathematical theory of perturbations in Hilbert space. There is a well-developed theory of regular perturbations due to Kato and Rellich (see, e.g., [1]). This theory which guarantees convergence of the Rayleigh–Schrödinger series for small coupling depends in application on $H(\beta)$ possessing either an invariant operator domain, $D[H(\beta)] = D[H(0)]$ where $D(H) = \{\psi \mid \| H\psi \| < \infty\}$, on which $H(\beta)$ is closed or an invariant from domain, $Q[H(\beta)] = Q[H(0)]$, where $Q(H) = \{\psi \mid \langle\psi, H\psi\rangle < \infty\}$, on which $H(\beta)$ is a closed form. For $H(\beta) = p^2 + x^2 + \beta x^4$, one finds $D[H(\beta)] = D(p^2) \cap D(x^4)$ if $\beta > 0$ and $D[H(0)] = D(p^2) \cap D(x^2)$, and similarly if all D's are changed to Q. Thus, x^4 is a singular perturbation in the sense that $D(H)$ and $Q(H)$ are radically changed as soon as the perturbation is turned on. We are thus studying here the most tractable of singular perturbations; it is a member of what appears to be a very distinguished class of singular perturbations; namely, those for which both the unperturbed Hamiltonian and the perturbation are positive.

We remark that since singular (in fact more singular) perturbations abound in physics, a real understanding of singular perturbations would be useful. For example, the nuclear corrections to the Born–Oppenheimer approximation in solid or molecular physics are singular perturbations and there may be physical phenomena hidden therein. And, of course, in quantum field theory, the interaction Hamiltonian is an ultrasingular perturbation. This brings us to the final pair of reasons for interest in this problem. Both come from the analogy of the problem to a $:\pi^2: + :\nabla\phi^2: + m^2:\phi^2: + \beta:\phi^4:$ field theory (we may view $p^2 + x^2 + \beta x^4$ as a zero-space dimensional theory or as a theory in a box with a stringent ultraviolet cutoff) and both have been discussed by Wightman [2].[1]

The perturbation series for both our anharmonic oscillator and the ϕ^4 theory [3] are known to diverge. However, divergent series are susceptable to various sumability techniques and these techniques are not unrelated to the analytic structure of the functions to which we would like to sum.

The final reason we are interested in this analytic structure involves the analogy between field theories and statistical mechanics in finite volume. Typically, correlation functions depend on various real quantities such as temperature, activity, or density and typically in finite volume theories, one can prove analyticity in a neighborhood of the real axis. One expects there to be complex singularities that pinch the real axis as $V \to \infty$ and produce phase transitions. In field theory, one expects phase transitions when new bound states appear and so this mechanism in the coupling constant is a candidate for their production.[2]

This problem has already been considered by C. Bender and T. T. Wu first in a brief note [4] and then in a lengthy paper [5]. The work of Bender and Wu is divided more or less into three parts:

1. A series of exact statements with proofs presented. The arguments in these proofs are not very careful; any technicalities are ignored. For example, they state $E_n(\beta)$ has a singularity if and only if $\int_C \Phi[z, E_n(\beta), \beta]^2\, dz = 0$, where Φ is the wave function and C a contour. They base their proof on the fact that the integral above is the denominator of an expression for $\partial E/\partial\beta$. They do not, however, consider the possibility that the numerator might vanish at the same time. (We

[1] This article, which made a preliminary announcement of some of the results reported here, complements our discussion in much of the additional material it presents. It makes a preliminary study of the vacuum expectation values $\langle 0; \beta \mid e^{iH(\beta)t_1} x e^{iH(\beta)(t_2-t_1)} x \ldots \mid 0; \beta\rangle$ and discusses one of the Bender–Wu results we do not recover; namely, that one can get from any even (odd) parity level to any other by analytic continuation in coupling.

[2] We will have nothing more to say about this possibility except to note that the summability method theorems are strongest when there are no singularities in a cut plane, so our desires viz. a viz. these last two reasons are in conflict.

prove this can't happen in Section II.8). One result they discuss is that $E_n(\beta)$ has a natural domain on a three sheeted surface.

2. A W. K. B. analysis of the singularities. Their main results are:

(i) On the three-sheeted β surface, $\beta = 0$ is a limit point of singularities.
(ii) All the singularities ($\beta \neq 0$) are square root branch points.
(iii) The asymptotic phase of the singularities $\arg \beta = 270°$. It is our goal in Section II to provide rigorous proofs of (i) and (iii). Modulo certain technical difficulties (e.g., we are unable to prove $E_n(\beta)$ doesn't develop natural boundaries[3]) we do provide these proofs. For two reasons, we feel a rigorous discussion is in order. First, the accuracy (or even validity) of the W.K.B method in the complex plane to a ground state problem is unknown. Second, their general procedure is of dubious validity as a general statement and probably is not an approximation to a correct theorem. This prcedure can be described as follows: One has a function $E(\beta)$ and knows E has singularities when $G(\beta, E(\beta)) = 0$, where G is a function of two variables. One finds approximations $\tilde{E}$ and $\tilde{G}$ for E and G and supposes the solutions of $\tilde{G}[\beta, \tilde{E}(\beta)] = 0$ approximate the singularities of E. Even if one had control over say $\sup_\beta | E(\beta) - \tilde{E}(\beta)|$ and $\sup_{z,w} | G(z, w) - \tilde{G}(z, w)|$, this assumption would be slightly shaky. Nevertheless, since we have been able to derive most of their results, we place credence in the ones we are unable to prove. In some sense, we view the W.K.B. analysis in a light similar to the way an axiomatic field theorist might view the Feynman series: as a reliable guide to what one should seek to prove and understand.

3. An analysis, both exact and numerical, of the perturbation coefficients. By following Jaffe's $:\phi^4:$ divergence proof [3], they are able to show the n-th term a_n, obeys $CD^n\Gamma(\frac{1}{2}n) < | a_n | < EF^n\Gamma(\frac{5}{2}n)$. The lower bound proves that the Rayleigh–Schrödinger series diverges for all coupling strengths (and provides the first rigorous proof of this fact to my knowledge). A direct use of the Rayleigh–Schrödinger formula (Bender and Wu use a Feynman series) will allow us to prove $| a_n | < P^nQn^n$ (Appendix V) which implies the crucial result $\sum a_n^{-1/2n+1} = \infty$, not obtainable from the $\Gamma(\frac{5}{2}n)$ bound. Bender and Wu also use a recursive formula for a_n to compute the first 75 a's to 16 place accuracy. They then perform an impressive numerical analysis to guess a_n has asymptotic form

$$a_n \sim \pi^{-3/2} \sqrt{6}\, 3^n\Gamma(n + \tfrac{1}{2})$$

(in their normalization, which will differ from ours).

Our derivation of the Bender–Wu results is found in Section II. Since much of the discussion is technical, we feel we should provide something of a guide to

[3] However, there are no natural boundaries on the first sheet by results in [38].

that section. (The heart of the argument may be found in Sections II.2, II.6, and II.10.) After establishing various domain properties of assorted operators in II.1, we turn to a crucial scaling argument suggested by Symanzik.[4] While the idea is simple, its ramifications are quite far-reaching. If one considers the scaling transformation $p \to \lambda p$; $x \to \lambda^{-1}x$ on $p^2 + x^2 + \beta x^4$, one obtains $\lambda^2[p^2 + \lambda^{-4}x^2 + \beta\lambda^{-6}x^4]$. Since the scaling is unitarily implementable, these two operators have identical eigenvalues. If we write $E_n(\alpha, \beta)$ for the n-th eigenvalue of $p^2 + \alpha x^2 + \beta x^4$ (α real, $\beta > 0$), we find ($\lambda = \beta^{1/6}$)

$$E_n(1, \beta) = \beta^{1/3} E_n(\beta^{-2/3}, 1).$$

We thus can study the analytic continuation of $E_n(\alpha, 1)$ at $\alpha = \infty$ in place of $E_n(1, \beta)$ at $\beta = 0$. The cube root nature of the singularity is immediately evident.

A. Martin pointed out to us,[5] that the relation $\text{Im } E_n(\alpha, 1) > 0$ if $\text{Im } \alpha > 0$ should play a crucial role. He remarked, for example, that such a function which is not linear cannot be entire so $E_n(\alpha, 1)$ must have some singularity. One can improve this argument to prove (II.6) that $\alpha = \infty$ is not an isolated singularity. One shows that a function with this Herglotz property (i.e., $\text{Im } E/\text{Im } \alpha > 0$) that has an isolated singularity at $\alpha = \infty$, must have only a pole there. Thus if $\alpha = \infty$ is isolated, $E_n(1, \lambda^3)$ has only a pole in λ^3 at $\lambda = 0$. But it known (Appendix II) that perturbation theory is asymptotic in the direction $\lambda^3 > 0$. This can only happen at a pole if the asymptotic series converges. Since perturbation theory diverges (II.4 or Bender–Wu [5]), we have a contradiction to $E_n(\alpha, 1)$ having an isolated singularity at $\alpha = \infty$.

In II.9, we show that the Hamiltonian $p^2 + \gamma x^2$ is an operator with pure point spectrum if γ lies in a plane cut along the negative axis and derive various properties of the Hamiltonians $p^2 + \gamma x^2, p^2 + \gamma x^2 + \beta x^4 (\beta > 0), p^2 + x^2 + \beta x^4 (|\text{Arg } \beta| < \pi)$.

In II.10, we show that as $|\beta| \downarrow 0$, the eigenvalues of $p^2 + \gamma x^2 + |\beta| x^4$ have asymptotic expansions and that the convergence of these expansions is uniform in compact subsets of the cut γ-plane. In particular, for any $\phi < \pi$ and any n, there is a B such that $E_n(\gamma, |\beta|)$ is analytic in $|\arg \gamma| < \phi$, $|\beta| < B$. Scaling arguments then show for any n and any $\theta < 3\pi/2$, there is a B so that $E_n(1, \beta)$ is analytic in $\{\beta \mid |\beta| < B, |\arg \beta| < \theta\}$. Since $E_n(1, \beta) = -E_n(1, e^{3\pi i}\beta)$, the behavior in $|\arg \beta| < 3\pi/2$ determines the behavior for all β. This fact, the analyticity just discussed and the infinite number of singularities prove that the singularities have asymptotic phase $\pm 3\pi/2$ (in the β variable).

In Section II.12, we discuss dispersion relations in coupling constant.

[4] Private communication via A. S. Wightman.

[5] Private communication via A. S. Wightman.

The remaining sections of II (II.3, II.5, II.7, II.8) are of a somewhat technical nature. We mention here only that II.3 provides counterexamples to a well-known folk theorem.

In Section III, we generalize the examples treated with most of the results of II carrying over. In Section III.1 we consider $p^2 + x^2 + \beta x^{2m}$ Hamiltonians, a set of theories treated by W.K.B. methods in a recent preprint of Bender [6]. In Section III.2, we consider multi-dimensional anharmonic oscillators. It is here that the rigorous approach pays dividends for W.K.B. techniques do not have any immediate extension to this case.

In Section IV, we use the analytic theory of Section II, to discuss a particular summability technique, the Padé approximants. The results of this section are the most exciting we present in this paper as they tend to support the conjecture of Bessis and co-workers [7, 8] that the diagonal Padé approximants for the Feynman series of a relativistic field theory converge.

Since the conclusion of the work described in this paper, Loeffel and Martin [37] have proven that none of the Bender–Wu branch points appear on the first sheet in β (which is true in the Bender–Wu approximation) and Loeffel *et al.* [38] have proven the absence of natural boundaries on the first sheet, thereby concluding the proof of analyticity on the first sheet.

II. Eigenvalues and Eigenvectors in one Degree of Freedom

II.1. *Hilbert Space Preliminaries*

As we have explained in the introduction, in studying the analyticity in β of $p^2 + x^2 + \beta x^4$, it is natural to treat the more general Hamiltonian

$$H(\alpha, \beta) \equiv p^2 + \alpha x^2 + \beta x^4.$$

We begin by the consideration of domain questions, etc. for $H(\alpha, \beta)$ when $\beta > 0$ and α is complex.

LEMMA II.1. *The operator $H(0, 1) = p^2 + x^4$ is a self-adjoint operator on the domain,*

$$D = D(p^2) \cap D(x^4),$$

i.e.,

$$D = \left\{\psi \in L^2 \;\middle|\; \int x^8 \mid \psi(x)|^2 \, dx < \infty; \int p^4 \mid \hat{\psi}(p)|^2 \, dp < \infty\right\},$$

where $\hat{\psi}$ is the Fourier transform of ψ. Moreover, H has compact resolvent and for all $\psi \in D$,

$$\| p^2\psi \|^2 + \| x^4\psi \|^2 \leqslant 2\|(p^2 + x^4)\, \psi \|^2 + b_0 \| \psi \|^2, \tag{II.1}$$

where b_0 is a constant.[6]

Proof. By a general theorem of Carleman [9],[7] H is essentially self-adjoint on C_0^∞ (the C^∞ functions of compact support) and thus on D.[8] By a standard closure argument, H will be closed on D if we can prove (II.1) for $\psi \in C_0^\infty$ and (II.1) will extend to all of D. To prove (II.1) we compute:

$$\begin{aligned}(p^2 + x^4)^2 &= p^4 + x^8 + p^2x^4 + x^4p^2 \\ &= p^4 + x^8 + (p, [p, x^4]) + 2px^4p \\ &= p^4 + x^8 - 12x^2 + 2px^4p.\end{aligned}$$

All manipulations are justified when the operators are applied to $\psi \in C_0^\infty$. Pick b_0 so that $\frac{1}{2}x^8 - 12x^2 + \frac{1}{2}b_0 \geqslant 0$ all x. Then:

$$\begin{aligned}(p^2 + x^4)^2 + \tfrac{1}{2}b_0 &= \tfrac{1}{2}(p^4 + x^8) + 2px^4p + \tfrac{1}{2}p^4 + \tfrac{1}{2}x^8 - 12x^2 + \tfrac{1}{2}b_0 \\ &\geqslant \tfrac{1}{2}(p^4 + x^8).\end{aligned}$$

Taking expectation values, we prove (II.1).

Finally, to prove H has a compact resolvent, we need only remark that $x^4 \geqslant x^2 - 1$ for all x, so that the min–max principle and the known spectrum for $p^2 + x^2 - 1$ imply H has discrete spectrum with eigenvalues converging to infinity, i.e., $p^2 + x^4$ has compact resolvent.[9] Q.E.D.

Remark. The double commutator technique used here is due to Jaffe [10].

THEOREM II.1.2. *x^2 is Kato tiny relative to $p^2 + x^4$, i.e., for any $a > 0$, there is b such that*

$$\| x^2\psi \| \leqslant a \| H(0, 1)\, \psi \| + b \| \psi \|. \tag{II.2}$$

Proof. Given a, find E so that

$$x^2 \leqslant \tfrac{1}{2}ax^4 + E, \qquad \text{for all } x.$$

Then

$$\| x^2\psi \| \leqslant \tfrac{1}{2}a \| x^4\psi \| + E \| \psi \| \leqslant a \|(p^2 + x^4)\, \psi \| + (E + b_0)\| \psi \|. \quad \text{Q.E.D.}$$

[6] By varying b_0, 2 may be replaced with any $a > 1$.

[7] For the special case needed here, see [10].

[8] For $p^2 + x^4 \restriction D$ is a symmetric extension of $p^2 + x^4 \restriction C_0^\infty$. Since the latter is essentially self-adjoint, it has the same closure as the former.

[9] For an alternate proof, see Titchmarsh [11], p. 113–114.

COROLLARY II.1.3. *For any complex* α, $H(\alpha, 1) = p^2 + \alpha x^2 + x^4$ *is defined as a closed operator on the domain D of Lemma* II.1.1. *It has the following properties*:

(1) $H(\alpha, 1)^* = H(\bar{\alpha}, 1)$.

(2) $H(\alpha, 1)$ *has compact resolvent.*

(3) *The resolvent is analytic in* α, *i.e., if* $z \notin \operatorname{spec}[H(\alpha_0, 1)]$, *then for all* α *near* α_0, $z \notin \operatorname{spec}[H(\alpha, 1)]$ *and the resolvent* $[H(\alpha, 1) - z]^{-1}$ *is analytic in* α.

(4) *For any* α_0 *and* $C > 1$, *there is a d so that*

$$\| H(0, 1)\, \psi \| \leqslant C \| H(\alpha_0, 1)\, \psi \| + d \| \psi \| \tag{II.3}$$

for all $\psi \in D$.

Proof. (1) is a direct consequence of (II.2) and Lemma II.1.

(2) and (3) also follow from (II.2), which implies that $H(\alpha, 1)$ is a holomorphic family of type (A) in the sense of T. Kato (see [1], p. 375–385). Explicitly, (II.2) implies

$$\| \alpha x^2 [H(0, 1) - z]^{-1} \| < 1,$$

for $| \operatorname{Im} z |$ large. Thus,

$$[H(\alpha, 1) - z]^{-1} = [H(0, 1) - z]^{-1}\{1 + \alpha x^2 [H(0, 1) - z\}^{-1})^{-1},$$

which yields the compactness (as the product of a compact and a bounded operator) and analyticity of the resolvent.

(4) follows from the computation,

$$\begin{aligned} \| H(0, 1)\psi \| &\leqslant \| H(\alpha_0, 1)\psi \| + | \alpha_0 | \, \| x^2 \psi \| \\ &\leqslant \| H(\alpha_0, 1)\psi \| + | \alpha_0 | (b \| \psi \| + a \| H(0, 1)\, \psi \|), \end{aligned}$$

from which follows

$$\| H(0, 1)\psi \| \leqslant (1 - a | \alpha_0 |)^{-1} \| H(\alpha_0, 1)\psi \| + (1 - a | \alpha_0 |)^{-1} | \alpha_0 | \, b \| \psi \|.$$

Q.E.D.

Kato uses (3) to show that the eigenvalues of a holomorphic family such as $H(\alpha, 1)$ are given by analytic functions $E_n(\alpha, 1)$ in the following sense. If $E_n(\alpha_0, 1)$ is a nondegenerate eigenvalue[10] of $H(\alpha_0, 1)$, then there exists a neighborhood of α_0

[10] Nondegenerate means more than $(H - E)\psi = 0$ has only one solution ($\equiv$ geometric multiplicity one). It requires $\int_{|z|=\epsilon} dz (H - E - z)^{-1}$ to be a one-dimensional projection. In case H has compact resolvent, it is equivalent to saying that there is only one solution of $(H - E)^n \psi = 0$ ($\equiv$ algebraic multiplicity one). When H is self-adjoint, however, nondegenerate has the usual meaning of $(H - E)\psi = 0$ having a unique solution (geometric multiplicity $=$ algebraic multiplicity).

and an analytic function $E_n(\alpha, 1)$ defined in that neighborhood so that $E_n(\alpha, 1)$ is a non degenerate eigenvalue for all α in that neighborhood and no other eigenvalue of $H(\alpha, 1)$ is near $E_n(\alpha_0, 1)$. (cf. Kato [1], p. 368–370). If there is an N-fold degeneracy,[11] then there are $\leqslant N$ eigenvalues $E_k(\alpha, 1)$ for α near α_0 which coalesce to $E(\alpha_0, 1)$. The E_k are analytic near α_0 and have at worst an algebraic branch point at α_0. (c.f. Kato [1], 370–371). Since $H(\alpha, 1)$ has compact resolvent, we conclude that all its eigenvalues are analytic functions with at worst algebraic branch points. This last statement is intended *as a local statement*, i.e., we do not rule out the inability to analytically continue the $E_n(\alpha, 1)$, nor the possibility of limit points of algebraic branch points (cf. [1], p. 371–372 for an elementary example of this latter pathology). We return to these questions in Sections II.3 and II.5.

COROLLARY II.1.4. *For any $\beta > 0$, $H(\alpha, \beta) = p^2 + \alpha x^2 + \beta x^4$ obeys Conditions* (1)–(3) *of Corollary* II.1.3. *Moreover $[H(\alpha_0, \beta) - z]^{-1}$ is analytic in β for β in a neighborhood of the positive real line.*

Proof. (1)–(3) follows as in the proof of II.1.3 from (II.1) and (II.2).

$$\| x^4\psi \| \leqslant 3\| H(\alpha_0, 1)\psi \| + d \| \psi \|.$$

Thus, $H(\alpha_0, \beta)$ is a holomorphic family of type (A) for β near 1. Similarly, it is holomorphic near any $\beta > 0$. Q.E.D.

The nondegenerate eigenvalues are thus analytic in α and β near (α_0, β_0) with $\beta_0 > 0$.

Finally, let us examine the number of eigensolutions:

THEOREM II.1.5. *For any $\beta > 0$, α and E, there is at most one (linearly independent) solution of*

$$H(\alpha, \beta)\psi = E\psi.$$

Proof (See also [12], p. 225–231). Let ψ_1, ψ_2 be two solutions. Since ψ_1, $\psi_2 \in D(p^2)$; ψ_1', $\psi_2' \in L^2$ and so $\psi_1'\psi_2 - \psi_1\psi_2' \in L^1$. But $\psi_1'\psi_2 - \psi_1\psi_2'$ is a constant. Thus, it must be zero and so $\psi_1 = (\text{const})\, \psi_2$. Q.E.D.

Remark. The care with domains which we used to conclude $\psi_1' \in L^2$ is essential; e.g., $-\psi'' - x^4\psi = E\psi$ has two L^2 solutions for any E (but $\psi_i' \notin L^2$ in this case). Bender and Wu [5] have an argument analogous to the one given above. In place of domain considerations, they use a demanded falloff at infinity.

[11] N-fold degeneracy means $\dim [\int_{|z|=\epsilon} dz\, (H - E - z)^{-1}] = N$, or alternately, in the compact resolvent case, that $(H - E)^m\psi = 0$ has at most N solutions (for any m).

In particular, there is, thereby, no degeneracy for $\beta > 0$ and α real,[12] a result we could have obtained by a Sturm Liouville nodal analysis. Thus, we can label the eigenvalues by an ordering $E_n(\alpha, \beta)$, $n = 0, 1, \ldots$, for α real, $\beta > 0$. The E_n's are analytic in a neighborhood of the region $\beta > 0$, α real.

II.2 *Scaling Transformations*[13]

As we have already remarked, the first important technique in studying $E_n(1, \beta)$ is to use scaling.

THEOREM II.2.1 (Symanzik). *Let* $\beta > 0$, $\lambda > 0$, α *real. Then*

$$E_n(\alpha, \beta) = \lambda E_n(\alpha\lambda^{-2}, \beta\lambda^{-3}); \tag{II.4}$$

in particular,

$$E_n(1, \beta) = \beta^{1/3} E_n(\beta^{-2/3}, 1). \tag{II.5}$$

Proof. Let $U(\lambda)$ be the unitary operator,

$$[U(\lambda) f](x) = \lambda^{1/4} f(\lambda^{1/2} x),$$

then $U(\lambda)$ leaves $D(x^m)$ and $D(p^m)$ invariant and

$$U(\lambda)\, x^n U(\lambda)^{-1} = \lambda^{n/2} x^n$$

and

$$U(\lambda)\, p^m U(\lambda)^{-1} = \lambda^{-m/2} p^m.$$

Thus, $U(\lambda)^{-1}(p^2 + \alpha x^2 + \beta x^4)\, U(\lambda) = \lambda(p^2 + \alpha\lambda^{-2} x^2 + \beta\lambda^{-3} x^4)$, which proves (II.4). Q.E.D.

Remarks. (1) By an analytic continuation argument (II.4) will hold for all complex α, β, λ if $E_n(\alpha, \beta)$ is defined for complex α and β by analytic continuation. Since E_n is many-sheeted, when dealing with λ complex, some care must be taken.

(2) (II.4) tells us that $E_n(\alpha, \beta)$ is really essentially a function of only one complex variable. As a result holomorphy envelope techniques, which might appear attractive at first glance, are not useful–any result obtainable by these techniques must be obtainable from (II.4).

[12] When α is real, $H(\alpha, \beta)$ is self-adjoint so nondegeneracy is equivalent to uniqueness of solutions of $(H - E)\psi = 0$. If α is complex, geometric multiplicity 1 (uniqueness of $(H - E)\,\psi = 0$) is not equivalent to nondegeneracy, see.[10]

[13] K. Symanzik (private communication via A. S. Wightman) first emphasized to us the importance of the scaling laws; in particular, he pointed out (II-5) to us. It is a pleasure to thank him for adding this essential tool to our arsenal.

(3) There is a proof of (II.5) that does not explicitly use scaling. For α, β physical, the virial theorem tells us that

$$E_n(\alpha, \beta) = \langle n \mid 2\alpha x^2 + 3\beta x^4 \mid n\rangle,$$

and the Feynman–Hellman theorem tells us:

$$\langle n \mid x^2 \mid n\rangle = (\partial/\partial\alpha)\, E_n(\alpha, \beta), \qquad \langle n \mid x^4 \mid n\rangle = (\partial/\partial\beta)\, E_n(\alpha, \beta).$$

Thus, E_n obeys the differential equation

$$E_n = 2\alpha(\partial/\partial\alpha)\, E_n + 3\beta(\partial/\partial\beta)\, E_n\, ,$$

which implies E is of the form $\beta^{1/3} f(\alpha\beta^{-2/3})$. Since the virial theorem is really a consequence of scaling,[14] this argument is closely related to our original proof.

As an immediate application of this scaling argument of Symanzik (and the continuity of $E_n(\alpha, 1)$ at $\alpha = 0$):

COROLLARY II.2.2.

$$\lim_{\alpha \downarrow 0} E_n(\alpha, \beta) = \beta^{1/3} E_n(0, 1).$$

The $\alpha = 0$ limit is of particular interest since it corresponds to the mass 0 limit in field theoretic models. We will see that for $\alpha_0 > 0$, $E_n(\alpha_0\, , \beta)$ as an analytic function of β has an infinite number of branch points near $\beta = 0$. What the corollary tells us is that as $\alpha \downarrow 0$, these singularities move in towards the origin and at $\alpha = 0$ they are *swallowed up*.

Another consequence of these two facts is:

COROLLARY II.2.3. *$E_n(1, \beta)$ has a convergent (strong-coupling) expansion in $\beta^{-2/3}$ convergent for large β.*[15]

The essential use of (II.5) is that it lets us shift the study of the analytic properties of $E(1, \beta)$ at $\beta = 0$ to those of $E(\alpha, 1)$ at $\alpha = \infty$. By the discussion in the previous section, the Hamiltonian $p^2 + \alpha x^2 + x^4$ in this subdominant case, makes sense as a Hilbert space operator for complex α, and, thus, we can hope to use Hilbert space methods to study it.

Armed with Theorem II.2.1, we can understand why Bender and Wu [4, 5] were able to study $E(1, \beta)$ by looking for functions which went to zero asymptotically

[14] The $x(d/dx)$ in the proof of the Virial theorem is the infinitesimal generator of scaling transformations.

[15] Such a strong coupling expansion has been conjectured by Frank [13], who did not guess the 2/3 explicitly. Frank also makes the reasonable conjecture that this converges for all $| \beta^{-2/3} | < \infty$. As we shall see, this latter conjecture is false.

in the region $|\arg(\pm x) + \frac{1}{6}(\arg \beta)| < \pi/6$. To obtain (II.5) we used the scaling transformation $x \to \beta^{-1/6}x$. As a result studying $E(1, \beta)$ with their boundary condition is equivalent to studying $E(\alpha, 1)$ with the boundary condition of vanishing for $|\arg(\pm x)| < \pi/6$. Since $E(\alpha, 1)$ is a Hilbert space eigenvalue, this is the correct condition for its study and so their *rotating sector* condition works.

II.3. *Continuation in the Subdominant Case*

The primary purpose of this subsection is to show that any analytic continuation of $E_n(\alpha, 1)$ is an eigenvalue of $p^2 + \alpha x^2 + x^4$. There is a general folk theorem about such behavior for any sort of Hamiltonian. This folk theorem is based in turn on a second folk theorem to the effect that any place where perturbation theory converges, its value is "physically meaningful."[16] Before discussing the specific problem at hand, we will demonstrate the falsity of the folk theorem in general by presenting simple counter examples:

EXAMPLE 1. $p^2 + \lambda x^4 \equiv H(\lambda)$.
As we have seen, $E_n(\lambda) = c_n \lambda^{1/3}$ for $\lambda > 0$. This function has an analytic continuation to a three-sheeted punctured plane. When we return to $\lambda > 0$ the first time, E_n is not real and, thus, fails to be an eigenvalue. We remark that this example is a little artificial in that $H(\lambda)$ is not a well-defined operator for all λ along the path of analytic continuation.

EXAMPLE 2. $H(\lambda) = p^2 - r^{-1} + \lambda r^{-1}$ (in three dimensions).
For λ real and $\lambda < 1$, the eigenvalues are, of course,[17]

$$E_n(\lambda - 1)^2 = E_n(\lambda^2 - 2\lambda + 1).$$

Thus, the Rayleigh–Schrödinger series is an entire function. For $\lambda > 1$, $H(\lambda)$ has no eigenvalues: thus, the perturbation series converges in places where there is no *direct* physical interpretation for the limit. We remark first that $H(\lambda)$ is a self-adjoint operator on $D(p^2)$ for all λ, so that the criticism of the first example is not relevant. We also note that there is an indirect physical interpretation of the values $E_n(\lambda - 1)^2$ for $\lambda > 1$. They are the energy of *antibound* states, i.e., the position of poles on the unphysical sheet of the scattering amplitude. Since this latter phenomena of antibound states is common, the phenomena of *bogus* convergence of the perturbation series is also going to be quite common.

[16] Such a folk theorem is behind Dyson's celebrated argument [14] for the divergence of perturbation theory for Q.E.D. Our remarks may be construed as a criticism of that argument.

[17] The identification of the exact answer with the perturbation series yields an infinite set of sum rules which we discuss in Appendix III.

This second example makes us expect the phenomena of bogus convergence is associated with continuous spectra and so it is:

THEOREM II.3.1. *Let $H(\alpha)$ be a holomorphic family of operators with compact resolvents. Let $f(\alpha)$ be an analytic function in some connected domain D, which agrees with an eigenvalue of $H(\alpha)$ in a small neighborhood of α_0. Then $f(\alpha)$ is an eigenvalue of $H(\alpha)$ throughout D.*

Proof. Let $S = \{\alpha \in D \mid f(\alpha)$ is an eigenvalue in a neighborhood of $\alpha\}$. Since $\alpha_0 \in S$, S is not empty and by definition S is open. If we can show S is closed, by the connectivity of D, we would have $S = D$. Let $\alpha_1, ..., \alpha_n, ..., \in S$; $\alpha_n \to \tilde{\alpha}$. Pick $z \notin \text{spec}\, H(\tilde{\alpha})$, and let $C(\alpha) = [H(\alpha) - z]^{-1}$ for $\alpha = \tilde{\alpha}, \alpha_N, \alpha_{N+1}, ...,$ where N is chosen so that $C(\alpha_i)$ exists for $i \geqslant N$. Let $g(\alpha) = (f(\alpha) - z)^{-1}$, so that $f(\alpha)$ is an eigenvalue of $H(\alpha)$ if, and only if, $g(\alpha)$ is an eigenvalue of $C(\alpha)$. Since $C(\tilde{\alpha})$ is compact, we can pick a finite rank operator F so that

$$\| C(\tilde{\alpha}) - F \| < \tfrac{1}{2} \sup | g(\alpha_n) |.$$

Let $D(\alpha) = C(\alpha) - F$. Then $[D(\alpha_n) - g(\alpha_n)]^{-1}$ exists for n sufficiently large (by the above condition). Thus $C(\alpha)\psi = g(\alpha)\psi$ ($\alpha = \alpha_N, ...; \tilde{\alpha}$) has a solution if, and only if, $F[g(\alpha) - D(\alpha)]^{-1}\phi = \phi$ has a solution. Since $F[g(\alpha) - D(\alpha)]^{-1}$ is a finite rank operator, this occurs when a certain determinant vanishes. This determinant vanishes for $\alpha = \alpha_N, \alpha_{N+1}, ...,$ and is continuous in α. Thus, it vanishes for $\alpha = \tilde{\alpha}$ and so $f(\tilde{\alpha})$ is an eigenvalue for $H(\tilde{\alpha})$. Since the resolvent is compact, $f(\tilde{\alpha})$ has finite multiplicity and so the eigenvalues of $H(\alpha)$ near $f(\tilde{\alpha})$ for α near $\tilde{\alpha}$ are branches of one (or several) function(s) with at worst algebraic singularities at $\tilde{\alpha}$. Since this function (or one of its branches) agrees with $f(\alpha_n)$ for n large, f is an eigenvalue in an entire neighborhood of $\tilde{\alpha}$. Thus, $\tilde{\alpha} \in S$ and S is closed. Q.E.D.

COROLLARY II.3.2. *Every analytic continuation of $E_n(\alpha, 1)$ is real on the real axis, and on every sheet* $\text{Im}\, E_n > 0$ *for* $\text{Im}\, \alpha > 0$.

Proof. We need only show the last result for eigenvalues. If $(H(\alpha) - E)\psi = 0$, then

$$0 = \text{Im}\langle \psi, [H(\alpha) - E]\psi \rangle = \text{Im}\, \alpha \int dx\, x^2 \mid \psi(x)|^2 - \text{Im}\, E \int \mid \psi(x)|^2\, dx$$

so $\text{Im}\, \alpha = C\, \text{Im}\, E$ with $C > 0$. Q.E.D.

Corollary II.3.2 will play a crucial role in our study of the analytic properties.[18]

[18] A. Martin, J. Loeffel, and H. Epstein (private communication via A. S. Wightman) first emphasized to us the wonderous properties of functions obeying Corollary II.3.2. We are indebted to them (and particularly to Prof. Martin) for this input to our attack.

II.4. *Divergence of Perturbation Theory*

As an application of the techniques we have developed, we show:

THEOREM II.4.1. *$E_n(1, \beta)$ is not analytic near $\beta = 0$.*

Proof. The basic idea behind the proof is the fact that (II.5) indicates E_n can have at best a third-order branch point. We define

$$\begin{aligned} f_n(\lambda) &= \lambda^2 E_n(\lambda^2, 1) \\ &= \lambda^3 E_n(1, \lambda^{-3}) \qquad \text{(by II.5)}. \end{aligned}$$

By general principles, $f_n(\lambda)$ is analytic near the real axis. If $E_n(1, \beta)$ is analytic near $\beta = 0$, then $f_n(\lambda)$ is also analytic near $\lambda = \infty$, and $f_n(\lambda e^{2\pi i/3}) = f_n(\lambda)$ for λ near ∞. Consider continuing f_n analytically along the path shown in Fig. 1, i.e., go out the real axis, C_1, and up along an arc C_2 out sufficiently far to be in the neighborhood of infinity for which $f_n(\lambda e^{2\pi i/3}) = f_n(\lambda)$. Finally, continue in along C_3; this can be accomplished by preserving the relation $f_n(\lambda e^{2\pi i/3}) = f_n(\lambda)$. When we get back to $\lambda = 0$, by our principle of analytic continuation, we must have $f_m(\lambda)$ for some m (m may not be n). Thus, near $\lambda = 0$,

$$f_m(\lambda) = f_n(\lambda e^{2\pi i/3}).$$

Identifying lowest powers:

$$E_m(0, 1) = e^{4\pi i/3} E_n(0, 1).$$

Since E_n and E_m are real and nonzero, this is impossible. Thus, our assumption that $E_n(1, \beta)$ is analytic near $\beta = 0$ is incorrect. Q.E.D.

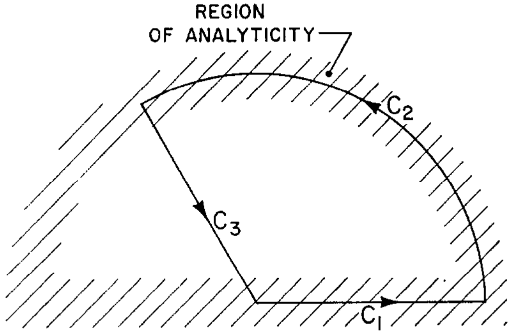

FIG. 1. A path of continuation used in proving Theorem II.4.1.

The fact established by Theorem II.4.1, that $E_n(1, \beta)$ is not analytic at $\beta = 0$, of course, implies that $E_n(1, \beta)$ cannot have a convergent Taylor series, which represents it in any neighborhood of $\beta = 0$. That does not prevent the Taylor coefficients from existing, for example as right-hand derivatives. (It is a standard bit of folklore that the Rayleigh–Schrödinger perturbation theory provides these Taylor coefficients. A proof of this assumption will be provided later). Theorem II.4.1 also does not eliminate the possibility that the formal Taylor series converges but not to the function [$f(z) = \exp(-z^{-1})$ has such a Taylor series]. Such behavior would fit in with the phenomena seen in II.3. That the perturbation series does in fact diverge follows from a considerably longer and more detailed analysis of the coefficients made by Bender and Wu [5].

II.5. *Global Analytic Structure*

In this section, we study several questions involving the function $E_n(\alpha, \beta)$ in the large and at singularities away from $\beta = 0$ (or $\alpha = \infty$). For clarity, we divide this study into smaller subsections.

A. *Nonisolated Singularities of* $E_n(\alpha, 1)$

We may as well begin by admitting that this question represents the most unsatisfactory element in our argument. While we believe that $E_n(\alpha, 1)$ can only have isolated singularities (in the finite α plane), we are not able to prove it at this time. However, Loeffel *et al.* [38] have shown this is true for $|\arg \alpha| < 2\pi/3$.

One can put this question of global analytic structure and isolated singularities into perspective by considering the completed Riemann surface of $E_n(\alpha, 1)$. Let us consider in $\mathbf{C}^2[=\{(\alpha, E)\}]$ the graph, Γ, of $E(\alpha, 1)$, i.e. all pairs (α, E) for which E is an eigenvalue of $p^2 + \alpha x^2 + x^4$. Our discussion in Section II.1 (i.e., the Kato–Rellich theory) can be considered as a local statement, explicitly given a point $(\alpha_0, E_0) \in \Gamma$, there is a product neighborhood, $D \times \Omega$, of (α_0, E_0) in $\mathbf{C}^2$ so that $\Gamma \cap (D \times \Omega)$ is the Riemann surface of a function defined on D with only isolated algebraic singularities. The problem of global structure is the classic analytic continuation question. One has a curve γ: $[0, 1] \to \mathbf{C}$ and a point $(\gamma(0), E_0) \in \Gamma$ and one would like to lift γ up to Γ, i.e., find $\tilde{\gamma}$: $[0, 1] \to \Gamma$, $\tilde{\gamma}(t) = [\gamma_0(t), \gamma_1(t)]$ so that $\gamma_0 = \gamma$ and $\gamma_1(0) = E_0$. In Theorem II.5.1, we will show that the only thing that can prevent us from lifting γ is the lifted curve running to ∞ in the E direction. This follows essentially from the local lifting allowable by our local regularity and the fact that Γ is closed. In the remainder of the section, we eliminate some of the more usual possibilities of infinities (such as poles). We are unable to eliminate some of the more bizarre types (like natural boundaries). We do not expect such pathologies to occur and so we will make the *working assumption* that any curve can be lifted to Γ, i.e., analytic continuation is possible along any path except that one may run into algebraic singularities (which will make the lifting non unique).

We remark that our lifting assumption does not preclude the possibility that the set of α at which level crossing takes place is dense on some line, curve or even the whole plane. For our analysis in II.1 tells us that if crossing takes place at (α_0, E_0), there is no other crossing nearby. But there can be crossing at (α, E) with α near α_0 without anything particularly pathological taking place (if E is not near E_0).

THEOREM II.5.1. *Let γ: $[0, 1] \to \mathbf{C}$ be a curve in the α plane. Let $E_n(\alpha, 1)$ have an analytic continuation along γ into the interval $[0, 1)$. If the values $E_n(\gamma(t), 1)$ have a finite limit point as $t \to 1$, then $E_n(\alpha, 1)$ has an isolated singularity at $\alpha_0 = \gamma(1)$.*

Remark. This statement tells us that if we have a nonisolated singularity, whenever we can approach it in some direction, the values of $E_n(\alpha, 1)$ must approach infinity.

Proof. Let $E_n[\gamma(t_i), 1] \to \lambda_0$ for a sequence $t_i \to 1$. Thus, as in the proof if Theorem II.3.1, λ_0 is an eigenvalue of $p^2 + \alpha_0 x^2 + x^4$. Since this operator has a compact resolvent, the eigenvalues which are near λ_0 for α near α_0 are branches of some analytic function with at worst a branch point at α_0. This completes the proof.

Any nonisolated singularity is either

(a) A limit point of isolated singularities. Since we will see isolated singularities occur only when eigenvalues cross, α_0 can only be a limit point of singularities of one level $E_n(\alpha, 1)$ goes to infinity as $\alpha \to \alpha_0$ crossing infinity many levels along the way.

(b) A limit point of nonisolated singularities is a part of a natural boundary. In this case $E_n(\alpha, 1)$ would blow up along an entire curve or some nonisolated subset of a curve such as a Cantor set on a curve.

Both of these possibilities seem very bizarre and so we cannot really conceive of their occurring. Two general sorts of approach seem possible for a successful proof of their non occurrence.

(i) *Complex analytic approach.* As we will show in Section II.7, the eigenvalues of $p^2 + \alpha x^2 + x^4$ are the implicit solutions of an equation $\psi(\alpha, E) = 0$, where ψ is an entire function of α and E. There is probably a general theorem that functions defined in such a manner cannot have natural boundaries.[19] However, such a theorem does not seem to have been proven in the complex analysis literature (and no explicit counter-example seems to be known).

[19] Complex analysis arguments cannot eliminate limits of branch points; the examples mentioned, in (ii) provide entire $\psi(\alpha, E)$ so that the implicit function $E(\alpha)$ has singularities of type (a); i.e., $\psi(\alpha, E) = E^{-1/2} \sin E^{1/2} + \alpha \cos E^{1/2}$ will exhibit type (a) behavior.

(ii) *Hilbert space approach.* There are very pathelogical examples of holomorphic families of operators with compact resolvent (see Kato [1], p. 371–372), examples which exhibit singularities of type (a) above. Thus, we cannot eliminate nonisolated singularities on the basis of general arguments alone; however, a detailed argument using special properties may be possible.

In any event, *we will ignore the possibility of nonisolated singularities. Technically, all our theorems should be modified to take into account this possibility.*[20] Thus, e.g., our statement that $E_n(1, \beta)$ has a limit of branch points at $\beta = 0$, should be replaced by a statement that $E_n(1, \beta)$ has a limit of branch points or a limit point of natural boundaries (or both).

B. *Isolated Singularities*

THEOREM II.5.2. *$E_n(\alpha, 1)$ has no poles or essential singularities. Algebraic branch points have no negative powers in their Puiseux series.*

Proof. Since $E_n(\alpha, 1)$ is analytic for α real there are no singularities on the real axis. If $E(\alpha, 1)$ had a pole near $\alpha = \alpha_0$ in the upper half-plane, then $E(\alpha, 1)$ would have a negative imaginary part for suitable α near α_0. This contradicts Corollary II.3.2. Similarly, we can eliminate the other singularities.

As with natural boundaries we cannot eliminate logarithmic branch points at this time; as in that case, there is an example in Kato's book [1] which tells us a general Hilbert space argument can't work. There may be an argument extending Theorem A.II.3 to allow us to use complex analysis to eliminate those beasts. *In any event, we suppose logarithmic branch points* (which correspond to infinitely many levels going to infinity at once) *do not occur.* Again, all our theorems should be suitably modified.

Before leaving the subject of branch points, we note that Loeffel and Martin [37] have proven that there are no branch points in the region $|\arg \alpha| < 2\pi/3$ (equivalently, $|\arg \beta| < \pi$).

C. *Global Nature of the $\alpha = \infty$ ($\beta = 0$) Singularity*

Since $\beta = 0$ is not an isolated singularity (as we shall prove) it doesn't make technical sense to say it is a third-order branch point. However, it does have such a branch point in a suitable sense as we shall now see.

LEMMA II.5.4. *Let γ: $[0, 1] \to \mathbf{C}$ be path in the α plane with the following properties*

(a) $\gamma(0) = \gamma(1)$ *is real.*

[20] We will however indicate which major results would be changed if such pathologies occurred.

(b) γ *is symmetric under complex conjugation, i.e.,* $\gamma(t) = \overline{\gamma(1-t)}$.

(c) $E_n(\alpha, 1)$ *can be continued along* γ. *Then* $E_n[\gamma(1), 1] = E_n[\gamma(0), 1]$,

i.e., continuation along γ *brings us back to where we started.*

Remark. 1. (b) essentially tells us that γ circles around complex conjugate branch points in complex conjugate ways.

2. This theorem tells us that if we draw branch cuts between complex conjugate points, we get a single-valued function on each sheet.

Proof. $\gamma(\frac{1}{2})$ is real and $E_n(\alpha, 1)$ is real for α real near $\gamma(\frac{1}{2})$. Thus, $E_n(\gamma(t), 1) = \overline{E_n[\gamma(1-t), 1]}$, by the Schwartz reflection principle. Since $E_n[\gamma(0), 1]$ is real, we are done.

THEOREM II.5.5. *Let* $\gamma[0, 1] \to \mathbf{C}$ *be a path in the* β *plane obeying* (a) *and* (b) *of Lemma* II.5.4. *Moreover, suppose* γ *winds about* $\beta = 0$ *three times and that* $E_n(1, \beta)$ *is continuable along* γ. *Then* $E_n(1, \gamma(0)) = E_n(1, \gamma(1))$. *If we continue along a circular path winding around three times, avoiding branch points in symmetric ways then*

(i) $E_n(1, \beta e^{3\pi i}) = -E_n(1, \beta)$;

(ii) *We never return to* $E_n(1, \beta)$ *after winding only once or twice around* $\beta = 0$.

Proof. Since $E(1, \beta) = \beta^{1/3} E(\beta^{-2/3}, 1)$, the general continuation claim follows from Lemma II.5.4 since γ can be lifted back to the α plane.

(i) follows from the $(\beta^{-1/3})^2$ in the scaling argument. If (ii) were false, we could derive a contradiction by an argument identical to the one we used to show non-analyticity at $\beta = 0$ (Theorem II.4.1.).

Remarks. 1. This theorem is precisely what we meant when we said $E_n(1, \beta)$ had a global third-order branch point.

2. A branch cut statement analogous to Remark 2 of Lemma II.5.4 is true.

3. The symmetry (i), remark 2 above and the general three-sheetedness are all results found by Bender and Wu.

II.6. *The Singularity at* $\alpha = \infty$ ($\beta = 0$)

This section contains the most crucial arguments in our discussion of the one-dimensional problem. Since we have not yet shown that the strong coupling expansion fails to be convergent for all $\beta^{-1/3}$, let us prove

THEOREM II.6.1. (Martin).[21] *$E_0(\alpha, 1)$ is not entire.*

Proof. Since $E_0(\alpha, 1)$ is Herglotz, if it is entire, $E_0(\alpha, 1) = E_0 + \alpha b$ (by Theorem A.I.1). By ordinary perturbative analysis, $b = \langle 0 \mid x^2 \mid 0 \rangle$, where $\mid 0 \rangle$ is the ground state for $p^2 + x^4$. Then

$$\langle 0 \mid p^2 + \alpha x^2 + x^4 \mid 0 \rangle = E_0(\alpha, 1),$$

the ground state energy, so $\mid 0 \rangle$ is also the perturbed ground state. Since this would imply $\mid 0 \rangle$ is an eigenfunction of x^2, this is impossible.

The crux of the argument is

THEOREM II.6.2. *$E_n(\alpha, 1)$ cannot have an isolated singularity at $\alpha = \infty$, i.e., ∞ is a limit point of singularities.*

Proof. Suppose $\alpha = \infty$ is isolated. By Theorem II.5.4, $E_n(\alpha, 1)$ cannot have a branch point at $\alpha = \infty$. By Theorem A.II.2, $E_n(\alpha, 1)$ has an expansion near ∞ of the form,

$$E_n(\alpha, 1) = \sum_{m=-\infty}^{1} a_n \alpha^n,$$

since $E(\alpha, 1)$ is Herglotz. Thus, near $\lambda = 0$, $E_n(1, \lambda^3)$ has an expansion

$$E_n(1, \lambda^3) = \sum_{m=-1}^{\infty} a_{-m} \lambda^{2m+1},$$

i.e., $E_n(1, \lambda^3)$ is meromorphic at $\lambda = 0$. But, by the asymptotic nature of perturbation theory (Kato [15], and Appendix II) $E_n(1, \lambda^3) \sim \sum_{k=0}^{\infty} b_k \lambda^{3k}$ as $\lambda \downarrow 0$. Since E is meromorphic, this can only happen if the asymptotic series is convergent to E_n, in which case $E_n(1, \beta)$ is analytic near $\beta = 0$. Since this contradicts Theorem II.4.1, $\alpha = \infty$ is not isolated.

COROLLARY II.6.3. *$E_n(1, \beta)$ has a global cubic branch point at $\beta = 0$, which is a limit point of algebraic branch points (or logarithmic branch points or natural boundaries).*

This is of course, the major result of Bender and Wu; here rigorously proven.

II.7. *Conditions for Crossing*

We have found that the general qualitative behavior of $E_n(1, \beta)$ on each sheet agrees precisely with that obtained by Bender and Wu with their vaguely suspicious

[21] A. Martin, unpublished. I should like to thank Dr. Martin for permission to include it here.

methods. It is also of great interest to verify their results on how the various sheets fit together,[22] particularly since they have studied only the first few sheets in detail. The obvious starting point for such a study (which we do not bring to fruition here) is to formulate a condition for crossing of levels, since Corollary II.1.3 assures us of analyticity at nondegenerate points. In this section, we will obtain a condition for crossing which is identical to the Bender and Wu condition [4, 5] modulo a scaling transformation (which has the effect of making all domains of integration the real axis). We will show that this condition is both necessary and sufficient for crossing to occur. In the next section, we prove that the eigenvalues are non-analytic at each of these crossing points.[23]

To understand better the significance of the conditions we will derive, let us first consider crossing in a two-dimensional matrix problem. Let

$$T(\lambda) = \begin{pmatrix} 1 & \lambda \\ \lambda & -1 \end{pmatrix},$$

for which the eigenvalues are $\pm\sqrt{1+\lambda^2}$. This family which is Hermitean analytic $[T(\lambda)^\dagger = T(\bar{\lambda})]$ has the typical behavior of branch points at values of λ ($\lambda = \pm i$) where crossing occurs. There are two other features at $\lambda = \pm i$ to note particularly:

(a) While the only eigenvalue of $T(\pm i)$ is 0, $T(\pm i) \neq 0$. In terms of a basis—$\binom{1}{\pm i}$, $\binom{1}{0}$, $T(\pm i)$ has the typical Jordan anomalous[24] form $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, i.e. there is only one eigenvector for T, but two for T^2.

(b) In terms of the inner product,

$$\left\langle \begin{pmatrix} \alpha \\ \beta \end{pmatrix}, \begin{pmatrix} \gamma \\ \delta \end{pmatrix} \right\rangle_R = \alpha\gamma + \beta\delta,$$

which is not positive definite (because of the appearance of α and β, instead of $\bar{\alpha}$ and $\bar{\beta}$), the eigenvectors $\binom{1}{\pm i}$ at crossing have 0 length. This is no coincidence! With respect to the inner product $\langle\ ,\ \rangle_R$, $T(\lambda)$ is symmetric and so its eigenvectors are

[22] On the basis of an examination of the first few sheets Bender and Wu conclude that E_n has infinitely many branch points coupling it to E_{n+1} and E_{n-1}, where E_n is the n th *even* (or odd) parity level. By E_n coupling to E_{n+1}, we mean continuation of E_n by a path circling only one branch point leads us to E_{n+1}. In particular, all the E_n's lie on one *coupling constant trajectory*.

[23] Bender and Wu [5] prove that their condition is necessary for nonanalyticity and that it is sufficient if a certain integral does not vanish. We prove that such vanishing does not take place in Section II.8.

[24] By Jordan anomalous form we mean that there are ones above the diagonal when the matrix is put into Jordan form. Alternately, we mean that the matrix cannot be put into a purely diagonal form. Equivalently the geometric and algebraic multiplicities are unequal (see footnote 9).

$\langle\ ,\ \rangle_R$, orthogonal. As $\lambda \to \pm i$ both eigenvectors must approach the single eigenvector and so the single eigenvector is R-orthogonal to itself.

Now let us return to the Hamiltonian $p^2 + \alpha x^2 + x^4$. We first note

LEMMA II.7.1. *There is a unique (up to a constant factor) solution of the differential equation,*

$$-\psi_\infty''(x, \alpha, E) + (\alpha x^2 + x^4 - E)\,\psi_\infty(x, \alpha E) = 0, \tag{II.6a}$$

satisfying the condition,[25]

$$\psi_\infty \sim \exp(-\tfrac{1}{3}x^3), \tag{II.6b}$$

in the sector $|\arg x| < \pi/6$.

Moreover,

(1) $\psi_\infty(x, \alpha, E)$ *is an entire function of* x, α *and* E;

(2) $\psi_\infty'(x, \alpha, E) \sim$ *(poly in* x*)* $\exp(-\tfrac{1}{3}x^3)$ *in the sector*;

(3) $\partial\psi_\infty/\partial E$ *and* $\partial\psi_\infty/\partial\alpha$ *are in* $L^2(0, \infty)$.

Proof. This result has been proven by Hsieh and Sibuya [16]. Their paper may be consulted for the exact asymptotic series. In Appendix IV, a simple proof of a weakened form of II.7.1 sufficient for our purposes is presented by A. Dicke. J. Loeffel[26] has independently proven a more general result valid for polynomial potentials of even degree.

THEOREM II.7.2. *E is an even (odd) parity eigenvalue of* $p^2 + \alpha x^2 + \beta x^4$ *if, and only if,*

$$\psi_\infty'(0, \alpha, E) = 0 \qquad [\psi_\infty(0, \alpha, E) = 0].$$

Proof. ψ_∞, which behaves nicely at $x \to \infty$, will be square integrable if, and only if, it is nicely behaved as $x \to -\infty$. This can only happen if ψ_∞ is odd or even (and will happen in that case). (II.7) is precisely an expression of the evenness (oddness) of ψ_∞.

Remarks. 1. This proof is equivalent to (and was suggested by) the Wronskian approach used by Titchmarsh [11] who notes ψ_∞ is an eigenvalue essentially if

[25] We do not write $\sim$ to mean $\psi_\infty/\exp(-1/3x^3) \to 1$, but only that $\psi_\infty/\exp[(-1/3 + \epsilon)x^3] \to 0$ and $\psi_\infty/\exp[(-1/3 - \epsilon)x^3] \to \infty$. In fact $\psi_\infty/\exp(-1/3x^3 - 1/2\,\alpha x - \log x) \to 1$.

[26] J. Loeffel, unpublished.

$W(\psi_\infty, \psi_{-\infty}) = 0$ where W is the Wronskian and $\psi_{-\infty}$ is the nice solution at $-\infty$; in our case $\psi_{-\infty}(x, \alpha, E) = \psi_\infty(-x, \alpha, E)$ and

$$W(\psi_\infty, \psi_{-\infty}) = 2\psi_\infty(0, \alpha, E)\, \psi_\infty'(0, \alpha, E).$$

2. Theorem II.7.2 immediately provides an alternate proof that the analytic continuations of $E_n(\alpha)$ are eigenvalues.

For simplicity of notation, let us work with the odd parity case henceforth. The odd parity eigenvalues are defined by the implicit functional relationship $\psi_\infty(0, \alpha, E) = 0$. The implicit function theorem immediately tells us:

LEMMA II.7.3. *If $\psi_\infty(0, \alpha_0, E_0) = 0$, $(\partial\psi_\infty/\partial E)(0, \alpha_0, E_0) \neq 0$, then there is a unique solution of $\psi_\infty(0, \alpha, E) = 0$ with E near E_0 when α is near α_0.*

Thus, we see $(\partial\psi_\infty/\partial E)(0, \alpha_0, E_0) = 0$ for an eigenvalue E_0 of $H(\alpha_0)$ is a necessary condition for crossing to take place. We also conclude, independent of the Kato–Rellich theory of analytic perturbations that the eigenvalues are analytic where no crossing occurs (i.e., we find an independent proof of Corollary II.1.3).

To show $\partial\psi_\infty/\partial E = 0$ is also sufficient for crossing, we find the property analogous to condition (a) that $T(\lambda)$ obeyed; i.e., we examine the Jordan anomalous behavior of $H - E$:

LEMMA II.7.4. *Let E_0 be an odd parity eigenvalue of $H(\alpha_0)$; then*

$$[-(d^2/dx^2) + \alpha_0 x^2 + x^4 - E_0]\, \phi(x) = \psi_\infty(x, \alpha_0, E_0) \tag{II.8}$$

has an L^2 solution ϕ if, and only if, $\partial\psi_\infty/\partial E\, (0, \alpha_0, E_0) = 0$. Moreover, in such a case, every such ϕ is of the form,

$$\phi(x) = (\partial\psi_\infty/\partial E)(x, \alpha_0, E_0) + a\psi_\infty(x, \alpha_0, E_0).$$

Proof. Differentiating (II.6a) with respect to E, we see that $(\partial\psi_\infty/\partial E)(x, \alpha_0, E_0)$ is a solution of (II.8), and this solution is L^2 at $x = +\infty$. If ϕ is an L^2 solution of (II.8), then $\phi - \partial\psi_\infty/\partial E$ is a solution of (II.6) is L^2 at $+\infty$ and thus $\phi = (\partial\psi_\infty/\partial E) + a\psi_\infty$. As a result (II.8) has an L^2 solution if, and only if, $\partial\psi_\infty/\partial E$ is an L^2 solution. $-(\partial\psi_\infty/\partial E)(-x, \alpha_0, E_0)$ is a solution of (II.8). If $(\partial\psi_\infty/\partial E)(0, \alpha_0, E_0) = 0$, then these two solutions are identical so that $\partial\psi_\infty/\partial E$ is an L^2 solution. Conversely, if $\partial\psi_\infty/\partial E$ is an L^2 solution, so is $-(\partial\psi_\infty/\partial E)(-x, \alpha, E)$ so that

$$(\partial\psi_\infty/\partial E)(x, \alpha, E) + (\partial\psi_\infty/\partial E)(-x, \alpha, E) = c\psi_\infty(x, \alpha, E),$$

since the left side is an L^2 solution of (II.6a). Putting $x = 0$, we see $(\partial\psi_\infty/\partial E)(0, \alpha, E) = 0$ in this case.

Thus, we see $\partial\psi_\infty/\partial E = 0$ is equivalent to the occurrence of Jordan anomalous behavior of H. Not surprizingly, this is equivalent to crossing of levels:

THEOREM II.7.5. *There is level crossing of odd parity levels at* α_0, E_0 *if, and only if, one of the following equivalent conditions holds*:

(a) $\psi_\infty(0, \alpha_0, E_0) = 0 = (\partial\psi_\infty/\partial E)(0, \alpha_0, E_0)$

(b) (II.8) *has an* L^2 *solution.*

Proof. Suppose first level crossing takes place. If

$$P = -\frac{1}{2\pi i}\int_c \frac{d\lambda}{H(\alpha_0) - \lambda},$$

where C is a small circle about E_0 [enclosing no other eigenvalues of $H(\alpha_0)$], then elementary arguments show that P is a (not necessarily orthogonal) projection which commutes with $H(\alpha_0)$ and that $\mathrm{spec}[H(\alpha_0) \restriction \mathrm{Ran}\, P] = \{E_0\}$ (see Kato [1]). If we define

$$P(\alpha) = -(2\pi i)^{-1}\int_c [H(\alpha) - \lambda]^{-1}\, d\lambda,$$

$\mathrm{Tr}\, P(\alpha) \geqslant 2$ for α near α_0, since there are two (or more) eigenvalues near E_0 for α near α_0. Thus, $\mathrm{Tr}\, P \geqslant 2$ by continuity. Since $\dim[\mathrm{Ran}\, P] < \infty$, $P[H(\alpha_0) - E_0]^n \equiv 0$, for some n. By theorem II.1.5, $[H(\alpha_0) - E_0]\,\psi = 0$ has a unique solution, so that the smallest such n is at least 2. As a result some ϕ exists for which

$$[H(\alpha_0) - E_0]^2\phi = 0$$

but $[H(\alpha_0) - E_0]\phi \neq 0$ from which it follows that (II.8) has a solution.

Conversely, let (II.8) have a solution. Then P as above has $\mathrm{Tr}\, P \geqslant 2$ so $\mathrm{Tr}\, P(\alpha_0) \geqslant 2$ for α near α_0. Then either there are two distinct levels near E_0 for α near α_0, or there is a single level near E_0, say $E(\alpha)$ which has multiplicity $\geqslant 2$, i.e., $\partial\psi_\infty/\partial E[0, \alpha, E(a)] = 0$. If we analytically continue $E(\alpha)$ to the real axis,[27] we would have Jordan anomalous behavior for a Hermitean operator. Since this cannot occur, we conclude that the degeneracy at α_0 splits, i.e., there is crossing at $\alpha = \alpha_0$.

We can also recover the condition for crossing obtained by Bender and Wu [5].

COROLLARY II.7.6. $\partial\psi_\infty/\partial E(0, \alpha, E) = 0$ *at a point for which* $\psi_\infty(0, \alpha, E) = 0$, *if and only if,*

$$\int_0^\infty \psi_\infty(x, \alpha, E)^2\, dx = \tfrac{1}{2}\int_{-\infty}^\infty \psi_\infty(x, \alpha, E)^2\, dx = 0.$$

[27] We are supposing here natural boundaries do not occur!

Proof. We have

$$\int_0^\infty \psi_\infty(x, \alpha, E)^2\, dx = \int_0^\infty dx\, \psi_\infty(x, \alpha, E)\left(\frac{-d^2}{dx^2} + \alpha x^2 + x^4 - E\right)\frac{\partial \psi_\infty}{\partial E}(x, \alpha, E)$$

$$= \int_0^\infty dx\, \frac{\partial \psi_\infty}{\partial E}\{[H(\alpha) - E]\psi_\infty\}$$

$$- \int_0^\infty dx \left[\left(\frac{\partial \psi_\infty}{\partial E}\right)'' \psi_\infty - \psi_\infty'' \frac{\partial \psi_\infty}{\partial E}\right]$$

$$= \left[\left(\frac{\partial \psi_\infty}{\partial E}\right)'(0, \alpha, E)\right][\psi_\infty(0, \alpha, E)]$$

$$- \left[\frac{\partial \psi_\infty}{\partial E}(0, \alpha, E)\, \psi_\infty'(0, \alpha, E)\right].$$

Under condition that $\psi_\infty(0, \alpha, E) = 0$, (so that $\psi_\infty'(0, \alpha, E) \neq 0$) we have the stated equivalence.

Remarks. 1. This is analogous to property (b) of our example $T(\lambda)$, since $H(\alpha)$ is symmetric with respect to the inner product $\langle f, g\rangle_r = \int_{-\infty}^{\infty} f(x)\, g(x)\, dx$. In fact, this symmetry was explicitly used above.

2. It would be an amusing (but presumably tedious) exercise to use W.K.B. techniques to analyze the $\partial\psi_\infty/\partial E = 0$ condition as opposed to the $\int \psi_\infty{}^2\, dx = 0$ condition as used by Bender and Wu. Since the methods are not identical (we use the differential equation which is not obyed by the W.K.B. solution to prove the conditions are equivalent for the exact solution), their comparison might give one an idea of the errors involved.

3. If we have a triple level crossing we find that

$$\int_{-\infty}^{\infty} \psi_\infty \frac{\partial \psi_\infty}{\partial E}\, dx = 0$$

and

$$\int_{-\infty}^{\infty} \left[\left(\frac{\partial \psi_\infty}{\partial E}\right)^2 - \psi_\infty \left(\frac{\partial^2 \psi_\infty}{\partial E^2}\right)\right] dx = 0.$$

In their discussion of level crossing, Bender and Wu found the first of these conditions (and then presented the argument that triple crossing wouldn't occur because it is unlikely that this first integral vanishes).

II.8. *Non-Analyticity at Crossing Points*

In the previous section, we have shown that crossing occurs if, and only if, $\int dx\, \psi_\infty^2 = 0$. We next turn to the question of showing that the eigenvalues are nonanalytic at the crossing point.[28] We first note

THEOREM II.8.1. *A sufficient condition for nonanalyticity at a crossing point is*

$$\int_{-\infty}^{\infty} x^2 \psi_\infty^2(x, \alpha, E)\, dx \neq 0. \tag{II.9}$$

Proof. Introduce the real inner product $\langle\ ,\ \rangle_r$ defined by

$$\langle f, g\rangle_r = \int_{-\infty}^{\infty} f(x)\, g(x)\, dx.$$

Then at a noncrossing point:

$$[H(\alpha) - E(\alpha)]\, \psi_\infty[x, \alpha, E(\alpha)] = 0;$$

so,

$$[H(\alpha) - E(\alpha)] \frac{d}{d\alpha} \{\psi_\infty[x, \alpha, E(\alpha)]\} = -\frac{dE}{d\alpha} \psi_\infty + x^2 \psi_\infty .$$

Taking the $\langle\ ,\ \rangle_r$ scalar product with ψ_∞, and using the R symmetry of H, we find

$$\frac{dE}{d\alpha} \int_{-\infty}^{\infty} \psi_\infty^2\, dx = \int_{-\infty}^{\infty} x^2\, \psi_\infty^2\, dx. \tag{II.10}$$

Thus, (II.9) and $\int \psi_\infty^2\, dx = 0$ at a crossing point imply $dE/d\alpha \to \infty$ at a crossing point obeying (II.9).

Remarks. 1. Bender and Wu [5], obtain a result analogous to (II.9) but in terms of x^4 moments. Since a virial theorem argument (see also below) implies

$$\int_{-\infty}^{\infty} (3x^4 + 2x^2)\, \psi_\infty^2(x)\, dx = E(\alpha) \int_{-\infty}^{\infty} \psi_\infty^2(x)\, dx,$$

the x^2 and x^4 moments vanish together.

2. An alternate proof of this last result in the multiplicity two case illuminates it and shows that (II.9) might not, in general, be a necessary condition. In

[28] $\begin{pmatrix} \lambda & 1 \\ 0 & -\lambda \end{pmatrix}$ is an example of a matrix with analyticity of the eigenvalues at crossing ($\lambda = 0$) in spite of the Jordan anomaly at $\lambda = 0$. Notice, however that the eigen*projections* are nonanalytic at $\lambda = 0$.

terms of the projections $P(\alpha)$ of the last section, the eigenvalues near $\alpha = \alpha_0$ are those of the two-dimensional matrix (see Kato [1]),

$$U(\alpha)\, P(\alpha)\, H(\alpha)\, U(\alpha)^{-1},$$

where the $U(\alpha)$ are unitary operator valued functions analytic in α with

$$U(\alpha)\, P(\alpha)\, U(\alpha)^{-1} = P(\alpha_0).$$

This two-dimensional matrix will have the form,

$$E(\alpha_0) \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} a(\alpha) & b(a) \\ c(\alpha) & d(\alpha) \end{bmatrix},$$

where $a(\alpha_0) = c(\alpha_0) = d(\alpha_0) = 0$; $b(\alpha_0) = 1$ when we use the basis

$$\begin{pmatrix} 1 \\ 0 \end{pmatrix} = \psi_\infty , \qquad \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \frac{\partial \psi_\infty}{\partial E} .$$

The eigenvalues for this matrix are a sum of a term linear in $a + d$ and a term proportional to $\sqrt{[a(\alpha) + d(\alpha)]^2 + 4b(\alpha)\, c(\alpha)}$. The $[a(\alpha) + d(\alpha)]^2$ has a second-order zero at $\alpha = \alpha_0$ and thus the square root will cause nonanalyticity if $c(\alpha)$ has a first-order zero at $\alpha = \alpha_0$. It is easy to see that $c'(\alpha_0)$ is proportional to $\int x^2 \psi_\infty{}^2$ so (II.9) is sufficient. (II.9) is not necessary, since the square root can be non-analytic even if $c'(\alpha_0) = 0$.

We now turn to showing that (II.9) always holds for the case at hand.[29] The crucial technique is to use a differential equation due to Grodnik and Sharp [17].[30] This equation is one for the quantity,

$$L(\lambda) = \int_{-\infty}^{\infty} \psi_\infty{}^2(x)\, e^{i\lambda x}\, dx.$$

We use Dirac notation for the $\langle\ ,\ \rangle_r$ inner product. We also suppose the potentials V below are nice enough to justify all commutation relations.

LEMMA II.8.2. *Let ψ be an eigenfunction of $p^2 + V$ (V may be complex). Then*

$$\langle \psi \mid e^{i\lambda x} p \mid \psi \rangle = -\frac{\lambda}{2} \langle \psi \mid e^{i\lambda x} \mid \psi \rangle. \tag{II.11}$$

[29] Bender and Wu do not consider the possibility that (II.9) might fail.

[30] These authors also obtain analogous *formal* differential equations in the field theoretic case. Their proofs however differ from the one we present. It is a pleasure to thank Dr. Sharp for communicating this technique to me before its publication.

Proof. Since $\langle\psi \mid H = E\langle\psi \mid; H \mid \psi\rangle = E \mid \psi\rangle$

$$0 = \langle\psi \mid [H, e^{ix}] \mid \psi\rangle = \langle\psi \mid [p^2, e^{i\lambda x}] \mid \psi\rangle$$

But

$$[p^2, e^{i\lambda x}] = 2\lambda e^{i\lambda x} p + \lambda^2 e^{i\lambda x}, \tag{II.12}$$

so the result is proven.

THEOREM II.8.3. *Let ψ be an eigenfunction of $p^2 + V$ of eigenvalue E. Then*

$$E\langle\psi \mid e^{i\lambda x} \mid \psi\rangle = \langle\psi \mid e^{i\lambda x} V \mid \psi\rangle + \frac{\lambda^2}{4}\langle\psi \mid e^{i\lambda x} \mid \psi\rangle - \frac{i}{2\lambda}\langle\psi \mid e^{i\lambda x} V' \mid \psi\rangle. \tag{II.13}$$

Proof. $[ip, p^2 + V] = V'$ so that

$$\begin{aligned}\langle\psi \mid e^{i\lambda x} V' \mid \psi\rangle &= \langle\psi \mid e^{i\lambda x} ipE \mid \psi\rangle - \langle\psi \mid e^{i\lambda x} Hip \mid \psi\rangle \\ &= \langle\psi \mid [H, e^{i\lambda x}]\, ip \mid \psi\rangle = \langle\psi \mid [p^2, e^{i\lambda x}]\, ip \mid \psi\rangle \\ &= 2\lambda\langle\psi \mid e^{i\lambda x} ip^2 \mid \psi\rangle + \lambda^2\langle\psi \mid e^{i\lambda x} ip \mid \psi\rangle \quad \text{(by II.12)} \\ &= 2i\lambda\langle\psi \mid e^{i\lambda x}(E - V) \mid \psi\rangle - \frac{i\lambda^3}{2}\langle\psi \mid e^{i\lambda x} \mid \psi\rangle.\end{aligned}$$

This proves (II.13).

Remark. The term linear in λ above yields the virial theorem. Thus, II.8.3 is a generalized viral theorem.

COROLLARY II.8.4. *Let $L(\lambda) = \int_{-\infty}^{\infty} \psi_\infty^2(x)\, e^{i\lambda x}\, dx$, where ψ_∞ is the eigenfunction at a fixed eigenvalue $E(\alpha, 1)$. Then*

$$\left[\left(\frac{d}{d\lambda}\right)^4 + \frac{2}{\lambda}\left(\frac{d}{d\lambda}\right)^3 - \alpha\left(\frac{d}{d\lambda}\right)^2 - \frac{\alpha}{\lambda}\left(\frac{d}{d\lambda}\right) + \frac{\lambda^2}{4}\right] L(\lambda) = EL(\lambda). \tag{II.14}$$

Proof. This is just (II.13) using

$$\frac{1}{i^n}\left(\frac{d}{d\lambda}\right)^n L(\lambda) = \langle\psi \mid x^n e^{i\lambda x} \mid \psi\rangle.$$

COROLLARY II.8.5. *If E is an eigenvalue for coupling constant α,*

$$\int_{-\infty}^{\infty} \psi_\infty^2(x, \alpha, E)\, dx = 0$$

implies

$$\int_{-\infty}^{\infty} x^2 \psi_\infty^2(x, \alpha, E)\, dx \neq 0.$$

Proof. Since ψ_∞ falls rapidly to zero as $x \to \infty$, a power series expansion of $e^{i\lambda x}$ in $L(\lambda)$ is justified. Plugging such a power series in (II.3) yields all the moments $\int x^{2n}\psi_\infty^2\, dx$ in terms of the zeroth and second moments. If these both vanish, $L(\lambda) \equiv 0$ so that $\psi_\infty^2(x, \alpha, E) \equiv 0$. Since this is impossible, the corollary is proven.

We thus conclude:

THEOREM II.8.6. *$E(\alpha)$ has a branch point whenever $\int \psi_\infty^2[x, \alpha, E(\alpha)]\, dx = 0$ or, equivalently*

$$\frac{\partial \psi_\infty}{\partial E}[0, \alpha, E(\alpha)] = 0, \quad \textit{odd parity levels,}$$

and

$$\frac{\partial^2 \psi_\infty}{\partial x \partial E}[0, \alpha, E(\alpha)] = 0, \quad \textit{even parity levels.}$$

At such a point $\partial E/\partial\alpha \to \infty$, e.g., in the multiplicity two case the Puiseux series

$$E(\alpha, 1) = a_0 + a_{1/2}(\alpha - \alpha_0)^{1/2} + \cdots$$

has $a_{1/2} \neq 0$.

II.9. *More Hilbert Space Preliminaries*

Our analysis thus far has been based on varying the subdominant coupling constant α, keeping β real. We next consider $p^2 + x^2 + \beta x^4$ and $p^2 + \gamma x^2$ where β and γ are allowed to wander off the real axis.

Consider $p^2 + x^2 + \beta x^4$. If β is a negative real, the classical force $2x - 4 \mid \beta \mid x^3$ is sufficiently large to send a particle to $\pm\infty$ in only a finite amount of time. This is reflected in the quantum mechanics: the Hamiltonian is not self-adjoint unless one adds boundary conditions at ∞ (the deficiency indices are $(+2, +2)$). This might lead one to suspect one can only define $H(1, \beta)$ naturally for Re $\beta > 0$. Actually, we will be able to define it for β in the entire complex plane cut along the negative real axis. There are two simple reasons for expecting this. Firstly, for β not a negative real, $H(1, \beta)$ defines a quadratic form whose expectation values lie in a sector of opening angle strictly less than π. Such objects are notoriously well-behaved[31] (See Kato [1]). Alternately, we have seen that

$$(-(d^2/dx^2) + \beta^{-2/3}x + x^4)\psi = E\psi$$

has solutions $\psi(z)$ which go to zero (as $e^{-z^3/3}$) for $\mid \arg z \mid < \pi/6$. If we make a scaling transformation $z \to \beta^{1/6}z$ which makes sense on entire functions but is not implementable in L^2, we see that $[-(d^2/dx^2) + x^2 + \beta x^4]\, \phi = E'\phi$ has solutions going to zero for $\mid \arg \beta^{-1/6}z \mid < \pi/6$. In particular, if $\mid \arg \beta \mid < \pi$, these solutions

[31] One can show directly using this nice behavior that $\beta^{-1/2}\, H(1, \beta)$ is a holomorphic family of type (b) in the sense of Kato.

go to zero on the real axis and so $p^2 + x^2 + \beta x^4$ has a large supply of candidates for eigenfunctions—it should thus make sense as nice operator.

We first establish a quadratic estimate of a form very similar to those used recently by Jaffe and Glimm in their treatment [18] of $(\phi^4)_2$ field theories:

LEMMA II.9.1. *Let β be complex with* $\operatorname{Im}\beta \neq 0$. *Then for a with*

$$a < 1 - |\beta|^{-1} |\operatorname{Re}\beta|,$$

there is a b so that

$$a[\|(p^2 + x^2)\psi\|^2 + |\beta|^2 \|x^4\psi\|^2] < \|(p^2 + x^2 + \beta x^4)\psi\|^2 + b\|\psi\|^2$$

for all $\psi \in D(p^2) \cap D(x^4)$.

Proof. By a closure argument, we need only consider ψ, C^∞ of compact support. Thus, we need only show

$$(p^2 + x^2 + \beta x^4)^\dagger(p^2 + x^2 + \beta x^4) \geqslant a[(p^2 + x^2)^2 + |\beta|^2 x^8] - b \quad \text{(II.15)}$$

on C_0^∞, where all formal manipulations with commutators are legitimate. The left side of (II.15) can be written as

$$\begin{aligned}
&(p^2 + x^2)^2 + |\beta|^2 x^8 + 2 \operatorname{Re}\beta[(p^2 + x^2)\, x^4 + x^4(p^2 + x^2)] + 2i \operatorname{Im}\beta[p^2, x^4] \\
&\quad = \frac{|\operatorname{Re}\beta|}{|\beta|}[p^2 + x^2 \pm |\beta|\, x^4]^2 + \left(1 - \frac{|\operatorname{Re}\beta|}{|\beta|}\right)[(p^2 + x^2)^2 \\
&\qquad + |\beta|^2 x^8] \pm 2i\, |\operatorname{Im}\beta|\, [p^2, x^4] \\
&\quad \geqslant (a + R)[(p^2 + x^2)^2 + |\beta|^2 x^8] \pm 8\, |\operatorname{Im}\beta|\, (px^3 + x^3 p) \text{ [for some } R > 0] \\
&\quad = (a + R)[(p^2 + x^2)^2 + |\beta|^2 x^8] - 8\, |\operatorname{Im}\beta|(p^2 + x^6) + 8\, |\operatorname{Im}\beta|(p \pm x^3)^2 \\
&\quad \geqslant a[(p^2 + x^2)^2 + |\beta|^2 x^8] - b + [R(p^2 + x^2)^2 - 8\, |\operatorname{Im}\beta|\, p^2 + \tfrac{1}{2}b] \\
&\qquad + [R\, |\beta|^2 x^8 - 8\, |\operatorname{Im}\beta|\, x^6 + \tfrac{1}{2}b].
\end{aligned}$$

By suitable choice of b we can make the last two terms positive. Q.E.D.

THEOREM II.9.2. *Let $H(1, \beta) = p^2 + x^2 + \beta x^4$ be defined with domain $D(p^2) \cap D(x^4)$ for β in the complex plane cut along the negative real axis. $H(1, \beta)$ is a holomorphic family of type* (A) *with compact resolvents.*

Proof. A direct consequence of (II.15) which implies that $H(1, \beta)$ is closed and that x^4 is a small perturbation of $H(1, \beta)$ in the technical sense of Kato [1]. Q.E.D.

COROLLARY II.9.3. *$E_n(1, \beta)$ continued from positive real β into the cut plane has* $\operatorname{Im} E_n(1, \beta) > 0$ *if* $\operatorname{Im} \beta > 0$.

Proof. By theorems II.9.2 and II.3.1 $E_n(1, \beta)$ is an eigenvalue of $H(1, \beta)$. If ϕ is the eigenvector, then

$$(\operatorname{Im} E_n) = \| \phi \|^{-2} (\operatorname{Im} \beta) \| x^2 \phi \|^2 > 0. \qquad \text{Q.E.D.}$$

An operator similar to $H(1, \beta)$ which will play a crucial role is $p^2 + \gamma x^2 = H(\gamma, 0)$. By a technique identical to that used in Lemma II.9.1 and Theorem II.9.2.:

THEOREM II.9.4. *Let γ be the cut plane. Then $p^2 + \gamma x^2$ defines a closed operator with domain $D(p^2) \cap D(x^2)$. Its eigenvalues are $(2n + 1)\gamma^{1/2}$; $n = 0, 1, \ldots$. For any compact subset Ω in the cut plane, there exists a and b so that*

$$\| x^2 \psi \| < a \| (p^2 + \gamma x^2) \psi \| + b \| \psi \| \qquad \text{(II.16)}$$

all $\psi \in D(p^2) \cap D(x^2)$; $\gamma \in \Omega$.

THEOREM II.9.5. *Let γ be in the cut plane. Then for any $| \beta | > 0$, $p^2 + \gamma x^2 + | \beta | x^4$ defines a closed operator on $D(p^2) \cap D(x^4)$. For any compact Ω in the cut plane, there exists a and b so that*

$$\| x^2 \psi \| \leqslant a \| (p^2 + \gamma x^2 + | \beta | x^4) \psi \| + b \| \psi \| \qquad \text{(II.17)}$$

all $\psi \in D(p^2) \cap D(x^4)$; $\gamma \in \Omega$; $0 < | \beta | \leqslant 1$.

II.10. *The Asymptotic Nature of the Singularities*

We next turn to investigate the position of the infinite number of singularities in the three-sheeted β plane; singularities whose existence we proved in II.6. We will show using the techniques of II.9 that these singularities that approach $\beta = 0$ do so by spiraling in at $\arg \beta = 270°$ (and $810° \equiv -270°$), i.e., the asymptotic phase of the branch points approach 270°. By a scaling of the phase of β we can study $p^2 + x^2 + \beta x^4$ by looking instead at $p^2 + \gamma x^2 + | \beta | x^4$ (where $| \gamma | = 1$, $\gamma = \exp(-\frac{2}{3} i \arg \beta)$. As $| \beta | \downarrow 0$, we expect $H(\gamma, | \beta |)$ to approach $H(\gamma, 0)$ when γ is fixed $| \gamma | = 1, \gamma \neq 1$. For negative β, $H(\gamma, \beta)$ is not a nice operator so $H(\gamma, | \beta |)$ cannot have a convergent power series about $| \beta | = 0$; the best we can hope for is that it have an asymptotic expansion. General folklore (see Kato [1], Chapter VIII) says that asymptotic perturbations are related to strong convergence of the resolvents while analytic perturbations are related to norm convergence of resolvents. Nevertheless, we have

LEMMA II.10.1. *Fix γ, γ not a negative real number. Let*

$$R_\gamma(|\beta|, E) = (H(\gamma, |\beta|) - E)^{-1}$$

Then

$$\| R_\gamma(|\beta|, E) - R_\gamma(0, E)\| \to 0 \quad \text{as} \quad |\beta| \downarrow 0$$

The convergence is uniform on compact subsets in γ.

Proof. Since $R_\gamma(|\beta|, E) = R_\gamma(|\beta|, E_0)[1 + (E_0 - E) R_\gamma(|\beta|, E_0)]^{-1}$, it is sufficient to prove the result for one E_0. Given a compact subset Γ in the cut plane, the union, U, of the numerical ranges (i.e., $\{\langle\phi, A\phi\rangle \mid \|\phi\| = 1\}$) for the $H(\gamma, |\beta|)$ over all $\gamma \in \Gamma$ is not all of $\mathbf{C}$, so we can choose E_0 with $\text{dist}(E_0, U) = C^{-1} > 0$. Then $\| R_\gamma(|\beta|, E_0)\| \leqslant C$ (see Kato, [1], p. 267–268). We have

$$R_\gamma(|\beta|, E_0) - R_\gamma(0, E_0) = -|\beta|[R_\gamma(0, E_0)x^2][x^2 R_\gamma(|\beta|, E_0)].$$

By Theorems II.9.4 and II.9.5, we can find a, b so that

$$\| x^2\psi \| \leqslant a \|[H(\gamma, |\beta|) - E_0]\psi\| + (b + a|E_0|)\|\psi\|$$

for all $\gamma \in \Gamma$ $1 \geqslant |\beta| \geqslant 0$. Thus,

$$\| x^2 R_\gamma(|\beta|, E_0)\| \leqslant a + C(b + a|E_0|)$$

and

$$\| R_\gamma(0, E_0)\, x^2 \| \leqslant a + C(b + a|E_0|);$$

so,

$$\| R_\gamma(|\beta|, E_0) - R_\gamma(0, E_0)\| \leqslant C' |\beta| \to 0 \quad \text{as} \quad |\beta| \downarrow 0. \qquad \text{Q.E.D.}$$

THEOREM II.10.2. *Let n be given and let Γ, a subset of the cut plane, be given. Then, there is a B so that for $|\beta| < B$ and $\gamma \in \Gamma$, $H(\gamma, |\beta|)$ has exactly one eigenvalue $E_n(\gamma, |\beta|)$ near $(2n + 1)\gamma^{1/2}$. As $|\beta| \downarrow 0$, $E_n(\gamma, |\beta|) - \gamma^{1/2}(2n + 1) \to 0$ uniformly for $\gamma \in \Gamma$.*

Proof. This is a direct consequence of the last lemma. Consider the projection P_γ onto the eigenvector of $H(\gamma, 0)$ with energy $(2n + 1)\gamma^{-1/2}$. Let

$$R_\gamma(|\beta|) = \frac{1}{2\pi i} \int_c R_\gamma(|\beta|, E)\, dE,$$

where C is a curve around $(2n+1)\gamma^{-1/2}$ enclosing no other eigenvalues of $H(\gamma, 0)$.

Then $P_\gamma(0) = P_\gamma$ and $P_\gamma(|\beta|) \to P_\gamma(0)$ in norm. Since $P_\gamma(0)$ is one-dimensional so is $P_\gamma(|\beta|)$ for $|\beta|$ small. Thus, $H(\gamma, |\beta|)$ has one eigenvalue inside C for $|\beta|$ small and this eigenvalue converges to $(2n + 1)\gamma^{1/2}$. The uniformity in Γ can be proven easily. Q.E.D.

Remark. Thus, all the eigenvalues of $H(\gamma, 0)$ are stable in the sense of Kato ([1], p. 437–438).

THEOREM II.10.3. *Let n be given and $\theta < 3\pi/2$. Then, there is a B so that $E_n(1, \beta)$ is analytic in $\{\beta$ on the three-sheeted surface $| 0 < |\beta| < B, |\arg \beta| < \theta\}$.*

Proof. We need only scale out arg β and apply the last theorem. Q.E.D.

Remark. These results are independent of the occurrence or nonoccurrence of natural boundaries, etc.

COROLLARY II.10.4. *The singularities of E_n at $\beta = 0$ have an asymptotic phase of* 270°.

Remarks. 1. The approximate branch points of Bender and Wu have this property.

2. Since there can't be any branch points with phase 270° (arg $\beta = 270°$ corresponds to α real under scaling), we should think of the branch points as spiraling into arg $\beta = 270°$.

These theorems allow us to develop an interesting picture of what produces the the singularities. Consider a fixed n. In Fig. 2, we draw a schematic picture of $E_n(\gamma, |\beta|)$ for $|\beta|$ fixed letting γ vary with $|\gamma| = 1, 0 \leqslant \arg < \pi$. When $\beta = 0$. we obtain a circular arc whose end is anchored at $i(2n + 1)$. For any $|\beta| \neq 0$, $E_n(-1, |\beta|)$ is real, so all these curves have their ends *anchored* on the real axis. Thus, as $|\beta| \downarrow 0$, the end of the curve shown gets stretched more and more.

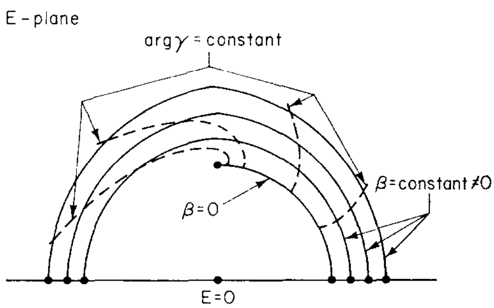

FIG. 2. A schematic picture of the behavior of $E_n(\gamma, |\beta|)$ for $|\gamma| = 1$; $0 < \arg \gamma < \pi$, discussed in Section II.10.

Superimposed on this picture is the fact that the curves for different n must tangle with each other.

II.11. *The Asymptotic Expansion for $E_n(1, \beta)$, Revisited*

In Appendix II, we show that $E_n(1, \beta)$ has an asymptotic expansion valid as $\beta \downarrow 0$ along the positive real axis. In this section, we wish to show that this expansion is valid as $\beta \downarrow 0$ in any sector $|\arg \beta| < \theta$ with $\theta < 3\pi/2$. We will first show that the eigenvalues and eigenvectors of $H(\gamma, |\beta|)$ have an asymptotic expansion for fixed γ. As preparation for this, we note:

LEMMA II.11.1. *Let P be a compact set in $\{\langle\gamma, E\rangle \mid (H(\gamma, 0) - E)^{-1}$ exists, $\gamma \in$ cut plane$\}$. Then $(H(\gamma, 0) - E)^{-1}$ is a continuous map of $\mathscr{S} \to \mathscr{S}$ for any $\langle\gamma, E\rangle \in P$, with the bounds uniform on P.*

Proof. The topology on $\mathscr{S}$ is generated by $\phi \to \|(p^2 + x^2)^n \phi\|$ $n = 0, 1, 2, \ldots$. Thus, we need only establish inequalities of the form,

$$\|(p^2 + x^2)^n (p^2 + \gamma x^2 - E)^{-1} \psi\| \leqslant \sum_{j=0}^{n} a_j^{(n)} \|(p^2 + x^2)^j \psi\| \tag{II.18}$$

with the $a_j^{(n)}$ independent of $\langle\gamma, E\rangle \in P$. We proceed inductively. For $n = 0$, (II.18) follows from the continuity of $(p^2 + \gamma x^2 - E)^{-1}$ on P as a map from L^2 to L^2. By Theorem II.9.4 we can find a and b so that

$$\|(p^2 + x^2)\phi\| \leqslant a \|(p^2 + \gamma x^2 - E)\phi\| + b \|\phi\|,$$

for all $\phi \in D(p^2) \cap D(x^2)$ and all $\langle\gamma, E\rangle \in P$. Thus,

$$\begin{aligned} &\|(p^2 + x^2)^n (p^2 + \gamma x^2 - E)^{-1} \psi\| \\ &\quad\leqslant a \|(p^2 + \gamma x^2 - E)(p^2 + x^2)^{n-1} (p^2 + \gamma x^2 - E)^{-1} \psi\| \\ &\quad\quad + b \|(p^2 + x^2)^{n-1} (p^2 + \gamma x^2 - E)^{-1} \psi\|. \end{aligned} \tag{II.19}$$

By the induction hypothesis, the second term is of form (II.18). To treat the first, we write

$$(p^2 + \gamma x^2 - E)(p^2 + x^2)^{n-1} = (p^2 + x^2)^{n-1} (p^2 + \gamma x^2 - E) + Q(a^{\#}), \tag{II.20}$$

where $Q(a^{\#})$ is a polynomial in a and a^+ of degree $2n - 2$, with coefficients bounded for $\langle\gamma, E\rangle \in P$. Using (II.20) and

$$\|(a^{\#})^{2m} \phi\| \leqslant (2m)^m \|(p^2 + x^2 + 1)^{2m} \phi\|,$$

we can bound the first term on the right side of (II.19) by a sum of the form $\|(p^2 + x^2)^m(p^2 + \gamma x^2 - E)^{-1}\psi\|(m < n - 1)$ plus $a\|(p^2 + x^2)^{n-1}\psi\|$. By the induction hypothesis (II.18) holds.

LEMMA II.11.2. *Let $R_\gamma(|\beta|, E) \equiv (p^2 + \gamma x^2 + |\beta| x^4 - E)^{-1}$ be viewed as a map of $\mathscr{S} \to L^2$. For any N, R has an expansion*

$$R_\gamma(|\beta|, E) = \sum_{n=0}^{N} A_{\gamma,n}(E)\,|\beta|^n + \mathscr{R}_{N+1,\gamma}(|\beta|, E)\,|\beta|^{N+1},$$

where

(a) *The $A_{\gamma,n}$ are independent of N and $|\beta|$ and are continuous maps of $\mathscr{S} \to L^2$,*

(b) *$A_{\gamma,0}(E) = R_\gamma(0, E)$, and*

(c) *$\mathscr{R}_{N+1,\gamma}(|\beta|, E)$ is bounded as a map of $\mathscr{S} \to L^2$ with the bounds uniform for $0 < |\beta| < 1$ on any compact P of the form considered in Lemma* II.11.1.

Proof. Consider the formal expansion

$$(A + B)^{-1} = \sum_{n=0}^{N} (-1)^n(A^{-1}B)^n + (-1)^{N+1}(A + B)^{-1}(BA^{-1})^{N+1}, \quad \text{(II.21)}$$

where $A = p^2 + \gamma x^2 - E$, $B = |\beta|\, x^4$. As a map of $L^2 \to L^2$, BA^{-1} is not bounded or everywhere defined so that this expansion is not meaningful in an $L^2 \to L^2$ sense. But A^{-1} and B are bounded maps of $\mathscr{S} \to \mathscr{S}$ and so (II.21) can be verified if B, A^{-1} are viewed as maps of $\mathscr{S} \to \mathscr{S}$ and $(A + B)^{-1}$ as a map of $\mathscr{S} \to L^2$. (a) and (b) follow immediately and (c) follows from Lemma II.11.1.

Remark. Using results of Jaffe [10], one can actually show $(A + B)^{-1}$ is a bounded map of $\mathscr{S} \to \mathscr{S}$ so that this last lemma can be strengthened to allow us to view everything as maps of $\mathscr{S} \to \mathscr{S}$.

THEOREM II.11.3. *Let $E_m(\gamma, |\beta|)$ be the unique eigenvalue of $H(\gamma, |\beta|)$ near $(2m + 1)\gamma^{1/2}$, and let $\psi_m(\gamma, |\beta|) = P_\gamma(|\beta|)\,\psi_m(\gamma, 0)$ be the corresponding eigenvector. Then E_m and ψ_m have asymptotic expansions*

$$E_m(\gamma, |\beta|) \sim \sum a_n(\gamma)\,|\beta|^n \quad \text{(II.22)}$$

$$\psi_m(\gamma, |\beta|) \sim \sum \phi_n(\gamma)\,|\beta|^n \quad \text{(II.23)}$$

valid for any γ fixed with remainder terms uniformly bounded as γ varies over compact subsets of the cut γ plane.

Remarks. 1. By remainder terms we mean

$$|\beta|^{-N-1}\left[E_m(\gamma, |\beta|) - \sum_{n=0}^{N} a_n(\gamma)|\beta|^n\right].$$

2. Using Jaffe's results, we can show that the asymptotic expansion of ψ_n has remainder terms bounded as an element of $\mathscr{S}$.

Proof. $P_\gamma(|\beta|) = (2\pi i)^{-1} \int_c R_\gamma(|\beta|, E)\, dE$ so that $P_\gamma(|\beta|)$ has an asymptotic expansion of the same type as Lemma II.11.2. Since the unperturbed eigenfunctions $\psi_m(\gamma, 0)$ are in $\mathscr{S}$, (II.23) follows. Using

$$E_m(\gamma, |\beta|) = [\langle\psi_m(\gamma, 0), P_\gamma(|\beta|)\,\psi_m(\gamma, 0)\rangle]^{-1}[\langle\psi_m(\gamma, 0), H(\gamma, |\beta|)\, P_\gamma(|\beta|)\,\psi_m(\gamma, 0)\rangle],$$

we obtain (II.22).

THEOREM II.11.4. *The asymptotic expansion,*

$$E_m(1, \beta) \sim \sum a_n^{(m)} \beta^n, \tag{II.24}$$

proven in Appendix II *for real positive β is valid uniformly in any sector* $|\arg \beta| < \theta$ *with* $\theta < 3\pi/2$.

Proof. Since $E_m(1, \beta) = \gamma^{-1/2} E_m(\gamma, |\beta|)$ with γ given by $\beta = |\beta|\, \gamma^{-3/2}$ it is sufficient to show that the asymptotic series for $E_m(\gamma, |\beta|)$ is of the form,

$$E_m(\gamma, |\beta|) \sim \sum a_n^{(m)} \gamma^{(1-3n)/2} |\beta|^n. \tag{II.25}$$

We establish this inductively. Since $E_m(\gamma, |\beta|) \to E_m(\gamma, 0) = \gamma^{1/2} E_m(1, 0)$, the asymptotic coefficients in (II.22) agrees with the form (II.25) in zeroth order. Suppose we have established this agreement to order N so that we know that

$$f_{|\beta|}(\gamma) = \left[E_m(\gamma, |\beta|) - \sum_{n=0}^{N} a_n^{(m)} \gamma^{(1-3n)/2} |\beta|^n\right] \Big/ |\beta|^{N+1}$$

is bounded uniformly in β on compact subsets of the cut γ plane. By the validity of (II.24) for real β, $f_{|\beta|}(\gamma) \to a_N^{(m)} \gamma^{(1-3N)/2}$ pointwise on the real axis. Thus, by the Vitali convergence theorem ([19], p. 168) $f_{|\beta|}(\gamma) \to a_N^{(m)} \gamma^{(1-3N)/2}$ uniformly on compact subsets of the γ plane. Therefore, (II.25) is correct to order $N + 1$ and the proof is complete.

COROLLARY II.11.5. *Consider $E_n(1, \beta)$ as a function in a small circle about $\beta = 0$ cut along the negative real axis. Then the discontinuity across the cut* $[= 2i \operatorname{Im} E_n(1, -\beta + i\epsilon)]$ *goes to zero faster than any power of* $|\beta|$.

II.12. *Dispersion Relations in Coupling Constant*

Let us summarize the situation with regard to functions $E_n(1, \beta)$ in a β-plane cut along the negative real axis. We have shown that $E_n(1, \beta)$ has continuation from a neighborhood of the real axis into the complement of a cut annulus, i.e., a region as shown in Fig. 3.

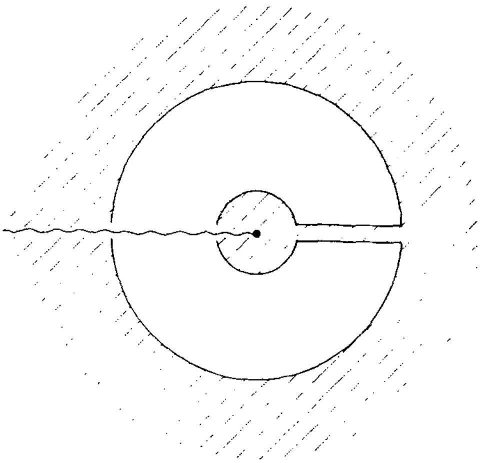

FIG. 3. The region of analyticity for $E_n(1, \beta)$ rigorously established in Section II.

Using results of Loeffel and Martin [37] and of Section II.5, Loeffel *et al.* [38] have proven analyticity in the entire cut plane.[32]

THEOREM II.12.1. *On the first β sheet, $E_n(1, \beta)$ has a representation,*

$$E_n(1, \beta) = (2n + 1) + \beta \int_0^\infty \rho_n(\gamma) \frac{d\gamma}{\gamma + \beta},$$

where $\rho_n(\gamma) = \lim_{\epsilon \downarrow 0} (+\pi\gamma + i\epsilon)^{-1} \operatorname{Im} E_n(1, -\gamma + i\epsilon)$ *has the properties,*

(a) $\rho_n(\gamma) \underset{\gamma \to \infty}{=} 0(\gamma^{-2/3})$,

(b) $\rho_n(\gamma) \underset{\gamma \to 0}{=} 0(\gamma^N)$ *all N, and*

(c) $\rho_n(\gamma) \geqslant 0$.

Proof. Since $E_n(1, \beta) = 0(\beta^{1/3})$ at ∞, a once subtracted dispersion relation holds.

[32] Bender and Wu also attribute a rigorous proof of the nonoccurrence of branch points for $| \arg \beta | < \pi$ to Jaffe. Such a proof is nonexistent. Jaffe has only shown analyticity of the resolvent, not of the eigenvalues.

III. The Eigenvalues in More Complex Cases

III.1. *Other One degree of Freedom Cases*

Our results carry over without any significant change to x^{2m} perturbations.[33]

Theorem III.1. *The eigenvalues of the operator $p^2 + \alpha x^2 + \beta x^{2m}$ for $\beta > 0$, α real with domain $D(p^2) \cap D(x^{2m})$ obey a scaling law,*

$$E_n(\alpha, \beta) = \lambda E_n(\alpha\lambda^{-2}, \beta\lambda^{-m-1}). \tag{III.1}$$

In particular,

$$E_n(1, \beta) = \beta^{1/m+1} E_n(\beta^{-2/m+1}, 1), \tag{III.2}$$

$E_n(1, \beta)$ has an analytic continuation with a global $(m + 1)$-st order branch point at $\beta = 0$. In addition, $\beta = 0$ is a limit point of singularities. It is Herglotz in $|\arg \beta| < \pi$. $E_n(1, \beta)$ has an extension to a sectorial neighborhood of 0 of the form $|\beta| < R$, $|\arg \beta| < \theta$ for any $\theta < \frac{1}{2}(m + 1)\pi$. In any such sector, the Rayleigh–Schrödinger series is asymptotic uniformly in angle.

The only part of the argument in Section II that fails to go through involves the discussion in II.8; for the Sharp differential equation is $2m$-th order and we only have $m + 2$ null initial conditions when $\int x^2\psi_\infty{}^2\, dx = \int \psi_\infty{}^2 = 0$ (all the odd moments vanish!)

There is another part of the argument of II which requires modification. In II.9, we showed that $(p^2 + \gamma x^2 + |\beta| x^4 - E)^{-1} - (p^2 + \gamma x^2 - E) \to 0$ as $\beta \to 0$; actually we showed it was $0(|\beta|)$. In the general case, the proof there needs modification. We can show however that

$$\Delta \equiv (p^2 + \gamma x^2 + |\beta| x^{2m} - E)^{-1} - (p^2 + \gamma x^2 - E)^{-1} = 0(|\beta|^{1/m})$$

for

$$\Delta = [-(p^2 + \gamma x^2 - E)^{-1} x^2][|\beta| x^{2(m-1)}(p^2 + \gamma x^2 + |\beta| x^{2m} - E)^{-1}].$$

The first term is bounded. Since $\| |\beta| x^{2m}\psi \| \leqslant a \|(p^2 + \gamma x^2 + |\beta| x^{2m})\psi\| + b\|\psi\|$ and $|\beta|^{(m-1)/m} x^{2(m-1)} < |\beta| x^{2m} + 1$, we have $|\beta|^{(m-1)/m} x^{2(m-1)}(p^2 + \gamma x^2 + |\beta| x^{2n} - E)^{-1}$ bounded. Thus, $\Delta = 0(|\beta|^{1/m})$.

We note also that the results of Loeffel and Martin on the absence of first sheet branch points [37] and of Loeffel *et al.* on the absence of natural boundaries on the first sheet [38] carry over to the x^{2m} case.

[33] W. K. B. techniques also extend to this case [6].

Without any change we can treat a general polynomial problem in which both the unperturbed and perturbed potentials have even degree with positive leading coefficient. If we denote the n-th eigenvalue of $p^2 + a_1x + a_2x^2 + \cdots + a_{2m}x^{2m}$ by $E_n(a_1, ..., a_{2m})$, a Symanzik scaling argument tells us that

$$E_n(a_1, ..., a_{2m}) = \lambda^2 E_n(a_1\lambda^{-3}, ..., ..., a_j\lambda^{-j-2}, ..., a_{2m}\lambda^{-2m-2})$$

for any λ. Thus,

$$E_n(a_1, ..., a_{2m}) = (a_{2m})^{1/m+1} E_n(a_1 a_{2m}^{-3/2m+2}, ..., a_j a_{2m}^{-j-2/2m+2}, ..., 1)$$

If

$$V_0 = a_1x + \cdots + a_{2m}x^{2m}$$

and

$$V_{\text{pert}} = b_1x + \cdots + b_{2k}x^{2k}, \quad k > m,$$

we have

$$E_n(H_0 + V_0 + \beta V_{\text{pert}}) = \beta^{1/k+1} E_n(c_1, ..., c_{2k}),$$

where

$$c_j = \begin{cases} a_j\beta^{-(j+2)/2k+2} + b_j\beta^{(2k-j)/2k+2} & j \leqslant 2m \\ b_j\beta^{(2k-j)/2k+2} & j > 2m. \end{cases}$$

Again, with the exception of the discussion in II.8, our entire argument goes through. In particular, for this problem, the perturbation series diverges; E_n has a global branch point of order $m + 1$, and an infinite number of singularities.

After scaling out a_{2m} from $E_n(a_i)$, we have a function of $2m - 1$ variables, so the branch points are actually branch varieties of codimension one. Very little is known about the structure of these varieties; an amusing structure could be produced since multidimensional complex varieties present many possibilities. For example, the branch points in $E_n(0, 1, 0, \beta)$ where E_0 and E_2 cross, may be continuable into one another by continuing in a_1 and a_3.

Finally, we remark that our methods will not handle potentials which misbehave at ∞ and which are not polynomials.[34]

III.2. *Finite Number of Degrees of Freedom*

It is only after the one degree of freedom problem is left behind that the advantage of abstract arguments over a W.K.B. method becomes apparent. While

[34] We note that the potential $V = \cos x$ mentioned by Bender–Wu [5] is not a singular potential. It is bounded (in x) and thus a bounded operator. For such an operator, the perturbation series converges by the Kato–Rellich theory.

it might be possible to use the multidimensional W.K.B. methods of Maslov [20] to treat the many dimensional problem, the details would clearly be formidable. However, since our arguments of Sections II.1 through II.6 didn't depend on being in one dimension, it is immediately clear that

THEOREM III.2.1. *Let*

$$H_0 = \sum_{n=1}^{N} p_n^2 + w_n^2 q_n^2$$

and

$$V = \sum_{i,j,k,l=1}^{N} a_{ijkl} q_i q_j q_k q_l$$

be potentials that are positive and that go to infinity as $q \to \infty$. *Then the eigenvalues of* $H_0 + \beta V$ *have a global third-order branch point at* $\beta = 0$ *and* $\beta = 0$ *is a limit point of singularities. In any sector* $|\arg \beta| < \theta$, $\theta < 3\pi/2$, $E_m(1, \beta)$ *is analytic for* β *small and the Rayleigh–Schrödinger series is asymptotic in the sector.*

Let us make one more detailed remark about the arguments leading up to Theorem II.2.1. An over-all scaling $q_n \to \beta^{-1/6} q_n$ (all n) is crucial in the proof. One can, however, consider the whole affine group, as generating a set of scaling transformations,

$$q_n \to (Aq)_n + a_n$$

and

$$p_n \to ((A^T)^{-1} p)_n ,$$

which relates various Hamiltonians and provides nontrivial symmetry groups for some of them [2].

The real distinction between the one and many degree problems is in the lack of a solution of the differential equation when E is not an eigenvalue which yields $E(\alpha)$ implicitly. While we did not use this implicit function explicitly, we think that the best hope for proving natural boundaries don't occur is through an implicit function argument. It is thus of interest to note we can obtain the eigenvalues in this case by an implicit definition. We will show that the eigenvalues are also eigenvalues of a Hilbert–Schmidt integral equation and thus given implicitly as zeros of a Fredholm determinant.

LEMMA III.2.2. *Let* V *be as in Theorem* III.2.1. *Let*

$$\tilde{H}_0 = \left(\sum_{n=1}^{N} p_n^2\right) + V$$

and

$$\tilde{V} = \sum_{n=1}^{N} w_n{}^2 q_n{}^2.$$

Then for any R, there is an E so that $(\tilde{H}_0 + \alpha\tilde{V} + E)^{-1}$ *is Hilbert–Schmidt for* $|\alpha| < R$.

Proof. We first note that the eigenvalues of $\tilde{H}_0$ obey $\sum_{n=0}^{\infty} E_n^{-2} < \infty$. This follows from the fact that $V > a(\sum_{n=1}^{N} q_n{}^2) - c$ so that $E_n > E_n^{(h\cdot 0)} - c$ where $E_n^{(h\cdot 0)}$ is the n eigenvalue of N harmonic oscillators. Thus,

$$\sum_{n=0}^{\infty} E_n^{-2} < \infty \qquad \text{if} \qquad \sum_{n=0}^{\infty} E_n^{(h\cdot 0)-2} < \infty$$

and this later is true. Thus, for any negative energy, E_0, $(\tilde{H}_0 - E_0)^{-1}$ is a Hilbert–Schmidt operator. Since $\tilde{V}$ is Kato tiny relative to $\tilde{H}_0$, given R, we can find E so that

$$|\tilde{V}(\tilde{H}_0 - E)^{-1}| < \frac{1}{2R}.$$

Then the Neumann series for $[(\tilde{H}_0 - E) + \alpha\tilde{V}]^{-1}$ converges for $|\alpha| < R$ and

$$[(\tilde{H}_0 - E) + \alpha\tilde{V}]^{-1} = (\tilde{H}_0 - E)^{-1}[1 + \alpha\tilde{V}(\tilde{H}_0 - E)^{-1}]^{-1}$$

so that the lemma is proven.

THEOREM III.2.3. *Let* $\tilde{H}_0$, $\tilde{V}$ *be as above. Then for any R, there is a function* $d(\alpha, \lambda)$ *entire in* λ *and analytic for* $|\alpha| < R$ *and a number E, so that* E_0 *is an eigenvalue of* $\tilde{H}_0 + \alpha_0\tilde{V}$ *with* $|\alpha| < R$ *if, and only if,* $d(\alpha_0, \lambda_0) = 0$, *where* $\lambda_0 = (E_0 + E)^{-1}$.

Proof. Pick E (as in Lemma III.2.2) so that $(\tilde{H}_0 + \alpha\tilde{V} + E)^{-1}$ is Hilbert–Schmidt for $|\alpha| < R$. It is analytic, so its (modified) Fredholm determinant [21] $d(\alpha, \lambda)$ is analytic for $|\alpha| < R$. E_0 is an eigenvalue of $\tilde{H}_0 + \alpha V$ if, and only if, $d(\alpha, \lambda) = 0$.

Thus, if an implicit argument eliminating natural boundaries exists, it will work in the many variable case.

We note that the Loeffel–Martin argument [37] on the absence of branch points on the first sheet does *not* carry over the several-dimensional case because it depends heavily on keeping track of the zeros of the wavefunction; these are a complicated affair in several variables. Since the Loeffel–Martin argument is used in the no-natural-boundaries proof of Loeffel *et al.* [38], this result is also unproven in the multivariable case.

IV. The Padé Approximants

IV.1. *Introduction*

As we explained in Section I, a prime reason for studying the problem of analytically continuing $E_n(1, \beta)$ is the connection between analyticity and summability methods. We have already seen that $E_n(1, \beta)$ has a perturbation series which diverges (II.4) but which is asymptotic uniformly in $|\arg \beta| < \theta$, any $\theta < 3\pi/2$ (II.11). This suggests one use an improper summability method on the Rayleigh–Schrödinger series. The recent work of Bessis *et al.* [7, 8] and of Copley–Masson [39] suggests that one try Padé approximants.

Given a formal power series $f = \sum a_n x^n$, the Padé table is defined by

(a) $f^{[N,M]}(x) = P^{[N,M]}(x)/Q^{[N,M]}(x)$,

where P is a polynomial in x of degree M and Q is a polynomial in x of degree N.

(b) $f^{[N,M]}(x) - \sum_{n=0}^{N+M} a_n x^n = 0(x^{N+M+1})$.

These conditions uniquely determine $f^{[N,M]}$ and determine P, Q up to one normalization constant (for each $[N, M]$). Thus, $f^{[0,M]}$ yields the usual Taylor approximants. The folklore tells us that the diagonal approximants $f^{[N,N+j]}$ converge better than the Taylor series. Bessis *et al.* considered these diagonal sequences (for $j = 0$, $N = 1, 2$) where the a_n are given as the partial wave projections of various Feynman series. They use the position of one resonance to determine a phenomenological coupling constant and then determine other resonances with rather spectacular results!

There is an explicit formula for $f^{[N,M]}$ [22]:

$$f^{[N,M]}(x) = \frac{\begin{vmatrix} a_{M-N+1} & a_{M-N+2} & \cdots & a_{M+1} \\ \vdots & \vdots & & \vdots \\ a_M & a_{M+1} & \cdots & a_{M+N} \\ \sum_{j=N}^{M} a_{j-N}x^j & \sum_{j=N-1}^{M} a_{j-N+1}x^j & \cdots & \sum_{j=0}^{M} a_j x^j \end{vmatrix}}{\begin{vmatrix} a_{M-N+1} & a_{M-N+2} & \cdots & a_{M+1} \\ \vdots & \vdots & & \vdots \\ a_M & a_{M+1} & \cdots & a_{M+N} \\ x^N & x^{N-1} & \cdots & 1 \end{vmatrix}}. \tag{IV.1}$$

We will show in Section IV.2 that the results of Loeffel *et al.* [37, 38] imply that *any diagonal Padé sequence* $f^{[N,N+j]}(\beta)$ *formed from the Rayleigh–Schrödinger series for* E_n *converges to* $E_n(1, \beta)$ *uniformly on compact subsets of the cut plane.*

In Section IV.4, we will study numerically the convergence of the Padé approximants to the ground state when β is small.

The result of these Padé arguments is to soften the effect of the Bender–Wu singularities. These later destroy the convergence of the strong coupling expansion, but since they stay in $|\arg \beta| > \pi$, they do not prevent the recovery of $E(1, \beta)$ from the perturbation series.

The convergence of these diagonal Padé approximants also suggests rather exciting possibilities for the Feynman series of a relativistic field theory. If we wish to be ultraconservative, one can merely conjecture convergence of the diagonal Padé approximants to the ground state energy density (=sum of connected vacuum graphs) in a two-dimensional ϕ^4 theory (i.e., one–space, one–time dimension). If that turned out to hold it would suggest a series of more and more far reaching conjectures: The renormalized energy density in an n-dimensional ϕ^4 theory has convergent Padé's, the renormalized Feynman amplitudes for the S matrix (equivalently, for the Green's functions) in a ϕ^4 theory have convergent Padé's, the renormalized Feynman amplitudes for any renormalizable theory have convergent Padé's. Such convergence would be particularly interesting since the Feynman series is known to diverge in the ϕ^4 cases (e.g., [3]) and is believed to diverge in the others.[35]

IV.2. *Convergence of the Padé Approximants*

THEOREM IV.2.1. *The negative of the Rayleigh–Schrödinger coefficients, are a series of Stieltjes for* $m > 1$, *i.e.*, $a_m = (-1)^{m+1} \int_0^\infty x^m \rho(x)\, dx (m > 1)$, *for a positive measure* $\rho(x)\, dx$. *In particular, the Padé approximants* $f^{[N,N+j]}(x)$ *converges as* $N \to \infty$ *uniformly on compacts if* j *is fixed. Since* $\sum |a_m|^{-1/2m+1} = \infty$, *the limits of these sequences are equal to each other and to* $E_n(1, \beta)$.

Notes. 1. We prove $\sum |a_m|^{-1/2m+1} = \infty$ in Appendix V.

2. We use the analyticity results of Loeffel *et al.* [37, 38], that E_n is analytic in the cut plane.

Technical Note. We will write $\rho(x)\, dx$ and $\operatorname{Im} E_n(1, -|\beta|)$ as if they were functions. Actually, they might only be measures in general (that $\operatorname{Im} E(1, -|\beta|)$ exists as a measure is a consequence of a theorem of Herglotz). If there are no natural boundaries at the cut, f is continuous at the cut.

For $j \geq 0$, this theorem is a special case of the more general result:

LEMMA IV.2.2. *Let* $f(z)$ *be a function with the following properties*:

(a) $f(z)$ *is analytic in the complex plane with the cut* $-\infty < z < 0$ *and is real on the positive real axis.*

[35] For a summary of the status of convergence results for field theories, see the introduction of [23].

(b) $|f(z)| < |z|^{\alpha}$ *for all* z *such that* $|z| > R_0$. *Here* $\alpha < k$, $k \geqslant 0$ *is integral, and* R_0 *is some positive number.*

(c) $f(z)$ *has an asymptotic expansion* $f(z) \sim \sum_{n=0}^{\infty} a_n z^n$ *valid uniformly in* $\arg(z)$.

(d) $\operatorname{Im} f$ *has limits above the cut along the negative real axis and* $\operatorname{Im} f \geqslant 0$ *there.*

Then

(i) *For* $n > k$, $-a_n$ *has Stieltjes form, i.e.,*

$$a_n = (-1)^{n+1} \int_0^{\infty} x^n \, d\phi \; n \geqslant k,$$

for a positive measure $d\phi$ *with finite moments when* $n \geqslant k$

(ii) *For any* $j > k - 1$, *the diagonal Padé approximants* $f^{[N,N+j]}$ *converge uniformly on compact subsets of the cut plane.*

(iii) *Moreover, if*

(e) $\sum_{n=k}^{\infty} a_n^{-1/2n} = \infty$, *then the limit of any of the diagonal sequences of* (ii) *is* f. *For* z *real and positive, the even* j *approximants converge monotonically upward to* $f(z)$ *and the odd* j *approximants converge monotonically downward to* $f(z)$.

Proof.

(1) *Reduction to* $k = 0$. Given f and $k > 0$, define

$$\tilde{f}(z) = (-z)^{-k}\left[f(z) - \sum_{n=0}^{k-1} a_n z^n\right].$$

Then $\tilde{f}$ obeys (a)–(d) with $k = 0$ and obeys (e) if f obeys (e). Since

$$f^{[N,N+j]}(z) = \sum_{n=0}^{k-1} a_n z^n + (-z)^k \tilde{f}^{[N,N+j-k]}(z)$$

for $j > k - 1$, the theorem need only be proven when $k = 0$. Henceforth, we thus suppose that $k = 0$.

(2) *Proof of the Stieltjes nature.* Consider

$$\frac{1}{2\pi i} \int_C z^{-n-1} f(z) \, dz = 0,$$

where C is the contour shown in Fig. 4, and n is an intger $n > 0$. Because of (b), the integrand on the large circle is bounded by $(R)^{-n-1+\alpha}$, so the contribution of the large circle goes to zero as $R \to \infty$ $(\alpha < 0)$. If we write $f(z) =$

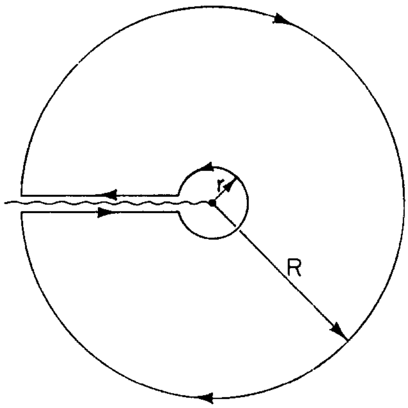

FIG. 4. A contour used in the proof of Lemma IV.2.2.

$[f(z) - \sum_{m=0}^{n} a_m z^m] + \sum_{m=0}^{n} a_m z^m$ for the integrand over the small circle and use the fact that the first term is $0(z^{n+1})$, we see that the little circle contributes a_n as $r \downarrow 0$. Thus,

$$a_n + \int_0^{-\infty} z^{-n-1}\pi^{-1} \operatorname{Im} f(z)\, dz = 0.$$

Letting $x = z^{-1}$ and

$$\rho(x) = (\pi x)^{-1} \operatorname{Im} f(-x^{-1}), \tag{IV.2}$$

we see $a_n = (-1)^{n+1} \int_0^\infty x^n \rho(x)\, dx$.

Note. (IV.2) *holds for* $n \geqslant k$ *even if* $k > 0$.

(3) (ii) *and the monotonicity parts of* (iii) *hold.* This is a general property of the Padé approximants of Stieltjes' series see, e.g., Baker [22].

(4) *If* (e) *holds, then all the Padé limits are* $f(z)$. The Padé approximants of $f^{[N,N+j]}(z)$ converge to functions $g_j(z)$, obeying (a)–(d). Thus,

$$a_n = (-1)^{n+1} \int_0^\infty x^n \rho_j(x)\, dx,$$

where $\rho_j(x) = (\pi x)^{-1} \operatorname{Im} g_j(-x^{-1})$. But, when (e) holds, the moment problem for $(-1)^{n+1} a_n$ has a unique solution by a theorem of Carleman [24]. Thus,

$$\operatorname{Im} g_j(-x^{-1}) = \operatorname{Im} f(-x^{-1})$$

all $x > 0$; this implies $f - g_j$ is an entire function with an asymptotic series $f(z) - g_j(z) \sim 0$ at $z = 0$; i.e., $f = g_j$. Q.E.D.

For $j < 0$, we note that the Padé's converge because $[E_n(\beta)]^{-1}$ obeys the conditions of the lemma with $k = 0$ and $(f^{-1})^{[N,M]} = (f^{[M,N]})^{-1}$.[36]

Many of the formal elements of this Padé proof carry over to a large class of singular perturbations. Let A and B be positive. Then $A + \beta B$ is a closed sectorial form for any β in the cut plane, i.e. in the cut plane $A + \beta B$ is a holomorphic family of type (b). Under additional assumptions one should be able to prove an asymptotic series exists. In any event eigenvalues are Herglotz. Thus, if one can show analytic continuation of some eigenvalue does not encounter natural boundaries or level crossing in the cut plane, one can prove the Padé approximants $f^{[N,N+j]}$ converge for j fixed $j \geq 0$ since any function Herglotz in the cut plane is bounded by $A \mid z \mid + B$.

IV.3. *Information about the Discontinuity*

We wish to collect here some information about the discontinuity. Since $\operatorname{Im} E_0(1, \beta) = \pi(-\beta)^{-1}\rho(-\beta^{-1})$, we can relate information about ρ to $\operatorname{Im} E$. We first recall that we have

PROPOSITION IV.3.1. $\operatorname{Im} E_0(1, \beta) \underset{\beta\to\infty}{\sim} C \mid \beta \mid^{1/3}$ *where* $C = E_0(0, 1)\sqrt{3}/2$.

Proof. As $\beta \to \infty$ in any direction, we know $E_0(1, \beta) \sim E_0(0, 1)\beta^{1/3}$ so $\operatorname{Im} E_0(1, \beta) \simeq E_0(0, 1) \mid \beta \mid^{1/3} \sin(\pi/3)$.

Note. Schwartz [25] has computed $E_0(0, 1)$ by a variational method and found

$$E_0(0, 1) = 1.0603621\ldots.$$

Thus,

$$C = .91830051\ldots.$$

Bender and Wu [5][37] on the basis of a numerical analysis, of the first 75 a_n conjecture an asymptotic form,

$$a_n \underset{n\to\infty}{\sim} (-1)^{n+1}\tfrac{1}{4}\pi^{-3/2}(\tfrac{3}{2})^{n+1/2}\Gamma(n + \tfrac{1}{2}), \tag{IV.3}$$

This has the integral form,

$$\begin{aligned} \mid a_n \mid &\sim \tfrac{1}{4}\pi^{-3/2}(\tfrac{3}{2})^{n+1/2} \int_0^\infty x^{n-1/2}e^{-x}\, dx \\ &= \tfrac{1}{4}\pi^{-3/2} \int_0^\infty y^n e^{-2y/3} y^{-1/2}\, dy. \end{aligned}$$

[36] We should like to thank Prof. D. Masson for pointing out this simple proof of convergence for the $j < 0$ case.

[37] Our normalization is different from that of Bender and Wu, explicitly $a_n^{us} = a_n^{BW}/2^{n-1}$.

Thus,

PROPOSITION IV.3.2. *If the asymptotic form* (IV.3) *of Bender–Wu is correct, then*

$$\operatorname{Im} E_0(1, \beta) \underset{(\beta \to 0^-)}{\longrightarrow} \tfrac{1}{4}\pi^{-1/2} \mid \beta \mid^{-1/2} e^{-2|\beta|^{-1}/3}.$$

Thus, solving for $E_0(1, \beta)$ in a dispersion form is equivalent to interpolating the $\beta \to -\infty$ and $\beta \to 0^-$ limits correctly.

IV.4. *Numerical Analysis*

Using Section IV.1 and the first 41 coefficients of the Rayleigh–Schrödinger series for $E_0(1, \beta)$, we have computed the first 20 Padé approximants of form $f^{[N,N]}$. In Table I, we list the $f^{[20,20]}$ for $\beta = 0.1, 0.2, ..., 1.0$ and compare it with rigorous upper and lower bounds as computed by variational methods by Bazley–Fox [26] and Reid [27]. In every case $f^{[20,20]}$ fits between the rigorous bounds.

In Table II, we list various Padé approximants to illustrate the rate of convergence. For $\beta = 0.1$, this rate is phenomenal; $f^{[5,5]}$ is accurate to one part in 10^5 (the figure of merit is $f^{[5,5]} - 1$ which is the total perturbation) and $f^{[12,12]}$ (which is not listed) to one part in 10^{11}! We note that the sum of the first 40 terms of the perturbation series at $\beta = 0.1$ (40 a_n's are involved in $f^{[20,20]}$) is $\sim 10^{24}$ so that we are not merely reproducing the asymptotic nature of the series. [We also note that for $\beta = 1$, the determinants of Section IV.1 for $f^{[20,20]}$ are of order 10^{377} (with ratio ~ 1!)].

To see this more explicitly, we compare, in Table V, the sum of the first $N + 1$ terms of the perturbation series with the Padé approximant that uses the same input coefficients. The convergence of one sequence and divergence of the other is dramatically demonstrated.

For $\beta > 1$, the rate of convergence is not as good. Since $f^{[N,N]} \to C_N$ as $\beta \to \infty$, while $E(1, \beta)/\beta^{1/3} =$ const, this is to be expected. But even for β as large as 15, there seems to be a limit approached for $N \sim 20$ (see Table III). In Table IV, we see that this limit is of the right size to be $E(1, \beta)$. For as $\beta \to \infty$,

$$E(1, \beta)/\beta^{1/3} \to 1.0603621...$$

monotonically from above.

Notes. 1. The astute reader will notice something awry in Table II. Since a_n is a Stieltjes series, the $f^{[N,N]}$ should increase monotonically as $N \uparrow$ and this is not true of $f^{[19,19]}, f^{[20,20]}$ for $\beta = 0.5$; 1.0. On the basis of further analysis, we attribute this to roundoff error in the evaluation of the determinants.

2. That $f^{[20,20]}(15)/15^{1/3} < 1.0603621$ while $E(1, \beta)/\beta^{1/3} > 1.0603621$ is not serious. For as $N \uparrow$ we expect $f^{[N,N]}(\beta) \uparrow E(1, \beta)$. Thus, we suspect

$$f^{[20,20]}(15) < E(1, 15)$$

by at least 5 %.

TABLE I

Comparison of Padé with Rigorous Bounds

β	Upper Bound[a]	Lower Bound[b]	$f^{[20,20]c}$
0.1	1.065286	1.065285	1.065285509543
0.2	1.118293	1.118292	1.1182926543(57)
0.3	1.164055	1.164041	1.164047156(234)
0.4	1.204848	1.204791	1.20481031(0603)
0.5	1.241957	1.241811	1.2418539(48135)
0.6	1.276195	1.275909	1.275983(105974)
0.7	1.308110	1.307324(†)	1.307747(246301)
0.8	1.338096	1.337397	1.33754(1726579)
0.9	1.366442	1.364349(†)	1.36566(2398911)
1.0	1.393371	1.392131	1.3923(37481861)

[a] From Bazley–Fox [26], Table I. A Rayleigh–Ritz method was used on the first five even-parity levels.

[b] From C. Reid [27], Table III except as noted by (†) which are taken from Bazley-Fox [26]. Reid uses the bracket method of Löwdin [31]. Bazley–Fox use the intermediate Hamiltonian method of Weinstein [35].

[c] We have thrown out the last 3 digits from a double precision answer assuming them insignificant because of roundoff error. The figures in parentheses represent digits that are not constant from $f^{[17,17]}$ on.

TABLE II

Rate of Convergence of Padé for β Small[a]

N	$\beta = 0.1$	$\beta = 0.5$	$\beta = 1.0$
1	1.063829787234	1.200000000000	1.272727272727
2	1.065217852490	1.231983691978	1.348289096707
3	1.065280680051	1.238985856539	1.373799864956
4	1.065285049128	1.240892502758	1.383756497228
5	1.065285455329	1.241496450913	1.388075603389
10	1.065285509535	1.241847634393	1.392102495074
15	1.065285509543	1.241853789165	1.392325157322
19	1.065285509543	1.241853988610	1.392341333864
20	1.065285509543	1.241853948135	1.392337481861

[a] We list $f^{[N,N]}$ with the last three double-precision figures suppressed.

TABLE III

Rate of Convergence for Intermediate β

β	$f^{[20,20]}$[a]	Δ (5)[b]	Δ (10)	Δ (15)
1	1.39234	0.0011	0.0006	0.00004
2	1.6071	0.0454	0.0056	0.00066
3	1.767	0.0850	0.0201	0.00157
4	1.897	0.1204	0.0256	0.00379
5	2.00(5)	0.1512	0.0348	0.00498
6	2.10(0)	0.1791	0.0473	0.00836
7	2.18(−2)	0.2034	0.0542	0.00763
8	2.25(0)	0.2240	0.0640	0.00920
9	2.31(3)	0.2290	0.0718	0.0107
10	2.37(0)	0.2555	0.0803	0.0124
11	2.4(21)	0.2660	0.0809	0.0134
12	2.4(68)	0.2779	0.0920	0.0150
13	2.5(11)	0.2886	0.0986	0.0159
14	2.5(57)	0.2999	0.1073	0.0218
15	2.5(90)	0.3088	0.1113	0.0220

[a] Figures not in ( ) appear to be convergent.

[b] $\Delta(x) = |f^{[x,x]} - f^{[20,20]}|/(f^{[20,20]} - 1)$ measures the rate of convergence relative to the perturbation.

TABLE IV

$E(0, 1)$ Test for β Intermediate

β	$f^{[20,20]}(\beta)/\beta^{1/3}$
1	0.1392
2	0.1276
3	0.1225
4	0.1195
5	0.1173
6	0.1155
7	0.1140
8	0.1127
9	0.1114
10	0.1102
11	0.1091
12	0.1081
13	0.1071
14	0.1062
15	0.1052

TABLE V

Comparison of the Perturbation Series and its Padé Approximants

N	$\sum_{n=0}^{N} a_n\beta^n$	$f^{[1/2N, 1/2N]}$
1	1.150000	
2	1.097500	1.111111
3	1.153750	
4	1.105372	1.117541
5	1.176999	
6	1.049024	1.118183
7	1.314970	
8	0.686006	1.118273
9	2.353090	
10	−2.442698	1.118288
11	13.253968	
12	−42.333586	1.118289
13	168.895730	
14	−796.466406	1.118289
15	3005.179546	
$\beta = 0.2$; $E(\beta) = 1.1182892...$		

APPENDIX I: Some Results from Complex Analysis

In this appendix we prove two results:

(a) An entire Herglotz function is linear.

(b) A Herglotz function analytic in a punctured neighborhood of $z = \infty$ has at worst a first-order pole there.

The first result is standard (c.f. [28]); we supply an elementary proof for the reader's convenience. While a cursory review of the complex analysis texts has not revealed a proof of (b), the result is sufficiently elementary that it is likely that it has have been previously discovered.

Definition. A function $f(z)$, analytic on a domain D, over **C** is called Herglotz if Im z and Im $f(z)$ always have the same sign.

Theorem A.1.1. *If $f(z)$ is Herglotz and entire, then $f(z) = az + b$.*

Proof. Since f is entire, it has an everywhere convergent expansion $f(z) = \sum_{n=0}^{\infty} a_n z^n$. Since f is Herglotz, it is real for z real and thus all the a_n are real. Thus, $g_r(\theta) \equiv \operatorname{Im} f(re^{i\theta}) = \sum_{n=0}^{\infty} a_n r^n \sin n\theta$. f is Herglotz also implies that

$$\begin{aligned} g_r(\theta) &\geqslant 0 \qquad 0 < \theta < \pi \\ &\leqslant 0, \qquad -\pi < \theta < 0. \end{aligned}$$

Consider the function $\sin m\theta/\sin\theta$ $(m > 1)$ defined on the interval $(-\pi, \pi)$. Since it is continuous, its absolute value is bounded by some constant C_m. Thus,

$$C_m \sin\theta \pm \sin m\theta \geqslant 0 \qquad 0 < \theta < \pi$$
$$\leqslant 0 \qquad -\pi < \theta < 0,$$

so that

$$0 \leqslant \frac{1}{2\pi}\int_{-\pi}^{\pi} g_r(\theta)[C_m \sin\theta \pm \sin m\theta] = C_m a_1 r \pm a_m r^m.$$

For this to hold for all r and for both $+$ and $-$, we must have $a_m = 0$ for $m > 1$, i.e.,

$$f(z) = a_0 + a_1 z. \qquad \text{Q.E.D.}$$

This result can be extended to limit the form of meromorphic Herglotz functions (cf. [28]). Wigner has used this extended form to study the R matrix [29]. We next extend our result to functions analytic near ∞.

THEOREM A.I.2. *If $f(z)$ is analytic in a punctured neighborhood of infinity and Herglotz, its Laurent series at ∞ is of the form,*

$$f(z) = \sum_{n=-\infty}^{1} a_n z^n.$$

Proof. If we expand f in Laurent series near ∞ to obtain $f(z) = \sum_{n=-\infty} a_n z^n$ and proceed analogously to our proof of Theorem A.I.1, we see

$$0 \leqslant C_m(a_1 r - a_{-1} r^{-1}) \pm (a_m r^m - a_{-m} r^{-m}),$$

for all r very large. This implies $a_m = 0$ for $m > 1$. Q.E.D.

APPENDIX II: THE ASYMPTOTIC NATURE OF PERTURBATION THEORY

In this appendix, we will prove that the ordinary Rayleigh–Schrödinger perturbation series for the eigenvalues of the Hamiltonian $H(\beta) = p^2 + x^2 + \beta x^4$ is an asymptotic series for the actual eigenvalues $E_n(1, \beta)$. Results of this sort were first derived by Kato [15], (see also [1] p. 443–451) under very general conditions. The Hamiltonian under consideration obeys Kato's conditions and so it is for two reasons we include this appendix at all:

(1) Kato's argument is quite complicated because it considers a very general case; for the problem under consideration, we can avoid relying on either spectral analysis, or lower bounds on eigenvalues—two advanced tools used by Kato.

(2) We wish to sketch a detailed analysis of the errors in the asymptotic expansion. We do not however suggest using these methods for practical computation. One can do much better with variational upper bounds coupled with Temple's inequality [30]. For truly impressive accuracy, one can use intermediate Hamiltonians [26] or *bracketing* techniques [27, 31].

We also remark that the question of asymptotic expansions and perturbation theory has been treated more recently by Kreiger [32]. He provides a clever proof of the asymptotic nature of the series *once it is known that the eigenvectors are continuous* (i.e., once it is known that the series are 0-th order asymptotic). Unfortunately, he never makes this continuity assumption explicit and thus his basic theorem is lacking a condition which would enable one to prove this assumption. As we shall see, it is sufficient (modulo some technical domain assumptions) for the potential to be bounded below for continuity to hold.

THEOREM A.II.1. *Let $\phi_n(\beta)$, $E_n(\beta)$ be the $n + 1^{\text{st}}$ normalized eigenvector and eigenvalue, respectively, where the phase of $\phi_n(\beta)$ is determined by requiring $\langle\phi_n(\beta), \phi_n(0)\rangle \geqslant 0$. Then as $\beta \downarrow 0$, $E_n(\beta) \to E_n(0)$ and $\phi_n(\beta) \to \phi_n(0)$, where the vector convergence is in the norm topology.*

Proof. Let us first consider the case $n = 0$; since $C_0 \equiv \langle\phi_0(0), x^4\phi_0(0)\rangle < \infty$, we have

$$\begin{aligned}
E_0(0) &= \langle\phi_0(0), H(0)\, \phi_0(0)\rangle \\
&\leqslant \langle\phi_0(\beta), H(0)\, \phi_0(\beta)\rangle \qquad \text{(by the variational principle)} \\
&\leqslant \langle\phi_0(\beta), H(\beta)\, \phi_0(\beta)\rangle = E_0(\beta) \qquad (\text{for } H(\beta) \geqslant H(0)) \\
&\leqslant \langle\phi_0(0), H(\beta)\, \phi_0(0)\rangle \qquad \text{(by the variation principle)} \\
&\leqslant E_0(0) + \beta C_0 .
\end{aligned}$$

Thus,

$$E_0(0) \leqslant \langle\phi_0(\beta), H(0)\, \phi_0(\beta)\rangle \leqslant E_0(\beta) < E_0(0) + \beta C_0 .$$

As a result $E_0(\beta) \to E_0(0)$ and

$$\langle\phi_0(\beta), H(0)\, \phi_0(\beta)\rangle \to E_0(0).$$

But

$$\langle\phi_0(\beta), H(0)\, \phi_0(\beta)\rangle \geqslant E_0(0)\, |\langle\phi_0(\beta) \mid \phi_0(0)\rangle|^2 + E_1(0)(1 - |\langle\phi_0(\beta) \mid \phi_0(0)\rangle|^2);$$

so, we must have $|\langle\phi_0(\beta) | \phi_0(0)\rangle|^2 \to 1$, which implies $\phi_0(\beta) \to \phi_0(0)$ when the phase condition is taken into account.

We next show that $E_m(\beta) \to E_m(0)$. Let us adopt the abbreviation $E_m \equiv E_m(0)$; $\phi_m = \phi_m(0)$. Let P_m be the projection onto the subspace $[\phi_0, ..., \phi_{m-1}]$. Since βx^4 is a positive perturbation, we have $E_m(\beta) \geqslant E_m$. On the other hand, the min-max principle assures us that $E_m(\beta) \leqslant \tilde{E}_m(\beta)$, where $\tilde{E}_m(\beta)$ is the largest eigenvalue of the matrix $(a_{ij})_{i,j=0,...,m}$ with

$$a_{ij} = E_i \delta_{ij} + \beta \langle \phi_i, x^4 \phi_j \rangle$$

[i.e., a_{ij} is the matrix of $P(H_0 + \beta x^4)P$]. Since $\tilde{E}_m(\beta) \to E_m$, we have $E_m(\beta) \to E_m$.

Finally, we prove $\phi_n(\beta) \to \phi_n$ under the inductive assumption that $\phi_i(\beta) \to \phi_i$ ($i = 0, ..., n-1$). Let $Q = 1 - P_n$ and define

$$\tilde{\phi}_n(\beta) = Q\phi_n(\beta) = \phi_n(\beta) - \sum_{i=0}^{n-1} \langle \phi_i, \phi_n(\beta) \rangle \phi_i .$$

Then, from the inductive hypothesis and the orthogonality relations

$$\langle \phi_i(\beta), \phi_n(\beta) \rangle = 0 \qquad (i \neq n),$$

we have

$$\lim_{\beta \downarrow 0} \| \tilde{\phi}_n(\beta) - \phi_n(\beta) \| = 0. \tag{A.II.1}$$

Since $Q\tilde{\phi}_n(\beta) = \tilde{\phi}_n(\beta)$, we have (by the variational principle)

$$E_n \| \tilde{\phi}_n(\beta) \|^2 \leqslant \langle \tilde{\phi}_n(\beta), H_0 \tilde{\phi}_n(\beta) \rangle \leqslant \langle \tilde{\phi}_n(\beta), H(\beta) \tilde{\phi}_n(\beta) \rangle \tag{A.II.2}$$

We will shortly show that

$$\lim_{\beta \downarrow 0} \{ \langle \tilde{\phi}_n(\beta), [H(\beta) - E_n] \tilde{\phi}_n(\beta) \rangle \} = 0. \tag{A.II.3}$$

From (A.II.2–3), we can conclude that $\langle \tilde{\phi}_n(\beta), H_0 \tilde{\phi}_n(\beta) \rangle \to E_n \| \tilde{\phi}_n(\beta) \|^2$. Then writing $\tilde{\phi}_n(\beta) = a(\beta) \phi_n + \phi_{\text{rem}}$ with $\langle \phi_{\text{rem}}, \phi_n \rangle = 0$, we obtain $\phi_{\text{rem}} \to 0$ from which it follows that $\tilde{\phi}_n(\beta) \to \phi_n$. Thus, (A.II.1) completes the proof that $\phi_n(\beta) \to \phi_n$.

All that remains is to prove (A.II.3). This follows from a computation

$$\begin{aligned} \langle \tilde{\phi}_n(\beta), [H(\beta) - E_n(\beta)] \tilde{\phi}_n(\beta) \rangle &= \langle \tilde{\phi}_n(\beta), H(\beta) \tilde{\phi}_n(\beta) \rangle - E_n(\beta) \langle \tilde{\phi}_n(\beta), \phi_n(\beta) \rangle \\ &= \langle \tilde{\phi}_n(\beta), H(\beta) [\tilde{\phi}_n(\beta) - \phi_n(\beta)] \rangle. \end{aligned}$$

Thus,

$$\begin{aligned} &| \langle \tilde{\phi}(\beta), [H(\beta) - E_n(\beta)] \tilde{\phi}_n(\beta) \rangle | \\ &\quad \leqslant \| H(\beta) [\tilde{\phi}_n(\beta) - \phi_n(\beta)] \| \\ &\quad \leqslant \sum_{i=0}^{n-1} | \langle \phi_i, \phi_n(\beta) \rangle | \, \| H(\beta) \phi_i \| \to 0. \end{aligned}$$

This last result along with $E_n(\beta) \to E_n$ implies (A.II.3) and, thus, our theorem. Q.E.D.

We remark that Theorem A.II.1 depended on

(a) V is bounded below;
(b) The eigenfunctions $\phi_n(0) \in D(V)$;
(c) Discreteness of the spectrum of $H(\beta)$ (if there are n discrete levels at the bottom, we can show that the first n eigenvalues and eigenvectors are continuous).

THEOREM A.II.2. *The Rayleigh–Schrödinger series for the eigenvalues (and eigenvectors) are asymptotic.*

Proof. Let us change notation and abbreviate $\phi_n(0) \equiv | n\rangle$ and let $\psi_m(\beta)$ be the multiple of $\phi_m(\beta)$ with $\langle \psi_m(\beta) | m\rangle = 1$ (this can be done for β sufficiently small). Let

$$a_n^{(m)}(\beta) = \langle n | \psi_m(\beta)\rangle, \qquad \text{so} \qquad \psi_m(\beta) = \sum_n a_n^{(m)}(\beta) \, | n\rangle.$$

Let us hold m fixed and suppress it as an index where no confusion will result. The basic equation,

$$E_m(\beta)\, \psi_m(\beta) = (H_0 + \beta V)\, \psi_m(\beta), \tag{A.II.4}$$

yields

$$E_m(\beta) = E_m + \beta\langle m | V | m\rangle + \beta \sum_{k\neq m} \langle m | V | k\rangle\, a_k(\beta) \tag{A.II.5}$$

and

$$a_n(\beta)\, E_m(\beta) = a_n(\beta)\, E_n + \beta\langle n | V | m\rangle + \beta \sum_{k\neq m} \langle n | V | k\rangle\, a_k(\beta). \tag{A.II.6}$$

Using (A.II.5) in (A.II.6) yields

$$a_n(\beta) = \frac{\beta}{E_m - E_n}\Big[\langle n | V | m\rangle + \sum_{k\neq m} \langle n | V | k\rangle\, a_k(\beta) - a_n(\beta)\langle m | V | m\rangle - a_n(\beta) \sum_{k\neq m} a_k(\beta)\langle m | V | k\rangle\Big]. \tag{A.II.7}$$

We remark that all the sums in (A.II.5–7) are finite since $\langle m | V | k\rangle = 0$ if $| k - m | > 4$. The familiar R–S series come from formal iteration of (A.II.7) and substitution into (A.II.5).

By iterating (A.II.7) we obtain

$$a_n(\beta) = \text{first } N \text{ terms of the R–S series} + \beta^N[R_N(\beta)], \tag{A.II.8}$$

where $R_N(\beta)$ is a finite sum of $a_k(\beta)$ $(k \neq m)$. Thus, $R_N(\beta) \to 0$ as $\beta \downarrow 0$ by Theorem A.II.1. As a result, the R–S series for the $a_n(\beta)$ and thus by (A.II.5) for $E_m(\beta)$ are asymptotic. Q.E.D.

Finally, let us briefly indicate how one can majorize the error in taking the first N terms of the asymptotic series. The method that we sketch is far from optimal, and it is clear that with some care it can be improved. For simplicity, we only consider the ground-state energy. By (A.II.5), the error in $E_0(\beta)$ can be majorized if we know the errors $a_k^{(0)}(\beta)$ to order $\beta^{N-1}(k = 2, 4)$. These, in turn, can be found if we know the zeroth order errors in $a_k^{(0)}(\beta)$ for $k \leqslant 4N$ since the iteration of (A.II.7) yields these errors in terms of $(a_k^{(0)}(\beta))$. But

$$| a_m^{(0)}(\beta)|^2 = |\langle \psi_0(\beta) \mid m\rangle|^2 = \frac{|\langle \phi_0(\beta) \mid m\rangle|^2}{\langle \phi_0\beta \mid \rangle|^2}$$
$$\leqslant \frac{1 - |\langle \phi_0(\beta) \mid 0\rangle|^2}{|\langle \phi_0(\beta) \mid 0\rangle|^2}.$$

This last factor can be majorized by using

$$\beta\langle 0 \mid x^4 \mid 0\rangle \geqslant (E_2 - E_0)(1 - |\langle \phi_0(\beta) \mid 0\rangle|^2),$$

a result implicit in the proof of Theorem A.II.1.

Note. After the completion of this appendix, we found a recent paper by Greenlee [36] containing similar techniques.

Appendix III: Some Sum Rules

In this appendix, we present a set of sum rules for the hydrogen (and any other) atom. They represent higher-order virial theorems in the following exact sense: The virial theorem is a statement about first-order changes under scaling transformations [the $x(d/dx)$ of the commutator in the derivation of the Virial theorem is the generator of scaling transformations]; these sum rules involve higher order changes under scaling. Since off-diagonal matrix elements of $1/r$ (or equivalently of p^2) between $\Delta l = 0$ states are involved, these sum rules do not appear to be of direct interest and are presented mainly as a curiosity.

Let

$$H = \frac{p^2}{2m} - \frac{ze^2}{r}; \qquad V = \frac{ze^2}{r}.$$

Let $E_{n,l}(\lambda)$ be the energy state of $H + \lambda V$ with quantum numbers n, l; and let $| n, l\rangle$; $E_{n,l}$ be the unperturbed states and energies. Then

$$E_{n,l}(\lambda) = -[C(\lambda - 1)^2/n^2], \qquad (\lambda < 1)$$

with $C = m(ze^2)^2/2h^2$. In terms of a perturbation expansion,

$$E_{n,l}(\lambda) = \sum_{m=0}^{\infty} a_{n,l}^{(m)} \lambda^m.$$

Thus, we conclude

$$a_{n,l}^{(0)} = -\frac{C}{n^2}, \tag{A.III.0}$$

$$a_{n,l}^{(1)} = \frac{2C}{n^2}, \tag{A.III.1}$$

$$a_{n,l}^{(2)} = -\frac{C}{n^2}, \tag{A.III.2}$$

$$a_{n,l}^{(m)} = 0, \qquad (m \geqslant 3). \tag{A.III.m}$$

(A.III.0) is a triviality and (A.III.1) is the virial theorem result. (A.II.m), $m \geqslant 2$ are the new sum rules. Explicitly,

$$\sum_{n' \neq n} \frac{|\langle n, l \mid r^{-1} \mid n', l\rangle|^2}{E_{n',l} - E_{n,l}} = \frac{m}{2h^2} \frac{1}{n^2} \tag{A.III.2}$$

$$\sum_{\substack{n' \neq n \\ n'' \neq n}} \frac{\langle n, l \mid r^{-1} \mid n', l\rangle\langle n', l \mid r^{-1} \mid n'', l\rangle\langle n'', l \mid r^{-1} \mid n, l\rangle}{(E_{n',l} - E_{n,l})(E_{n'',l} - E_{n,l})}$$

$$= \frac{2}{n^2}\left(\frac{mze^2}{2h^2}\right) \sum_{n' \neq n} \frac{|\langle n, l \mid r^{-1} \mid n', l\rangle|^2}{(E_{n',l} - E_{n,l})^2} \qquad \text{etc.,} \tag{A.III.3}$$

APPENDIX IV: Construction of the Subdominant Solutions (by A. Dicke)

We wish to show the existence of a solution to the differential equation,

$$L(\alpha, \lambda)\psi = -\psi'' + (x^4 + \alpha x^2 - \lambda)\psi = 0 \qquad x \in R; \quad \alpha, \lambda \in \mathbf{C}, \tag{A.IV.1}$$

which falls off fast (together with its derivative) as $x \to \infty$ and which is entire in α and λ for any x. Sibuya and Hsieh [16] have found the behavior of *subdominant solutions* at ∞ for a more general class of equations. The present approach, suggested by V. Bargmann,[38] provides a simpler route to the result for the present case, and more readily displays the analyticity in the parameters.

[38] V. Bargmann, private communication.

We rewrite (A.IV.1) in matrix form,

$$y' = A(\alpha, \lambda)\, y; \quad y = \begin{pmatrix} \psi \\ \psi' \end{pmatrix}; \quad A = \begin{pmatrix} 0 & 1 \\ x^4 + \alpha x^2 - \lambda & 0 \end{pmatrix}. \qquad \text{(A.IV.2)}$$

An equation,

$$w' = -Mw, \qquad \text{(A.IV.3)}$$

can be solved by a Neumann series,

$$w(x) = \sum_N w_N(x)$$

and (A.IV.4)

$$w_N(x) = \int_x^{+\infty} M w_{N-1}\, dx'; \quad w_0 = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

provided the integrals and series converge. This is not the case for (A.IV.2) as it stands. However, let us make an x-dependent change of basis choosing $\binom{\phi_\pm}{\phi_\pm'}$ as a new basis where $\binom{\phi_\pm}{\phi_\pm'}$ are analytic and linearly independent when the variables and parameters are in a domain G. Let

$$U = \begin{pmatrix} \phi_- & \phi_+ \\ \phi_-' & \phi_+' \end{pmatrix}$$

Then U^{-1} and U' exist on G and Eq. (A.IV.2) transforms to

$$a' = -Ra, \qquad \text{(A.II.5)}$$

where $a = \binom{a_-}{a_+} = U^{-1}y$ and $R = U^{-1}(U'U^{-1} - A)U$; a and R are analytic in G. A suitable choice of $\phi_\pm$ may lead to the convergence of the Neumann series; in this case,

$$\psi(x) = a_-(x)\, \phi_-(x) + a_+(x)\, \phi_+(x)$$

and (A.IV.6)

$$\psi'(x) = a_-(x)\, \phi_-'(x) + a_+(x)\, \phi_+'(x)$$

is a solution of (A.IV.2). If, in addition

$$a_-(x) = 1 + o(1)$$
$$a_+(x) = 0(|\, \phi_-/\phi_+ \,|) \qquad x \to \infty, \qquad \text{(A.IV.7)}$$

then

$$\psi(x) \sim \phi_-(x) \qquad x \to \infty \qquad \text{(A.IV.8)}$$

This suggests we take for ϕ_- a guess for the asymptotic behavior of the small solution. Since the exact solutions have $R = 0$, we take for ϕ_+ a guess for the asymptotic behavior of the large solution. Letting $y = \exp[\int g(x)\,dx]$, (A.IV.1) becomes

$$g' + g^2 = x^4 + \alpha x^2 - \lambda, \tag{A.IV.9}$$

which is solved by

$$g = \pm x^2 \pm \tfrac{1}{2}\alpha - x^{-1} + 0(x^{-2}). \tag{A.IV.10}$$

Thus, we make the choices

$$\phi_\pm(x) = \exp(\pm\tfrac{1}{3}x^3 \pm \tfrac{1}{2}\alpha x - \log x). \tag{A.IV.11}$$

Let $G_m = \{(x, \alpha, \lambda) \mid x > m, \mid \alpha \mid < m, \lambda \in \mathbf{C}\}$ with $m = 1, 2, \ldots$. Straightforward calculation show that $(\phi_\pm, \phi_\pm')$ are independent on G_m and that (on G_m):

$$\mid R_{--} \mid < Cx^{-2}; \qquad \mid R_{-+} \mid < Cx^{-2} \exp(\tfrac{2}{3}x^3 + \mathrm{Re}\,\alpha x)$$

and

$$\mid R_{+-} \mid < Cx^{-2} \exp(-\tfrac{2}{3}x^3 - \mathrm{Re}\,\alpha x); \qquad \mid R_{++} \mid < Cx^{-2}, \tag{A.IV.12}$$

where C is larger than

$$\tfrac{1}{2}[\tfrac{1}{4} \mid \alpha \mid^2 + \mid \lambda \mid + \mid \alpha \mid + 2].$$

The integrals (A.IV.4) of the Neumann series will exist if $a_{+,0} = 0$ (which is needed to kill R_{-+} contributions). In fact,

$$\begin{aligned} \mid a_{-,n}(x) \mid &< \tfrac{1}{2}(n!)^{-1}(2Cx^{-1})^n \\ \mid a_{+,n}(x) \mid &< \tfrac{1}{2}(2Cx^{-1})^n \exp(-\tfrac{2}{3}x^3 - \mathrm{Re}\,\alpha x). \end{aligned} \tag{A.IV.13}$$

Thus,

$$\mid a_- \mid < 1 + 0\left(\frac{1}{x}\right)$$

and

$$\mid a_+ \mid < 0\left(\frac{1}{x}\right) \exp(-\tfrac{2}{3}x^3 - \mathrm{Re}\,\alpha x) = 0\left(\frac{1}{x}\right) \mid \phi_-/\phi_+ \mid, \tag{A.IV.14}$$

as desired.

Thus, for any $m = 1, 2, \ldots$, we have constructed a solution, $\psi_\infty(x, \alpha, \lambda)$, of (A.IV.1) so that $\psi_\infty(x, a_0, \lambda_0) = 0\,[\exp(-\tfrac{1}{3}x^3 - \tfrac{1}{2}x - \log x)]$ $(x \to \infty)$ and which is analytic (in α and λ) on G_m. To extend ψ_∞ to all x, α, λ, we proceed as follows:

Let $\psi_0(x, \alpha, \lambda)$ and $\psi_E(x, \alpha, \lambda)$ be the odd and even solutions of (A.IV.1), i.e.,

$$\psi_0(0) = 0, \qquad \psi_0'(0) = 1,$$
$$\psi_E(0) = 1, \qquad \psi_E'(0) = 0.$$

An elementary computation shows that the power series solutions for ψ_0 and ψ_E converge to functions entire in α, λ and x. Moreover, since $W(\psi_0, \psi_E) = 1$,

$$\psi_\infty = W(\psi_\infty, \psi_E)\, \psi_0 - W(\psi_\infty, \psi_0)\, \psi_E . \tag{A.IV.15}$$

By the analyticity in G_m, $W[\psi_\infty, \psi_{E(0)}]$ is an entire function in α and λ (it is independent of x!) so ψ_∞ is entire in α, λ, x.

APPENDIX V: Upper Bounds on the Perturbation Series for $E_0(1, \beta)$

We wish to establish here the following.

THEOREM A.IV.1. *Let $f(\beta)$ be the ground-state energy for $p^2 + x^2 + \beta x^4$. Let $f(\beta) = \sum_{n=0}^{\infty} E_n \beta^n$ be the (formal) Rayleigh–Schrödinger series. Then*

$$| E_n | < P^n Q n^n, \tag{A.V.1}$$

for some real numbers P and Q. In particular,

$$\sum_{n=0}^{\infty} | E_n |^{-1/2n+1} = \infty. \tag{A.V.2}$$

Remarks. 1. This is an improvement of the bound $| E_n | < CD^n \Gamma(\frac{5}{2}n)$ established by Bender and Wu [5],[39] which is not good enough to prove (A.V.2).

2. A numerical analysis of the first $75 E_n$ by Bender and Wu [5] suggest E_n is asymptotically of the form $CD^n n^{n+1/2}$ [with C, D, explicitly computed].

The proof is based on the iterative formulas for E_n, namely (A.II.5) and (A.II.7). If we expand $a_m(\beta) = \sum_n \beta^n a_m^{(n)}$, we find

$$E_n = \sum_{k \neq 0} \langle 0 \mid V \mid k \rangle\, a_k^{(n-1)}; \qquad n > 2, \tag{A.V.3}$$

$$a_m^{(n)} = \sum_{k \neq 0} a_k^{(n-1)} \frac{\langle m \mid V \mid k \rangle}{E_0 - E_m} - a_m^{(n-1)} \frac{\langle 0 \mid V \mid 0 \rangle}{E_0 - E_m}$$

$$- \sum_{j=1}^{n-2} a_m^{(j)} \sum_{k \neq 0} a_k^{(n-1-j)} \frac{\langle 0 \mid V \mid k \rangle}{E_0 - E_m}; \qquad n > 2, \tag{A.V.4}$$

[39] Bender and Wu announced a $CD^n\Gamma(n)$ bound [equivalent to (A.V.1)] in their letter [4] but their lengthier paper only establishes the $\Gamma(\frac{5}{2}n)$ result.

and

$$a_m^{(0)} = 0; \quad a_m^{(1)} = [\langle m \mid V \mid 0\rangle(E_0 - E_m)^{-1}]. \tag{A.V.5}$$

For the case at hand, we notice that

$$\langle m \mid V \mid k\rangle = 0 \qquad \textit{unless} \quad \mid m - k \mid = 0, 2, 4 \tag{A.V.6}$$

and

$$\langle m \mid V \mid k\rangle(E_k - E_0)^{-1} < Ck \qquad (k \neq 0), \tag{A.V.7}$$

for all m. It follows from (A.V.6) and (A.V.4) that

$$a_k^{(n)} = 0, \qquad \text{if} \qquad k > 4n. \tag{A.V.8}$$

We then have

LEMMA A.V.2. *There is a P and an L so that*

$$\mid a_k^{(n)} \mid \leqslant LP^n n^n \qquad (\textit{all } k). \tag{A.V.9}$$

Proof. Pick L so that $\mid a_k^{(1)} \mid \leqslant L$, $k \leqslant 4$ (and thus for all k). Choose $P > 20 + D + 2DL$ where D is chosen so that $|\langle 0 \mid V \mid m\rangle(E_0 - E_k)^{-1} \mid < D$, all m, k. Suppose (A.V. 9) has been established for $n < N$. Then the induction hypothesis and (A.V.6–7) imply $a_m^{(n)} = 0$, or

$$\left| \sum_{k \neq 0} a_k^{(n-1)}[\langle m \mid V \mid k\rangle(E_0 - E_m)^{-1}] \right| < (5C)(m)\, LP^{n-1}(n-1)^{n-1}$$
$$< (5C)(4n)\, LP^{n-1}(n-1)^{n-1} < 20LP^{n-1}n^n.$$

We have used (A.V.8) to conclude $m < 4n$.

We next have that

$$a_m^{(n-1)}\langle 0 \mid V \mid 0\rangle(E_0 - E_m)^{-1} \langle LDP^{n-1}n^n;$$

and finally, that

$$\sum_{j=1}^{n-2} a_m^{(j)} \sum_{k \neq 0} a_k^{(n-1-j)}\langle 0 \mid V \mid k\rangle(E_0 - E_m)^{-1}$$
$$\leqslant 2(n-3)\, L^2P^{n-1}n^{n-1}D \leqslant 2DL^2P^{n-1}n^n,$$

since we have $(n - 3)$ terms each bounded by $2D(LP^j j^j)[L(n-1-j)^{n-1-j}]$. These last three inequalities and (A.V.4) imply

$$\mid a_m^{(n)} \mid \leqslant LP^{n-1}n^n(20 + D + 2DL) \leqslant LP^n n^n. \qquad \text{Q.E.D.}$$

Proof of Theorem A.IV.1. By (A.V.3) and Lemma A.V.2,

$$E_n \leqslant ([\langle 0 \mid V \mid 2\rangle + \langle 0 \mid V \mid 4\rangle] L) P^n n^n.$$

To prove (A.V.2), we note that (A.V.1) implies

$$\mid E_n \mid^{-1/2n+1} > P^{-n/2n+1} Q^{-1/2n+1} n^{-n/2n+1} > \tfrac{1}{2} P^{-1/2} n^{-1/2}$$

for n sufficiently large. Since $\sum n^{-1/2} = \infty$, (A.V.2) is proven. Q.E.D.

Notes. 1. This proof holds for excited states, as well as the ground states.

2. For a Hamiltonian $p^2 + x^2 + \beta x^{2m}$, this method yields $\mid E_n \mid \leqslant P^n Q n^{(m-1)n}$. Thus, we can prove (A.V.2) for x^6 perturbations. If the bound analogous to (A.V.1) is "best possible" in this x^{2m} case, then (A.V.2) is false for x^8, x^{10},..., perturbations and the proof of uniqueness of the Padé approximants breaks down in these cases.

Notes added in proof: 1. The author should like to thank J. Loeffel for pointing out an error in the original manuscript.

2. Computations (but no proofs!) similar to those made in Section IV.4 (although to lower order in N) have appeared in the chemical physics literature: C. Reid, *Int. J. Quan. Chem.* **1** (1967), 521; C. Rousseau, *Bull. A. P. S.* **13** (1968), 25.

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